Contents

E. L. Pop, D. Duca, A. Rațiu, Properties of the intermediate point from a mean value theorem of the integral calculus .................. 3
S. Deshwal, P.N. Agrawal, Miheșan-Kantorovich operators of blending type ................................................................. 11
A. E. Bără, Some families of rational Heun functions and combinatorial identities .......................................................... 29
F. Nasaireh, Voronovskaja-type formulas and applications ........ 37
D. Foukrach, On nonexistence of solutions to a nonlinear Cauchy problem for a higher order hyperbolic equation ............... 45
P. Agarwal, C. K. Goel, Extremal sets in a topological space .... 53
A. Hassanzadeh, A Young’s inequality for the Sugeno integrals .. 61
A. Kajla, Blending type approximation by summation-integral operators based on Polya distribution ................................. 69
E. P. Mazi, T. O. Opoola, On some subclasses of bi-univalent functions associating pseudo-starlike functions with Sakaguchi type functions ................................................................. 85
V. Neagose, A note on the Pompeiu-Stamate mean-value theorem .. 97
M. Iranmanesh, A. G. Sanatee, Lattice $g$-2-normed spaces and 2-best approximation properties of their downward subsets .......... 105
E. Szatmari, Á. O. Páll-Szabó, Differential subordination results obtained by using a new operator ................................. 119
A. Vernescu, About the sequence of general term $\Omega_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$ .................................................. 133
M. F. Causley, P. Morell, Sequences which converge to e: New insights from an old formula .............................................. 159
V. Lokesha, S. Jain, T. Deepika, K.M. Devendraiah, Some computational aspects of polycyclic aromatic hydrocarbons ....... 175
Properties of the intermediate point from a mean value theorem of the integral calculus

Emilia-Loredana Pop, Dorel Duca, Augusta Rațiu

Abstract

If the functions \( f, g : [a, b] \to \mathbb{R} \) satisfy the conditions:

(a) \( f \) and \( g \) are continuous on \( [a, b] \),
(b) the function \( f \) is decreasing on \( [a, b] \),
(c) \( f(x) \geq 0 \), for all \( x \in [a, b] \),

then there exists a point \( c \in [a, b] \) such that

\[
\int_a^b f(x) g(x) \, dx = f(a) \int_a^c g(x) \, dx.
\]

In this paper, we study the approaching of the point \( c \) towards \( a \), when \( b \) approaches \( a \).

2010 Mathematics Subject Classification: 26A24.

Key words and phrases: intermediate point, mean-value theorem.

1 Introduction and Preliminaries

In the proof of the second mean-value theorem of integral calculus (Bonnet’s theorem) one uses the following theorem.

**Theorem 1** Let \( a, b \in \mathbb{R} \) with \( a < b \) and \( f, g : [a, b] \to \mathbb{R} \). If

(a) the function \( f \) is decreasing on \( [a, b] \),
(b) \( f(x) \geq 0 \), for all \( x \in [a, b] \),
(c) the function \( g \) is Riemann integrable on \( [a, b] \),

then the function \( fg \) is Riemann integrable on \( [a, b] \) and there exists a point \( c \in [a, b] \) such that

\[
\int_a^b f(x) g(x) \, dx = f(a) \int_a^c g(x) \, dx.
\]

1Received 1 September, 2017
Accepted for publication (in revised form) 2 October, 2017
For the proof of this theorem, one can see [2].

It follows that if the continuous functions \( f, g : [a, b] \to \mathbb{R} \) satisfy the following properties:

(a) the function \( f \) is decreasing on \([a, b]\),  
(b) \( f(x) \geq 0 \), for all \( x \in [a, b] \),

then, for each \( x \in (a, b) \), there exists a point \( c_x \in [a, x] \) such that

\[
\int_{a}^{x} f(t) g(t) \, dt = f(a) \int_{a}^{c_x} g(t) \, dt.
\]

If for each \( x \in (a, b) \) there exists a unique point \( c_x \in [a, x] \) such that (1) holds, we can define the function \( c : (a, b) \to [a, b] \) by

\[
(1) \quad c(x) = c_x, \text{ for all } x \in (a, b).
\]

The function \( c \) has the property that

\[
(3) \quad \int_{a}^{x} f(t) g(t) \, dt = f(a) \int_{a}^{c(x)} g(t) \, dt, \text{ for all } x \in (a, b).
\]

If for some \( x \in (a, b) \), there exist more points \( c_x \in [a, x] \) such that (3) is true, then for each \( x \in (a, b) \) choose \( c_x \in [a, x] \) which satisfies (3). It follows that we can also define the function \( c : (a, b) \to [a, b] \) by formula (2). This function \( c \) satisfies (3), too.

Consequently, the following statement is true.

\textbf{Theorem 2} Let \( a, b \in \mathbb{R} \) with \( a < b \). If the continuous functions \( f, g : [a, b] \to \mathbb{R} \) satisfy properties:

(a) the function \( f \) is decreasing on \([a, b]\),  
(b) \( f(x) \geq 0 \), for all \( x \in [a, b] \),

then there exists a function \( c : (a, b) \to [a, b] \) such that (3) is true.

If \( x \in (a, b) \) tends to \( a \), because \( |c(x) - a| \leq |x - a| \), we have

\[
\lim_{x \to a} c(x) = a.
\]

Then the function \( \overline{c} : [a, b] \to [a, b] \) defined by

\[
\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in (a, b) \\ a, & \text{if } x = a. \end{cases}
\]

is continuous at \( x = a \).

The purpose of this paper is to establish under which circumstances the function \( \overline{c} \) is the third order differentiable at the point \( x = a \) and to compute its derivatives \( \overline{c}^{(1)}(a) \), \( \overline{c}^{(2)}(a) \) and \( \overline{c}^{(3)}(a) \). Do the derivatives \( \overline{c}^{(1)}(a) \), \( \overline{c}^{(2)}(a) \) and \( \overline{c}^{(3)}(a) \) depend upon the functions \( f \) and \( g \)? If there exist several functions \( \overline{c} \) which satisfy (3), do the derivatives of the function \( \overline{c} \) at \( x = a \) depend upon the function \( \overline{c} \) we choose?
The intermediate point from a mean value theorem

Since for \( x \in (a, b) \),
\[
\frac{v(x) - v(a)}{x - a} = \frac{c(x) - a}{x - a},
\]
if we denote by
\[
\theta(x) = \frac{c(x) - a}{x - a},
\]
then
\[
\theta(x) \in [0, 1]
\]
and hence
\[c(x) = a + (x - a)\theta(x)\]
and hence
\[\int_a^x f(t)g(t)\,dt = f(a)\int_a^{a + (x - a)\theta(x)} g(t)\,dt, \text{ for all } x \in (a, b).\]

Consequently, the following statement is true.

**Theorem 3** Let \( a, b \in \mathbb{R} \) with \( a < b \). If the continuous functions \( f, g : [a, b] \rightarrow \mathbb{R} \) satisfies properies:

(a) the function \( f \) is decreasing on \([a, b]\),
(b) \( f(x) \geq 0 \), for all \( x \in [a, b] \),

then there exists a function \( \theta : (a, b) \rightarrow [0, 1] \) such that (4) is true.

Another purpose of this paper is to establish in which conditions the function \( \theta : (a, b) \rightarrow [0, 1] \) has limit in the point \( x = a \) and also in which conditions the extension through continuity of the function \( \theta \), denoted by \( \overline{\theta} \), is derivable of first and second order in the point \( x = a \). The study of these functions, for other mean value theorems, can be find in papers like [3], [4], [5], [6], [7].

## 2 Main results

First, we remark that \( F \) is a primitive of the function \( fg \) and \( G \) is a primitive of the function \( g \), then, from the Leibniz Newton Theorem, the equality (4) becomes
\[F(x) - F(a) = f(a)[G(a + (x - a)\theta(x)) - G(a)], \text{ for all } x \in (a, b).\]

In this section we present some results related to the intermediate point functions \( c \) and \( \theta \).

**Theorem 4** Let \( a, b \in \mathbb{R} \) with \( a < b \) and let \( f, g \) be two continuous functions on \([a, b]\). If

(a) the function \( f \) is decreasing on \([a, b]\),
(b) \( f(x) \geq 0 \), for all \( x \in [a, b] \),
(d) \( f(a)g(a) \neq 0 \),
then

1° The function \( \theta : (a,b] \rightarrow [0,1] \) has limit at the point \( x = a \) and

\[
\lim_{x \to a} \theta(x) = 1.
\]

2° The function \( \tau \) is differentiable at \( x = a \) and

\[
\tau'(a) = 1.
\]

Proof. 1° Equality (5) can be write as follows

\[
\frac{F(x) - F(a)}{x - a} = f(a) \frac{G(a + (x-a)\theta(x)) - G(a)}{(x-a)\theta(x)} \theta(x),
\]
for all \( x \in (a,b] \).

Obviously,

\[
\lim_{x \to a} \frac{F(x) - F(a)}{x - a} = F'(a) = f(a)g(a).
\]

Because the function \( \theta \) is bounded, we have that

\[
\lim_{x \to a} (x-a)\theta(x) = 0,
\]
and hence

\[
\lim_{x \to a} \frac{G(a + (x-a)\theta(x)) - G(a)}{(x-a)\theta(x)} = \lim_{x \to a} \frac{G(a + (x-a)\theta(x)) - G(a)}{a + (x-a)\theta(x) - a} = G'(a) = g(a).
\]

It follows that, if \( f(a)g(a) \neq 0 \) then there exists \( \lim \theta(x) \) and from (6) we obtain that

\[
f(a)g(a) = f(a)g(a) \lim \theta(x),
\]
so

\[
\lim_{x \to a} \theta(x) = 1.
\]

2° The statement 2° follows from the statement 1°.

In what follows, we denote by \( \overline{\theta} : [a, b] \rightarrow [0,1] \) de function defined by

\[
\overline{\theta}(x) = \begin{cases} 
\theta(x), & \text{if } x \in (a, b] \\
1, & \text{if } x = a.
\end{cases}
\]

Theorem 5 Let \( a, b \in \mathbb{R} \) with \( a < b \) and let \( f, g \) be differentiable functions on \( [a, b] \).
If

(a) the function \( f \) is decreasing on \( [a, b] \),
(b) \( f(x) \geq 0 \), for all \( x \in [a, b] \),
(c) \( f' \) and \( g' \) are continuous at \( x = a \),
The intermediate point from a mean value theorem

(d) If \( f(a)g(a) \neq 0 \), then

1° The function \( \overline{\theta} \) is differentiable at \( x = a \) and
\[
\overline{\theta}'(a) = \frac{f'(a)}{2f(a)}.
\]

2° The function \( \overline{\xi} \) is twice differentiable at \( x = a \) and
\[
\overline{\xi}''(a) = \frac{f'(a)}{2f(a)}.
\]

Proof. 1° By Taylor’s theorem, for each \( x \in (a, b) \), there exist \( \xi_x, \eta_x \in (a, x) \) such that
\[
F(x) = F(a) + \frac{F'(a)(x-a)}{1!} + \frac{F''(\xi_x)(x-a)^2}{2!},
\]
\[
G(x) = G(a) + \frac{G'(a)(x-a)}{1!} + \frac{G''(\eta_x)(x-a)^2}{2!}.
\]

We replace these formulas in (5) and get
\[
\frac{F'(a)}{1!} + \frac{F''(\xi_x)}{2!}(x-a) = f(a)\left[ \frac{G'(a)}{1!}\theta(x) + \frac{G''(\eta_x)}{2!}(x-a)\theta^2(x) \right].
\]

For each \( x \in [a, b] \), we have \( F'(x) = f(x)g(x) \), \( F''(x) = f'(x)g(x) + f(x)g'(x) \), \( G'(x) = g(x) \), \( G''(x) = g'(x) \), so the last equality becomes
\[
\frac{f(a)g(a)}{1!} + \frac{f'(\xi_x)g(\xi_x) + f(\xi_x)g'(\xi_x)}{2!}(x-a) = \frac{f(a)g(a)}{1!}\theta(x) + \frac{f(a)g'(\eta_x)}{2!}(x-a)\theta^2(x),
\]

which is equivalent with
\[
f(a)g(a)\left[ \frac{\theta(x)-1}{x-a} \right] = \frac{1}{2!}\left[ f'(\xi_x)g(\xi_x) + f(\xi_x)g'(\xi_x) - f(a)g'(\eta_x)\theta^2(x) \right],
\]
for all \( x \in (a, b) \)

Since \( \lim_{x \to a} \theta(x) = 1 \) and \( f, f', g, g' \) are continuous at \( x = a \), it follows that
\[
\lim_{x \to a} \frac{1}{2!}\left[ f'(\xi_x)g(\xi_x) + f(\xi_x)g'(\xi_x) - f(a)g'(\eta_x)\theta^2(x) \right] = \frac{1}{2} f'(a)g(a),
\]
and then there exists
\[
\overline{\theta}'(a) = \lim_{x \to a} \frac{\overline{\theta}(x) - \overline{\theta}(a)}{x-a} = \lim_{x \to a} \frac{\theta(x) - 1}{x-a}
\]
and
\[
f(a)g(a)\overline{\theta}'(a) = \frac{1}{2} f'(a)g(a).
\]

Hence
\[
\overline{\theta}'(a) = \frac{f'(a)}{2f(a)}.
\]

2° The statement 2° follows from the statement 1°.
Theorem 6 Let \( a, b \in \mathbb{R} \) with \( a < b \) and let \( f, g \) be two twice differentiable functions on \([a, b]\). If
(a) the function \( f \) is decreasing on \([a, b]\),
(b) \( f(x) \geq 0 \), for all \( x \in [a, b]\),
(c) \( f'' \) and \( g'' \) are continuous at \( x = a \),
(d) \( f(a)g(a) \neq 0 \),
then
1° The function \( \overline{G} \) is twice differentiable at \( x = a \) and
\[
\overline{G}''(a) = \frac{f''(a)g(a) - f'(a)g'(a)}{3f(a)g(a)}.
\]
2° The function \( \overline{\tau} \) is third differentiable at \( x = a \) and
\[
\overline{\tau}'''(a) = \frac{f''(a)g(a) - f'(a)g'(a)}{3f(a)g(a)}.
\]

Proof. 1° By Taylor’s theorem, for each \( x \in (a, b) \), there are \( \xi_x, \eta_x \in (a, x) \) such that
\[
F(x) = F(a) + \frac{F'(a)}{1!} (x - a) + \frac{F''(a)}{2!} (x - a)^2 + \frac{F'''(\xi_x)}{3!} (x - a)^3
\]
\[
G(x) = G(a) + \frac{G'(a)}{1!} (x - a) + \frac{G''(a)}{2!} (x - a)^2 + \frac{G'''(\eta_x)}{3!} (x - a)^3.
\]
We replace these formulas in (5) and get
\[
\frac{F'(a)}{1!} + \frac{F''(a)}{2!} (x - a) + \frac{F'''(\xi_x)}{3!} (x - a)^2 =
\]
\[
f(a) \left[ \frac{G'(a)}{1!} \theta(x) + \frac{G''(a)}{2!} (x - a) \theta^2(x) + \frac{G'''(\eta_x)}{3!} (x - a)^2 \theta^3(x) \right].
\]
For each \( x \in [a, b] \), we have \( F'(x) = f'(x)g(x) \), \( F''(x) = f''(x)g(x) + f'(x)g'(x) \),
\( F'''(x) = f'''(x)g(x) + 2f''(x)g'(x) + f'(x)g''(x) \), \( G'(x) = g'(x) \), \( G''(x) = g''(x) \),
\( G'''(x) = g'''(x) \), so the last equality becomes
\[
f(a)g(a) \left[ \frac{\theta(x) - 1}{x-a} \right] = \frac{1}{x} \left[ f''(a)g(a) + f(a)g'(a) - f(a)g'(a) \theta^2(x) \right] +
\]
\[
+ \frac{1}{x} \left[ f'''(\xi_x)g(\xi_x) + 2f''(\xi_x)g'(\xi_x) + f'(\xi_x)g''(\xi_x) - f(a)g''(\eta_x) \theta^3(x) \right] (x - a),
\]
or, equivalently
\[
f(a)g(a) \left[ \frac{\theta(x) - \theta(a)}{x-a} - \overline{G}'(a) \right] = -\frac{f''(a)g(a)}{2} +
\]
\[
+ \frac{1}{x} \left[ f'(a)g(a) + f(a)g'(a) - f(a)g'(a) \theta^2(x) \right] +
\]
\[
+ \frac{1}{x} \left[ f'''(\xi_x)g(\xi_x) + 2f''(\xi_x)g'(\xi_x) + f'(\xi_x)g''(\xi_x) - f(a)g''(\eta_x) \theta^3(x) \right] (x - a),
\]
and then the last equality becomes
\[ f(a)g(a) \frac{\overline{\tau}(x)-\overline{\tau}(a)}{x-a} - \frac{\overline{\tau}(x)-\overline{\tau}(a)}{x-a} \frac{f(a)g'(a)}{2} \left[ \theta(x) + 1 \right] + \]
\[ + \frac{1}{3!} \left[ f''(\xi_x)g(\xi_x) + 2f'(\xi_x)g'(\xi_x) + f(\xi_x)g''(\xi_x) - f(a)g''(\eta_x)\theta''(x) \right]. \]

Since
\[ \lim_{x \to a} \theta(x) = \overline{\theta}(a) = 1, \quad \lim_{x \to a} \frac{\overline{\theta}(x) - \overline{\theta}(x)}{x-a} = \overline{\theta}'(a) = \frac{f'(a)}{2f(a)}, \]

it follows that the function \( \overline{\theta} \) is twice differentiable at \( x = a \), and
\[ f(a)g(a) \frac{\overline{\theta}'(a)}{2} = -\frac{f(a)g'(a)}{2} \frac{f'(a)}{2f(a)} (1 + 1) + \]
\[ + \frac{1}{3!} \left[ f''(a)g(a) + 2f'(a)g'(a) + f(a)g''(a) - f(a)g''(a) \right]. \]

Hence,
\[ f(a)g(a)\overline{\theta}''(a) = \frac{f''(a)g(a) - f'(a)g'(a)}{3}, \]
or
\[ \overline{\theta}'''(a) = \frac{f''(a)g(a) - f'(a)g'(a)}{3f(a)g(a)}. \]

2° The statement 2° follows from the statement 1°.

3 Conclusions and further challenges

In this paper we gave conditions for the functions \( f \) and \( g \) such that the intermediate point functions \( \overline{\tau} \) and \( \overline{\theta} \) to be derivable in the point \( a \) and also we provided the derivative \( \overline{\tau}'(a) \), \( \overline{\theta}'(a) \), \( \overline{\tau}''(a) \), \( \overline{\theta}''(a) \) and \( \tau'''(a) \). In future we want to verify in which conditions the functions \( \tau \) and \( \theta \) are differentiable of order \( n \) in the point \( a \) and to calculate the corresponding derivative.

References


**Emilia-Loredana Pop**  
Babeș-Bolyai University  
Faculty of Mathematics and Computer Science  
Department of Computer Science  
Street Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania  
e-mail: pop_emilia_loredana@yahoo.com

**Dorel Duca**  
Babeș-Bolyai University  
Faculty of Mathematics and Computer Science  
Department of Mathematics  
Street Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania  
e-mail: dorelduca@yahoo.com

**Augusta Rațiu**  
Lucian Blaga University of Sibiu  
Faculty of Science  
Department of Mathematics and Computer Science  
Street Dr. I. Rațiu 5-7, 550012 Sibiu, Romania  
e-mail: augu2003@yahoo.com
Abstract

Miheșan (2008) generalized the Szász operators by introducing a general class of linear positive operators. We introduce a Kantorovich generalization of Miheșan operator based on a real parameter $\rho$. We study the approximation properties of these operators including weighted Korovkin theorem, the rate of convergence in terms of the modulus of continuity, second order modulus of continuity via Steklov-mean, the degree of approximation for the weighted spaces. Furthermore, we obtain the rate of convergence of the considered operators with the aid of the Ditzian-Totik modulus of smoothness and K-functional. At the last we investigate quantitative Voronovskaja and Grüss-Voronovskaja type theorems.


Key words and phrases: Linear positive operators, Miheșan operators, unified Ditzian-Totik modulus of smoothness, weighted spaces.

1 Introduction

In 2008, Miheșan ([20],Theorem 4.2) obtained a vital generalization of the Szász operators by using Gamma transformation depending on real parameter $\eta$ as

$$(1.1) \quad (M_\alpha^n f)(x) = \sum_{k=0}^{\infty} m_{\alpha,k}^\eta f\left(\frac{k}{\alpha}\right), \quad x \in [0, \infty),$$

with $\eta + \alpha x > 0$, $m_{\alpha,k}^\eta(x) = \frac{(\eta)_k}{(1 + \frac{\alpha x}{\eta})^{\eta+k}}$ where $\eta$ may depend on $\alpha$ and $(a)_k$ is well known Pochhammer symbol defined as

---

$^1$Received 5 June, 2017
Accepted for publication (in revised form) 3 August, 2017
Many researchers studied Szász-Mirakyan operators and their modifications. Varma and Taşdelen [25], introduced a generalization of Szász operators involving Charlier polynomials. In 2016, Ozarslan and Duman [21], proposed a modified Bernstein-Kantorovich operator based on a parameter $\rho > 0$,

$$ K_{\alpha,\rho}(f; x) = \sum_{k=0}^{\infty} p_{\alpha,k}(x) \int_0^1 f\left(\frac{k+t^\rho}{\alpha+1}\right) dt, \quad x \in [0, 1], $$

where $p_{\alpha,k}(x) = \binom{\alpha}{k} x^k (1-x)^{\alpha-k}$.

In view of operators (1.1) and (1.2), for each $h \in C_\gamma[0, \infty) := \{ f \in C[0, \infty) : |f(t)| \leq M_f (1+t^\gamma), (M_f$ is a constant depending only on function $f)\}$ we construct the following sequence of linear positive operators of blending type

$$ M_{\alpha,\rho}^\eta(h; x) = \sum_{k=0}^{\infty} m_{\alpha,k}^\eta(x) \int_0^1 f\left(\frac{k+u^\rho}{\alpha+1}\right) du, $$

The operator defined by (1.3) is named as Miheşan-Kantorovich operator of blending type.

The approximation properties of different mixed hybrid operators has been studied by many researchers in last few years. In [22], Srivastava and Gupta investigated approximation properties of certain family of summation integral-type operators. After that, Yüksel and Ispir [26] studied weighted approximation properties of certain family of summation integral-type operators. Agrawal et al. [5], introduced the Baskakov-Szász type operator depending on a nonnegative parameter and studied rate of convergence and simultaneous approximation properties. In [15], Gupta and Russians studied approximation properties of Durrmeyer type generalization of Szász type operators. In [14], Gupta defined a sequence of blending type operators involving the weights of Păltânea basis function and investigated the rate of convergence of these operators. In 2016, Tuncer and Ulusoy [3], studied approximation properties of mixed Szász-Durrmeyer type operators. Very recently Kajla et al. [17], introduced the hybrid operators based on inverse Polya-Eggenberger distribution and studied their rate of convergence. Kajla and Araci [18], studied approximation properties of blending type Stancu-Kantorovich operators based on Polya-Eggenberger distribution.

In this present work, our focus is to study the approximation properties of Miheşan-Kantorovich operator (1.3) via Bohman-Korovkin theorem and in terms of first and second order modulus of continuity and Steklov mean. We estimate the rate of convergence of these operators in terms of Ditzian-Totik modulus of smoothness and K-functional. Furthermore, we investigate weighted approximation theorems. Lastly we study quantitative Voronovskaja and Grüss-Voronovskaja type theorem.
2 Preliminaries

Throughout this paper, we assume that \( \eta = \eta(\alpha) \to \infty \), as \( \alpha \to \infty \) and \( \lim_{\alpha \to \infty} \frac{\alpha}{\eta(\alpha)} = l ( \in \mathbb{R} ) \). In the following lemma we obtain the values of the moments for the operators \( M_{\alpha,\rho}^n \).

Lemma 1 The operators \( M_{\alpha,\rho}^n \) satisfies following equalities:

(i) \( M_{\alpha,\rho}^0(u; x) = \frac{\alpha x}{\alpha + 1} + \frac{1}{(\rho + 1)(\alpha + 1)} \),

(ii) \( M_{\alpha,\rho}^1(u^2; x) = \frac{\alpha x^2(\eta + 1)}{\eta(\alpha + 1)^2} + \frac{\alpha x(\rho + 3)}{(\rho + 1)(\alpha + 1)^2} + \frac{1}{(2\rho + 1)(\alpha + 1)^2} \),

(iii) \( M_{\alpha,\rho}^2(u^3; x) = \frac{\alpha x^3(1 + \eta)(2 + \eta)}{\eta(\alpha + 1)^3} + \frac{3\alpha x^2(1 + \eta)(\rho + 2)}{\eta(\rho + 1)(\alpha + 1)^3} + \frac{1}{(3\rho + 1)(\alpha + 1)^3} \),

(iv) \( M_{\alpha,\rho}^3(u^4; x) = \frac{\alpha x^4(1 + \eta)(2 + \eta)(3 + \eta)}{\eta^3(\alpha + 1)^4} + \frac{1}{\eta(\rho + 1)(\alpha + 1)^4} \),

(v) \( M_{\alpha,\rho}^4(u^5; x) = \frac{\alpha x^5(1 + \eta)(2 + \eta)(3 + \eta)(4 + \eta)}{\eta^4(\alpha + 1)^5} \times (10\rho + 15) + \frac{\alpha^2 x^2(1 + \eta)}{\eta^2(\alpha + 1)^5(\rho + 1)(2\rho + 1)^3} \),

(vi) \( M_{\alpha,\rho}^5(u^6; x) = \frac{\alpha^3 x^3(1 + \eta)(2 + \eta)}{\eta^3(\alpha + 1)^6(\rho + 1)(2\rho + 1)^5} \).
Lemma 2 are proved. Hence we skip the details.

**Proof.** Using Lemma 2.1 from [16] and simple calculations, the identities (i)-(vii) are proved. Hence we skip the details.

As a consequence of Lemma 1, we obtain:

**Lemma 2** The operator $M_{q,\rho}^\eta(u^6; x)$ verifies the following equalities:

(i) \[ M_{q,\rho}^\eta(u^6; x) = \frac{x^6}{\eta^4(\alpha + 1)^6(\rho + 1)} \]

(ii) \[ M_{q,\rho}^\eta((u - x)^2; x) = \frac{x^2(\alpha^2 + \eta)}{\eta^4(\alpha + 1)} + \frac{x(\alpha\rho + \alpha - 2)}{(\alpha + 1)^2(\rho + 1)} + \frac{1}{(2\rho + 1)(\alpha + 1)^2}; \]

(iii) \[ M_{q,\rho}^\eta((u - x)^4; x) = \frac{x^4}{\eta^4(\alpha + 1)^4} \sum \frac{\alpha^4(3\eta + 6) - 8\eta\alpha^3 + 6\alpha^2\eta^2 + \eta^3}{\eta^4(\alpha + 1)^4} \]

\[ + \frac{x^3}{\eta^2(\alpha + 1)^4(\rho + 1)} \sum \frac{\alpha^3(6\eta\rho + 6\eta + 12\rho + 20) - 12\eta\alpha^2(\rho + 1) + 6\eta^2(\rho + 1)}{\eta^4(\alpha + 1)^4(\rho + 1)(2\rho + 1)} \]

\[ + \frac{x^2}{\eta^4(\alpha + 1)(\rho + 1)(2\rho + 1)} \sum \frac{\alpha^2\{\rho^2(6\eta + 14) + \rho(9\eta + 51) + (3\eta + 25)\}}{\eta^4(\alpha + 1)^4(\rho + 1)(2\rho + 1)(3\rho + 1)} \]

\[ - \alpha\{8\eta(\rho + 1)(\rho + 2) + 6\eta(\rho + 1)\} + \frac{x}{(\alpha + 1)^4(\rho + 1)(2\rho + 1)(3\rho + 1)} \]

\[ + \frac{1}{(\alpha + 1)^4(4\rho + 1)} \sum \frac{\alpha\{6\rho^3 + 53\rho^2 + 50\rho + 11 - 4(2\rho^2 + 3\rho + 1)\}}{\eta^4(\alpha + 1)^4(\rho + 1)(2\rho + 1)(3\rho + 1)} \]

(iv) \[ M_{q,\rho}^\eta((u - x)^6; x) = \frac{x^6}{\eta^4(\alpha + 1)^6} \sum \frac{\alpha^6(15\eta^2 + 130\eta + 120) - \alpha^5(120\eta^2 + 144\eta)}{\eta^4(\alpha + 1)^6(\rho + 1)} + \frac{x^5}{(\alpha + 1)^6}\]

\[ + \alpha^4(45\eta^3 + 90\eta^2) - 40\alpha^2\eta^3 + 15\alpha^2\eta^2 + \eta^5 \]
process for functions in $C$.

In the following theorem we show that the operators $M_{\alpha,\rho}$ are an approximation process for functions in $C_\gamma(\mathbb{R}^+)$. 

### Lemma 3

For every $x \in [0, \infty)$, we have

(i) $\lim_{\alpha \to \infty} \alpha M_{\alpha,\rho}^\eta((u-x);x) = -x + \frac{1}{\rho + 1}$;

(ii) $\lim_{\alpha \to \infty} \alpha M_{\alpha,\rho}^\eta((u-x)^2;x) = x(lx + 1)$;

(iii) $\lim_{\alpha \to \infty} \alpha^2 M_{\alpha,\rho}^\eta((u-x)^4;x) = 3x^2(lx + 1)^2$;

(iv) $\lim_{\alpha \to \infty} \alpha^3 M_{\alpha,\rho}^\eta((u-x)^6;x) = 15x^3(lx + 1)^3$.

Consequently, for every $x \in [0, \infty)$ and sufficiently large $\alpha$, we can find a constant $c = c(l) > 0$ such that

$M_{\alpha,\rho}^\eta((u-x)^2;x) \leq \frac{c(1+x^2)}{\alpha}$, $M_{\alpha,\rho}^\eta((u-x)^4;x) \leq \frac{c(1+x^2)^2}{\alpha^2}$ and

$M_{\alpha,\rho}^\eta((u-x)^6;x) \leq \frac{c(1+x^2)^3}{\alpha^3}$.

**Proof.** The proof of Lemma 3 follows easily from Lemma 2, so the details are omitted.

### 3 Main results

In the following theorem we show that the operators $M_{\alpha,\rho}$ is an approximation process for functions in $C_\gamma(\mathbb{R}^+)$. 

Theorem 1 Let \( h \in C_\gamma(\mathbb{R}^+) \). Then,

\[
\lim_{\alpha \to \infty} M_{\alpha,\rho}(^\eta h; x) \to h(x),
\]

uniformly on each compact subset \( A \) of \([0, \infty)\).

Proof. In view of Lemma 1,

\[
M_{\alpha,\rho}(^\eta u^i; x) \to x^i, \quad \text{as} \quad \alpha \to \infty,
\]

uniformly on \( A \), for \( i = 0, 1, 2 \).

Hence, the required result follows on applying the Bohman-Korovkin criterion.

Let \( C_B[0, \infty) \) denote the space of bounded and uniformly continuous functions on \([0, \infty)\) endowed with the sup norm,

\[
||f|| = \sup_{x \in [0,\infty)} |f(x)|.
\]

The first and second order modulus of continuity are respectively defined as

\[
\omega(f, \delta) = \sup_{x,u,v \in [0,\infty), |u-v| \leq \delta} |f(x+u) - f(x+v)|
\]

and

\[
\omega_2(f, \delta) = \sup_{x,u,v \in [0,\infty), |u-v| \leq \delta} |f(x+2u) - 2f(x+u+v) + f(x+2v)|, \quad \delta > 0.
\]

In our next result we obtain the degree of approximation in terms of the modulus of continuity.

Theorem 2 Let \( h \in C_B[0, \infty) \) and \( \omega(h; \delta), \delta > 0 \), be its first order modulus of continuity. Then the operator \( M_{\alpha,\rho}(^\eta h; x) \) satisfies the inequality

\[
|M_{\alpha,\rho}(^\eta h; x) - h(x)| \leq (1 + \sqrt{x^2(\alpha^2 + \eta)} + \frac{x(\alpha \rho + \alpha - 2)}{(\alpha + 1)^2(\rho + 1)} + \frac{1}{(2\rho + 1)(\alpha + 1)^2}) \times \omega(h; 1/\sqrt{\alpha}).
\]

Proof. By definition of \( \omega(h; \delta) \), Lemma 2 and Cauchy-Schwarz inequality, we may get

\[
|M_{\alpha,\rho}(^\eta h; x) - h(x)| \leq M_{\alpha,\rho}(^\eta |h(u) - h(x)|; x)
\]

\[
\leq \left(1 + \frac{1}{\delta} M_{\alpha,\rho}(^\eta |u-x|; x)\right) \omega(h; \delta)
\]

\[
\leq \left(1 + \frac{1}{\delta} \sqrt{M_{\alpha,\rho}(^\eta (u-x)^2; x)}\right) \omega(h; \delta)
\]

\[
\leq \left(1 + \frac{1}{\delta} \sqrt{\frac{x^2(\alpha^2 + \eta)}{\eta(\alpha + 1)^2} + \frac{x(\alpha \rho + \alpha - 2)}{(\alpha + 1)^2(\rho + 1)} + \frac{1}{(2\rho + 1)(\alpha + 1)^2}}\right) \times \omega(h; \delta).
\]

Now, choosing \( \delta = \alpha^{-1/2} \), we arrive to result.

For \( f \in C_B[0, \infty) \), the Steklov mean is defined as

\[
f_h(x) = \frac{4}{h^2} \int_0^{\frac{x}{2}} \int_0^{\frac{x}{2}} [2f(x+u+v) - f(x+2(u+v))] dudv.
\]
Lemma 4 [13] The Steklov mean $f_h(x)$ satisfies the following properties:

(i) $\|f_h - f\| \leq \omega_2(f, h)$,

(ii) $f'_h, f''_h \in C_B[0, \infty)$ and

$$\|f'_h\| \leq \frac{5}{h^2}\omega_2(f, h), \quad \|f''_h\| \leq \frac{9}{h^2}\omega_2(f, h).$$

By using Steklov mean, in our next theorem we establish the rate of convergence in terms of the first and second order modulus of continuity.

Theorem 3 Let $h \in C_B[0, \infty)$. Then for each $x \in [0, \infty)$, we have

$$|M^\eta_{\alpha, \rho}(h; x) - h(x)| \leq \frac{5}{\sqrt{\alpha + 1}} \left(\frac{x + 1}{(\rho + 1)}\right)\omega(h; (\alpha + 1)^{-1/2})$$

$$+ \left(2 + \frac{9}{2}\frac{x^2(\alpha^2 + \eta)}{\eta(\alpha + 1)} + \frac{x(\alpha \rho + \alpha - 2)}{(\alpha + 1)(\rho + 1)} + \frac{1}{(2\rho + 1)(\alpha + 1)}\right)\omega_2(h; (\alpha + 1)^{-1/2}).$$

Proof. Applying Lemma 1 and Lemma 4, one has

$$|M^\eta_{\alpha, \rho}(h - f_h; x)| \leq \|h - f_h\| \leq \omega_2(h; h).$$

Since $f''_h \in C_B[0, \infty)$, by Taylor’s expansion,

$$f_h(u) = f_h(x) + (u - x)f'_h(x) + \int_x^u (u - s)f''_h(s)ds.$$ 

Applying operator $M^\eta_{\alpha, \rho}$ on the above equality, we get

$$|M^\eta_{\alpha, \rho}(f_h(u) - f_h(x); x)| \leq \|f'_h\| |M^\eta_{\alpha, \rho}(u - x; x)| + \frac{\|f''_h\|}{2}M^\eta_{\alpha, \rho}((u - x)^2; x).$$
Hence using Lemma 2 and Lemma 4, we have

\[
|M_{\alpha, \rho}^{\eta}(h; x) - h(x)| \leq |M_{\alpha, \rho}^{\eta}(h - f_h; x)| + |M_{\alpha, \rho}^{\eta}(f_h - f_h(x); x)| + |f_h(x) - h(x)|
\]

\[
\leq \frac{\omega_2(h; h)}{h} + \frac{||f_h'||}{(\alpha + 1)(\rho + 1)} + \frac{||f_h'||}{2}
\]

\[
\times \left(\frac{x^2(\alpha^2 + \eta)}{\eta(\alpha + 1)^2} + \frac{x(\alpha \rho + \alpha - 2)}{(\alpha + 1)(\rho + 1)} + \frac{1}{(2\rho + 1)(\alpha + 1)^2}\right) + ||f_h - h||
\]

\[
\leq \frac{5}{h} \left(\frac{-x}{\alpha + 1} + \frac{1}{(\alpha + 1)(\rho + 1)}\right) \omega(h; h) + \left(2 + \frac{9}{2h^2}\right)
\]

\[
\times \left(\frac{x^2(\alpha^2 + \eta)}{\eta(\alpha + 1)^2} + \frac{x(\alpha \rho + \alpha - 2)}{(\alpha + 1)(\rho + 1)} + \frac{1}{(2\rho + 1)(\alpha + 1)^2}\right) \omega(h; h).
\]

Finally, choosing \( h = (\alpha + 1)^{-1/2} \), the required result is obtained.

In the following theorem, we obtain the estimate of error in approximation for continuously differentiable functions.

**Theorem 4** For \( h' \in C_B[0, \infty) \), we have

\[
|M_{\alpha, \rho}^{\eta}(h; x) - h(x)| \leq M \left|\frac{-x}{\alpha + 1} + \frac{1}{(\alpha + 1)(\rho + 1)}\right| + \omega(h'; M_{\alpha, \rho}^{\eta}, ((u - x)^2; x)
\]

\[
\times (1 + \sqrt{M_{\alpha, \rho}^{\eta}((u - x)^2; x)}),
\]

where \( M \) is some positive constant, and \( \omega(h'; \delta) \) denotes the modulus of continuity of \( h' \).

**Proof.** Since \( h' \in C_B[0, \infty) \) \( \exists M > 0 \) such that \( |h'(x)| \leq M, \forall x \geq 0 \). Using mean value theorem, one may write

\[
h(u) = h(x) + (u - x)h'(\zeta)
\]

\[
= h(x) + (u - x)h'(x) + (u - x)(h'(x) - h'(x)),
\]

where \( \zeta \) lies between \( u \) and \( x \).

Applying the operator \( M_{\alpha, \rho}^{\eta} \) on both sides of the above equality and using Lemma 2, we get

\[
|M_{\alpha, \rho}^{\eta}(h; x) - h(x)| \leq |h'(x)||M_{\alpha, \rho}^{\eta}(u - x; x)| + M_{\alpha, \rho}^{\eta}(|u - x||h'(\zeta) - h'(x)|; x)
\]

\[
\leq M \left|\frac{-x}{\alpha + 1} + \frac{1}{(\alpha + 1)(\rho + 1)}\right| + M_{\alpha, \rho}^{\eta}(|u - x||h'(\zeta) - h'(x)|; x).
\]

(3.2)
Now, applying Cauchy-Schwarz inequality, we can get
\[
M^{\eta, \alpha, \rho}(|u - x| \|h'(\zeta) - h'(x)||; x) \leq M^{\eta, \alpha, \rho}(|u - x| \omega(h', \delta) \left(1 + \frac{|u - x|}{\delta}\right); x)
\]
\[
\leq \omega(h', \delta) M^{\eta, \alpha, \rho}(\|u - x\|^2; x)
\]
\[
\leq \omega(h', \delta) \sqrt{M^{\eta, \alpha, \rho}((u - x)^2; x)}
\]
\[
+ \frac{\omega(h', \delta)}{\delta} M^{\eta, \alpha, \rho}((u - x)^2; x).
\]
(3.3)

Choosing \(\delta = M^{\eta, \alpha, \rho}((u - x)^2; x)\) and combining (3.2)-(3.3), we arrive to conclusion.

Now, we recall the definition of second order Ditzian-Totik modulus (see[8]) as:
\[
\omega^2(h, \delta) = \sup_{0 \leq t \leq \delta} \sup_{x + t\phi(x) \in [0, \infty)} |(h(x + t\phi(x))) - 2h(x) + h(x - t\phi(x))|, \ \delta \geq 0,
\]
where \(\phi: [0, \infty) \rightarrow \mathbb{R}\) is an admissible step-weight function.
The corresponding K-functional is
\[
K_{2, \phi}(h, \delta) = \inf\{||h - g|| + \delta ||\phi^2 g''||, g \in W^2_{\infty}(\phi), \ \delta \geq 0, \}
\]
where
\[
W^2_{\infty}(\phi) = \{g \in C_B[0, \infty): g' \in AC_{loc}[0, \infty), \phi^2 g'' \in C_B[0, \infty]\}.
\]

Also, the Ditzian-Totik modulus of the first order is given by
\[
\overline{\omega}_\phi(h, \delta) = \sup_{0 \leq t \leq \delta} \sup_{x + t\phi(x) \in [0, \infty)} |h(x + t\phi(x)) - h(x)|.
\]

From [8], for an absolute constant \(C\), the following relation between second order
Ditzian-Totik modulus and K-functional is well known
\[
C^{-1} \omega^2(h, \sqrt{\delta}) \leq K_{2, \phi}(h, \delta) \leq C \omega^2(h, \sqrt{\delta}).
\]

In order to prove our next result, we consider \(\phi^2(x) = 1 + x^2\).
In the following theorem, we establish the rate of convergence with the aid of the
Ditzian-Totik modulus of the first and second order.

**Theorem 5** Let \(h \in C_B[0, \infty), \ \eta(\alpha) \rightarrow \infty \text{ as } \alpha \rightarrow \infty, \text{ and } \frac{\alpha}{\eta(\alpha)} \rightarrow l \in \mathbb{R}\)
as \(\alpha \rightarrow \infty\). Then for sufficiently large \(\alpha\), there is an absolute constant \(c\) such that
\[
|M^{\eta, \alpha, \rho}(h; x) - h(x)| \leq 4K_{2, \phi}\left(h, \frac{c}{2\alpha}\right) + \overline{\omega}_\phi\left(h, \frac{\sqrt{\alpha}}{2}\right) \leq C \omega^2(h, \sqrt{\frac{c}{2\alpha}}) + \overline{\omega}_\phi\left(h, \frac{\sqrt{c}}{\alpha}\right).
\]
Proof. First we define auxiliary operator as
\[
\vec{M}_{\eta,\alpha,\rho}(h; x) = M_{\eta,\alpha,\rho}(h; x) - h \left( \frac{\alpha x}{\alpha + 1} + \frac{1}{(\rho + 1)(\alpha + 1)} \right) + h(x).
\]
Using definition of the operator \( M_{\eta,\alpha,\rho} \), it is easy to verify that \( \vec{M}_{\eta,\alpha,\rho}(1; x) = 1 \) and \( \vec{M}_{\eta,\alpha,\rho}(u - x; x) = 0 \).

Now, for any \( g \in W_{\infty}(\phi) \), using Taylor’s formula, we have
\[
g(u) = g(x) + g'(x)(u - x) + \int_x^u g''(s)(s - x)ds.
\]
Applying operator \( \vec{M}_{\eta,\alpha,\rho} \) on above equality, we obtain
\[
|\vec{M}_{\eta,\alpha,\rho}(g; x) - g(x)| \leq |g'(x)\vec{M}_{\eta,\alpha,\rho}(u - x; x)| + |\vec{M}_{\eta,\alpha,\rho}\left( \int_x^u g''(s)(s - x)ds; x \right)|
\]
\[
\leq M_{\eta,\alpha,\rho}\left( \left| \int_x^u g''(s)(s - x)ds \right|; x \right)
\]
\[
\leq M_{\eta,\alpha,\rho}\left( \left| \int_x^u g''(s)(s - x)ds \right|; x \right)
\]
\[
\leq \frac{|\phi^2g''|}{\phi^2(x)} M_{\eta,\alpha,\rho}(u - x)^2; x) + \frac{|\phi^2g''|}{\phi^2(x)} \left( \frac{\alpha x}{\alpha + 1} + \frac{1}{(\rho + 1)(\alpha + 1) - x} \right)^2
\]
\[
\leq \frac{|\phi^2g''|}{\phi^2(x)} \left( M_{\eta,\alpha,\rho}(u - x)^2; x) + (M_{\eta,\alpha,\rho}(u - x))^2 \right).
\]
From Lemma 3,
\[
\frac{M_{\eta,\alpha,\rho}(u - x)^2; x)}{\phi^2(x)} \leq \frac{c}{\alpha} \text{ and } \frac{(M_{\eta,\alpha,\rho}(u - x); x))^2}{\phi^2(x)} \leq \frac{c}{\alpha^2},
\]
therefore,
\[
|\vec{M}_{\eta,\alpha,\rho}(g; x) - g(x)| \leq \frac{2c}{\alpha}||\phi^2g''||.
\]
Now, we have
\[
|M_{\eta,\alpha,\rho}(h; x) - h(x)| \leq |\vec{M}_{\eta,\alpha,\rho}(h - g; x) - (h - g)(x)| + |\vec{M}_{\eta,\alpha,\rho}(g; x) - g(x)|
\]
\[
+ |h\left( \frac{\alpha x}{\alpha + 1} + \frac{1}{(\rho + 1)(\alpha + 1)} \right) - h(x)|
\]
\[
\leq 4||h - g|| + \frac{2c}{\alpha}||\phi^2g''||
\]
\[
+ |h\left( \frac{\alpha x}{\alpha + 1} + \frac{1}{(\rho + 1)(\alpha + 1)} \right) - h(x)|
\]
(3.5)
Also, we have
\[
\left| \frac{\alpha x}{\alpha + 1} + \frac{1}{(\rho + 1)(\alpha + 1)} \right| - h(x)
\]
\[
= \left| \frac{x + \phi(x) \left( \frac{\alpha}{\alpha + 1} - 1 \right) x + \frac{1}{(\rho + 1)(\alpha + 1)} \phi(x)}{\phi(x)} \right| - h(x)
\]
\[
\leq \sup \left| \frac{x + \phi(x) \frac{M_{\alpha,\rho}(u-x; x)}{\phi(x)} - h(x)}{\phi(x)} \right|
\]
\[
\leq \overline{\omega}_\phi \left( \frac{h}{x^2} \right)
\]
(3.6)

Therefore, from (3.5) and (3.6), it follows that
\[
|M_{\alpha,\rho}(h; x) - h(x)| \leq 4 \left\{ ||h - g|| + \frac{c}{2\alpha} ||\phi^2 g''|| \right\} + \overline{\omega}_\phi \left( \frac{h}{\alpha} \right).
\]

Now using the definition of $K_{2,\phi}$, we immediately arrive to the required result.

Next, we define some weighted spaces on $[0, \infty)$ to investigate the weighted approximation results for the operators defined by (1.3).

\[
B_\sigma(\mathbb{R}^+) := \{ f : ||f(x)|| \leq M f \sigma(x) \},
\]
\[
C_\sigma(\mathbb{R}^+) := \{ f : f \in B_\sigma(\mathbb{R}^+) \cap C[0, \infty) \},
\]
and
\[
C^k_\sigma(\mathbb{R}^+) := \left\{ f : f \in C_\sigma(\mathbb{R}^+) \text{ and } \lim_{x \to \infty} \frac{f(x)}{\sigma(x)} = k \text{ (some constant)} \right\},
\]
where $\sigma(x) = 1 + x^2$ is a weight function and $M f$ is a constant depending only on the function $f$. From [6], it is noted that $C_\sigma(\mathbb{R}^+)$ is a normed linear space endowed with the norm $||f||_\sigma := \sup_{x \geq 0} \frac{|f(x)|}{\sigma(x)}$.

It is well known that the classical modulus of continuity $\omega(f; \delta)$ does not tend to zero if $f$ is continuous on an infinite interval. Therefore, in order to study the approximation of functions in the weighted space $C^k_\sigma(\mathbb{R}^+)$, Ispir and Atakut [6] introduced the following weighted modulus of continuity

\[
\Omega(f; \delta) = \sup_{x \in [0, \infty), \delta} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)},
\]
(3.7)
and proved that $\lim_{\delta \to 0} \Omega(f; \delta) = 0$, $\omega(f; \lambda \delta) \leq 2(1 + \lambda)(1 + \delta^2)\Omega(f; \delta)$, $\lambda > 0$, and
\[ |f(u) - f(x)| \leq 2 \left( 1 + \frac{|u - x|}{\delta} \right) (1 + \delta^2)(1 + x^2)(1 + (u - x)^2) \Omega(f; \delta), \]
\[ u, x \in [0, \infty). \]
\[ (3.8) \]

In the following theorem we show that the operator \( M_{a,\rho}^n \) is an approximation method for functions belonging to the weighted space \( C_\sigma^k(\mathbb{R}^+) \):

**Theorem 6** For each \( h \in C_\sigma^k(\mathbb{R}^+) \), the sequence of linear positive operators \( \{ M_{a,\rho}^n \} \) satisfies following equality

\[ \lim_{\alpha \to \infty} || M_{a,\rho}^n(h; x) - h(x) ||_\sigma = 0. \]

**Proof.** From Lemma 1, clearly \( \lim_{\alpha \to \infty} || M_{a,\rho}^n(1; x) - 1 ||_\sigma = 0. \)

Now,

\[ \sup_{x \geq 0} \frac{|M_{a,\rho}^n(u; x) - x|}{1 + x^2} \leq \frac{1}{\alpha + 1} \sup_{x \geq 0} \frac{x}{1 + x^2} + \frac{1}{(\rho + 1)(\alpha + 1)} \sup_{x \geq 0} \frac{1}{1 + x^2} \]
\[ \leq \frac{1}{2(\alpha + 1)} + \frac{1}{(\rho + 1)(\alpha + 1)}. \]

Therefore, \( \lim_{\alpha \to \infty} || M_{a,\rho}^n(u; x) - x ||_\sigma = 0. \) Again,

\[ \sup_{x \geq 0} \frac{|M_{a,\rho}^n(u^2; x) - x^2|}{1 + x^2} \leq \frac{|\alpha^2 - 2\alpha \eta - \eta|}{\eta(\alpha + 1)^2} \sup_{x \geq 0} \frac{x^2}{1 + x^2} + \frac{\alpha(\rho + 3)}{\eta(\alpha + 1)^2} \sup_{x \geq 0} \frac{1}{1 + x^2} \]
\[ \times \sup_{x \geq 0} \frac{x}{1 + x^2} + \frac{1}{(\rho + 1)(\alpha + 1)^2} \sup_{x \geq 0} \frac{1}{1 + x^2} \]
\[ \leq \frac{|\alpha^2 - 2\alpha \eta - \eta|}{\eta(\alpha + 1)^2} + \frac{\alpha(\rho + 3)}{2(\rho + 1)(\alpha + 1)^2} \]
\[ + \frac{1}{(2\rho + 1)(\alpha + 1)^2}, \]

we obtain \( \lim_{\alpha \to \infty} || M_{a,\rho}^n(u^2; x) - x^2 ||_\sigma = 0. \) Hence, applying weighted Korovkin-type theorem given by Gadzhiev [9], we reach the desired result.

In the following theorem, the rate of convergence is obtained by means of the weighted modulus of continuity.

**Theorem 7** Let \( h \in C_\sigma^k(\mathbb{R}^+) \). Then for sufficiently large \( \alpha \), the following inequality is verified

\[ \sup_{x \in [0, \infty)} \frac{|M_{a,\rho}^n(h; x) - h(x)|}{(1 + x^2)^{5/2}} \leq K \Omega \left( h; \frac{1}{\sqrt{\alpha}} \right), \]

where \( K \) is a constant not dependent on \( h \) and \( \alpha \).
Proof. Using the definition of $\Omega(f; \delta)$, Lemma 3 and Cauchy-Schwarz inequality, one can easily see that

$$|M_{\eta, \alpha, \rho}^\eta(h; x) - h(x)| \leq M_{\eta, \alpha, \rho}^\eta(|h(u) - h(x)|; x)$$

$$\leq 2(1 + \delta^2)(1 + x^2)\Omega(h; \delta)$$

$$\times M_{\eta, \alpha, \rho}^\eta \left(1 + \frac{|u - x|}{\delta} \right)(1 + (u - x)^2); x)$$

$$\leq 2(1 + \delta^2)(1 + x^2)\Omega(h; \delta)\left(M_{\eta, \alpha, \rho}^\eta(1; x)ight)$$

$$+ M_{\eta, \alpha, \rho}^\eta((u - x)^2; x) + \frac{1}{\delta}(M_{\eta, \alpha, \rho}^\eta((u - x)^2; x))^{1/2}$$

$$+ \frac{1}{\delta}(M_{\eta, \alpha, \rho}^\eta((u - x)^2; x))^{1/2} \times (M_{\eta, \alpha, \rho}^\eta((u - x)^4; x))^{1/2}$$

Now, choosing $\delta = \frac{1}{\sqrt{\alpha}}$, we arrive to conclusion immediately.

In the following result, we prove a quantitative Voronovskaja type theorem by utilizing the weighted modulus of continuity.

**Theorem 8** Let $h \in C_k^k(\mathbb{R}^+) \cap C_k^k([0, \infty))$ and $u < \zeta < x$, by Taylor’s expansion, we have

$$h(u) = h(x) + h'(x)(u - x) + \frac{h''(\zeta)}{2!}(u - x)^2$$

(3.9)

where $\Lambda(u, x)$ is given by

$$\Lambda(u, x) = \frac{h''(\zeta) - h''(x)}{2!}(u - x)^2.$$ 

Applying operator $M_{\eta, \alpha, \rho}^\eta$ on equation (3.9), we obtain

$$\left| M_{\eta, \alpha, \rho}^\eta(h; x) - h(x) - h'(x)M_{\eta, \alpha, \rho}^\eta(u - x; x) - \frac{h''}{2!}M_{\eta, \alpha, \rho}^\eta((u - x)^2; x) \right|$$

$$\leq |M_{\eta, \alpha, \rho}^\eta(\Lambda(u, x); x)|.$$ 

(3.10)
By the definition (3.7) of weighted modulus of continuity,
\[
\left| \Lambda(u, x) \right| \leq \frac{1}{2!} \Omega(h''; |\zeta - x|)(1 + (\zeta - x)^2)(1 + x^2)(u - x)^2
\]
\[
\leq \frac{1}{2!} \Omega(h''; |\zeta - x|)(1 + (\zeta - x)^2)(1 + x^2)(u - x)^2
\]
\[\leq \left( 1 + \frac{|u - x|}{\delta} \right)(1 + \delta^2)\Omega(h''; \delta)(1 + (u - x)^2)(1 + x^2)(u - x)^2, \quad \delta > 0\]
\[
\leq \begin{cases} 
2(1 + \delta^2)^2(1 + x^2)\Omega(h''; \delta)(u - x)^2, & |u - x| < \delta; \\
2(1 + \delta^2)^2(1 + x^2)\frac{(u - x)^4}{\delta^4}\Omega(h''; \delta)(u - x)^2, & |u - x| \geq \delta.
\end{cases}
\]
(3.11)
\[
\leq 2(1 + \delta^2)^2(1 + x^2)\Omega(h''; \delta)\left( 1 + \frac{(u - x)^4}{\delta^4} \right)(u - x)^2.
\]

Now, selecting \( \delta < 1 \), from (3.11), we obtain
\[
|\Lambda(u, x)| \leq 8(1 + x^2)\Omega(h''; \delta)\left( (u - x)^2 + \frac{(u - x)^2(u - x)^4}{\delta^4} \right)
\]
(3.12)

Applying operator \( M_{\alpha, \rho}^n \) on above inequality and considering Lemma 3, we obtain
\[
|\alpha M_{\alpha, \rho}^n(\Lambda(u, x); x)| \leq 8\alpha(1 + x^2)\Omega(h''; \delta)\left( M_{\alpha, \rho}^n((u - x)^2; x) + \frac{1}{\delta^4}M_{\alpha, \rho}^n((u - x)^6; x) \right) = 8\alpha(1 + x^2)\Omega(h''; \delta)\left( O\left( \frac{1}{\alpha} \right) + \frac{1}{\delta^4}O\left( \frac{1}{\alpha^2} \right) \right),
\]
(3.13)
as \( \alpha \to \infty \).

Now, choosing \( \delta = \frac{1}{\sqrt{\alpha}} \), we obtain
\[
|\alpha M_{\alpha, \rho}^n(\Lambda(u, x); x)| \leq 8\alpha(1 + x^2)\Omega(h''; \frac{1}{\sqrt{\alpha}})O(1).
\]
(3.14)

On collecting (3.10), (3.14)and using Lemma 2, we arrive to required result.

In our next result we discuss Grüss-Voronovskaja type theorem for the operator defined by (1.3). Grüss inequality [12] measures the difference of integral of two functions with the product of integral of the two functions. Acu et al. [4], were first to show the application of Grüss inequality in approximation theory. In [11], Gonska and Tachev discussed Grüss-type inequality using second order modulus of smoothness. Gal and Gonska [10], proved Grüss-Voronovskaya estimates for the first time using Grüss inequality for Bernstein operators and for a class of

**Theorem 9** For $\hat{h}, g, h', g', h'', (hg)', (hg)'' \in C_k^1(\mathbb{R}^+)$, we have the following equality

$$\lim_{\alpha \to \infty} \alpha \{M_{\alpha,\rho}^n(hg; x) - M_{\alpha,\rho}^n(h; x)M_{\alpha,\rho}^n(g; x)\} = x(l x + 1)h'(x)g'(x).$$

**Proof.** By a straightforward calculation, we may write

$$\alpha \{M_{\alpha,\rho}^n(hg; x) - M_{\alpha,\rho}^n(h; x)M_{\alpha,\rho}^n(g; x)\}$$

as

$$\alpha \left\{ M_{\alpha,\rho}^n(hg; x) - h(x)g(x) - M_{\alpha,\rho}^n(u - x; x)(hg)'(x) - \frac{M_{\alpha,\rho}^n((u - x)^2; x)}{2!}(hg)''(x) - g(x) \left[ M_{\alpha,\rho}^n(h; x) - h(x) \right] ight. \left. \times \left[ M_{\alpha,\rho}^n(g; x) - g(x) - M_{\alpha,\rho}^n(u - x; x)g'(x) - \frac{M_{\alpha,\rho}^n((u - x)^2; x)}{2!}g''(x) \right] ight. \right.$$

$$\left. + 2 \frac{M_{\alpha,\rho}^n((u - x)^2; x)}{2!}h'(x)g'(x) + g''(x) \frac{M_{\alpha,\rho}^n((u - x)^2; x)}{2!} \right\}.$$ 

Now, in view of Theorem 6, it follows that $M_{\alpha,\rho}^n(h; x) \to h(x)$, as $\alpha \to \infty$ and using Theorem 8, we have

$$\alpha \{M_{\alpha,\rho}^n(h; x) - h(x) - M_{\alpha,\rho}^n(u - x; x)h'(x) - \frac{M_{\alpha,\rho}^n((u - x)^2; x)}{2!}h''(x)\} \to 0, \text{ as } \alpha \to \infty,$$

since $h', h'' \in C_k^1(\mathbb{R}^+)$. Thus, using Theorem 6, 8 and Lemma 3, we obtain the required result

$$\lim_{\alpha \to \infty} \alpha \{M_{\alpha,\rho}^n(hg; x) - M_{\alpha,\rho}^n(h; x)M_{\alpha,\rho}^n(g; x)\} = x(l x + 1)h'(x)g'(x).$$

Acknowledgement: The first author expresses her sincere thanks to “The Ministry of Human Resource and Development”, India for the financial assistance without which the above work would not have been possible.
References


[12] G. Grüss, *Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$*, Math Z., vol. 39, 1935, 215-226.


Sheetal Deshwal
Indian Institute of Technology Roorkee, India
Research Scholar
Department of Mathemtics
IIT Roorkee, Uttarakhand-247667, India
e-mail: shetald1990@gmail.com

P. N. Agrawal
Indian Institute of Technology Roorkee, India
Faculty
Department of Mathematics
IIT Roorkee, Uttarakhand-247667, India
e-mail: pnappfma@gmail.com
Some families of rational Heun functions and combinatorial identities

Bără Adina - Elena

Abstract

Continuing some previous investigations, we present new families of rational and polynomial Heun functions, and new combinatorial identities.

2010 Mathematics Subject Classification: 33E30, 05A19.

Key words and phrases: Heun functions, combinatorial identities

1 Introduction

The general Heun equation is

\[ y''(x) + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right)y'(x) + \left( \frac{\alpha\beta x - q}{x(x-1)(x-a)} \right)y(x) = 0, \]

where

\[ a \notin \{0, 1\}, \gamma \notin \{0, -1, -2, \ldots\} \]

and

\[ \alpha + \beta + 1 = \gamma + \delta + \epsilon. \]

Its solution \( y(x) \) normalized by \( y(0)=1 \) is called the (local) Heun function and is denoted by \( Hl(a, q; \alpha, \beta; \gamma, \delta; x) \). See, e.g., [10],[8],[7] and the references therein.

It was proved in [9] that

\[ Hl\left(\frac{1}{2}, -n; -2n, 1, 1; x \right) = \sum_{k=0}^{n} \binom{n}{k}x^k(1-x)^{n-k}^2, \]

\[ Hl\left(\frac{1}{2}, n; 2n, 1, 1; -x \right) = \sum_{k=0}^{\infty} \binom{n+k-1}{k}x^k(1+x)^{-n-k}^2. \]

Received 10 August, 2017
Accepted for publication (in revised form) 1 October, 2017
These functions are related to the Rényi entropy and Tsallis entropy; see, e.g., [9] and [3].

Closed forms of the functions

\[ H_l\left(\frac{1}{2}, -2n\theta; -2n, 2\theta; \gamma, \gamma; x\right) \]

and

\[ H_l\left(\frac{1}{2}, 2n\theta; 2n, 2\theta; \gamma, \gamma; x\right) \]

were presented in [2] and [1]. The corresponding results were used in order to derive several combinatorial identities, generalizing some formulas from [4].

Closed forms of Heun functions (in particular, rational and polynomial Heun functions) are important in applications; see, e.g., [7], [5], [11] and the references therein.

In this paper we continue the investigations from [1]. We present new families of rational and polynomial Heun functions, and new combinatorial identities.

We shall use the notation

\[ (a)_0 := 1, (a)_k := a(a+1)...(a+k-1), k \geq 1, \]

\[ a_{nk} := 4^{-n} \binom{2k}{k} \binom{2n-2k}{n-k}, r_{nk} := \binom{n}{k}^{-1} a_{nk}. \]

2 Heun functions

In what follows we need the following results.

**Theorem 1** ([6],[2,Prop1], [2;(14)], [2;(15)], [1]) Let \( \alpha \beta \neq 0 \). Then

(1)

\[ H_l\left(\frac{1}{2}, \frac{1}{2}(\alpha + 2)(\beta + 2); \alpha + 2, \beta + 2; \gamma + 1, \gamma + 1; x\right) = \]

\[ = \frac{\gamma}{\alpha \beta} (1 - 2x)^{-1} \frac{d}{dx} H_l\left(\frac{1}{2}, \frac{1}{2} \alpha \beta; \alpha, \beta; \gamma, \gamma; x\right), \]

(2)

\[ H_l\left(\frac{1}{2}, \frac{1}{2}(2\gamma - \alpha)(2\gamma - \beta); 2\gamma - \alpha, 2\gamma - \beta; \gamma + 1, \gamma + 1; x\right) = \]

\[ = \frac{\gamma}{\alpha \beta} (1 - 2x)^{\alpha + \beta + 1 - 2\gamma} \frac{d}{dx} H_l\left(\frac{1}{2}, \frac{1}{2} \alpha \beta; \alpha, \beta; \gamma, \gamma; x\right). \]

**Theorem 2** ([2;Th.1],[2;Cor.2])

(i) Let \( n \geq 0 \) be an integer and \( \theta \in R \). Then

(3)

\[ H_l\left(\frac{1}{2}, -2n\theta; -2n, 2\theta; \gamma, \gamma; x\right) = \sum_{k=0}^{n} 4^k \binom{n}{k} \frac{(\theta)_k}{(\gamma)_k} (x^2 - x)^k. \]
(ii) Let $\gamma$ and $n$ be integers, $0 < \gamma \leq n$, and $\theta \in \mathbb{R}$. Then

\begin{equation}
Hl\left(\frac{1}{2}, 2n\theta; 2n, 2\theta; \gamma, \gamma; x\right) =
\end{equation}

\begin{equation}
= (1 - 2x)^{2(\gamma - n - \theta)} \sum_{k=0}^{n-\gamma} 4^k \binom{n - \gamma}{k} \frac{(\gamma - \theta)_k (x^2 - x)^k}{(\gamma)_k}.
\end{equation}

Now we can state the main results of this section.

**Theorem 3** Let $0 \leq m \leq n - 1$. Then

\begin{equation}
Hl\left(\frac{1}{2}, n + 1; 2n + 2, 1; m + 2, m + 2; x\right) =
\end{equation}

\begin{equation}
= \frac{(m + 1)2^{2m+1}}{(n - m)(2m + 1)} \binom{n}{m}^{-1} \binom{2m}{m}^{-1} \times
\end{equation}

\begin{equation}
\sum_{k=0}^{n-m-1} (k + 1) \binom{m + k + 1}{m} a_{n,m+k+1}(1 - 2x)^{2m-2n+1+2k}
\end{equation}

\begin{equation}
= (1 - 2x)^{2m-2n+1} \sum_{k=0}^{n-m-1} 4^k \binom{n - m - 1}{k} \left(\frac{m + \frac{3}{2}}{m + 2}\right)_k (x^2 - x)^k.
\end{equation}

**Proof.** According to [1, Th2.1] we have

\begin{equation}
Hl\left(\frac{1}{2}, (2m + 1)(m - n); 2(m - n), 2m + 1; m + 1, m + 1; x\right) =
\end{equation}

\begin{equation}
= 4^m \binom{n}{m}^{-1} \binom{2m}{m}^{-1} \sum_{j=0}^{n-m} \binom{m + j}{m} a_{n,m+j}(1 - 2x)^{2j}.
\end{equation}

By using (2) and (6) we get

\begin{equation}
Hl\left(\frac{1}{2}, n + 1; 2n + 2, 1; m + 2, m + 2; x\right) =
\end{equation}

\begin{equation}
= \frac{m + 1}{2(m - n)(2m + 1)} (1 - 2x)^{2m-2n} \times
\end{equation}

\begin{equation}
d\frac{d}{dx} Hl\left(\frac{1}{2}, (2m + 1)(m - n); 2(m - n), 2m + 1; m + 1, m + 1; x\right) =
\end{equation}

\begin{equation}
= \frac{m + 1}{2(m - n)(2m + 1)} (1 - 2x)^{2m-2n} 4^m \binom{n}{m}^{-1} \binom{2m}{m}^{-1} \times
\end{equation}
Now the first equality in (5) follows by setting \(j = k + 1\).

In order to prove that the first member and the last member of (5) are equal, it suffices to use (4) with \(\gamma = m + 2, \theta = \frac{1}{2}\) and \(n\) replaced by \(n+1\).

**Remark 1** Since \(2m - 2n + 1 \leq -1\), (5) shows that \(Hl(\frac{1}{2}, n+1; 2n+2, 1; m+2, m+2; x)\) is a rational function for which \(\frac{1}{2}\) is a pole of order \(2n - 2m - 1 \geq 1\). In particular, for \(m = n - 1\) we get

\[
Hl(\frac{1}{2}, n+1; 2n+2, 1; n+1, n+1; x) = \frac{1}{1 - 2x}.
\]

**Theorem 4** Let \(0 \leq k \leq n - 1\). Then

\[
Hl(\frac{1}{2}, (k - n + 1)(2k + 3); 2(k - n + 1), 2k + 3; 2k + 2, 2k + 2; x) =
\]

\[
= 2^{2k+1} \left(\frac{n}{n-k}\right)^{-1} \left(\frac{n}{k}\right)^{-1} \times
\]

\[
\sum_{j=0}^{n-k-1} (j+1) \binom{2n-2j-2}{2k} \binom{n}{j+1} r_{n,k+j+1}(1 - 2x)^{2j} =
\]

\[
= \sum_{j=0}^{n-k-1} 4^j \binom{n-k-1}{j} \binom{k + \frac{3}{2}}{j} (x^2 - x)^j.
\]

**Proof.** From [1;2.12] we know that

\[
Hl(\frac{1}{2}, (k - n)(2k + 1); 2(k - n), 2k + 1; 2k + 1, 2k + 1; x) =
\]

\[
= 4^k \left(\frac{n+k}{n}\right)^{-1} \left(\frac{n}{k}\right)^{-1} \sum_{i=0}^{n-k} \binom{2n-2i}{2k} \binom{n}{i} r_{n,k+i}(1 - 2x)^{2i}.
\]

Combining (1) and (8) we get

\[
Hl(\frac{1}{2}, (k - n + 1)(2k + 3); 2(k - n + 1), 2k + 3; 2k + 2, 2k + 2; x) =
\]

\[
= \frac{1}{2(k-n)}(1 - 2x)^{-1} \frac{d}{dx} Hl(\frac{1}{2}, (k - n)(2k + 1); 2(k - n), 2k + 1; 2k + 1, 2k + 1; x) =
\]
Some families of rational Heun functions and combinatorial identities

\[
\frac{2^{2k+1}}{n-k} \binom{n+k}{k}^{-1} \binom{n}{k}^{-1} \sum_{i=1}^{n-k} i \binom{2n-2i}{2k} \binom{n}{i} r_{n,k+i}(1-2x)^{2i-2}.
\]

In order to prove the first equality in (7), it suffices to replace \(i\) by \(j+1\). The second equality is a consequence of (3), if we put \(\gamma = 2k + 2, \theta = k + \frac{3}{2}\), and replace \(n\) by \(n-k\).

**Remark 2** From Theorem 4 we see that

\[
\text{Hl}\left(\frac{1}{2}, (k - n + 1)(2k + 3); 2(k - n + 1), 2k + 3; 2k + 2, 2k + 2; x\right)
\]

is a polynomial of degree \(2(n-k-1)\). In particular,

\[
\text{Hl}\left(\frac{1}{2}, 1 - 2n; -2, 2n - 1; 2n - 2, 2n - 2; x\right) = 1 + \frac{2n - 1}{n - 1}(x^2 - x).
\]

**Theorem 5** Let \(0 \leq k \leq n - 1\). Then

(9) \[
\text{Hl}\left(\frac{1}{2}, (2k + 3)(k - n + 1); 2(k - n + 1), 2k + 3; 2k + 3, 2k + 3; x\right) =
\]

\[
= \frac{4^{k+1}}{n+1} \binom{n+k+1}{n+1}^{-1} \binom{n}{k+1}^{-1} \times
\]

\[
\sum_{j=0}^{n-k-1} \binom{2k+2j+2}{2j} \binom{n+1}{k+j+1} r_{nj}(n-k-j)(1-2x)^{2n-2k-2j-2} =
\]

\[
= \sum_{j=0}^{n-k-1} 4^j \binom{n-k-1}{j} \left(\frac{k+\frac{3}{2}}{2k+3}\right)^j (x^2-x)^j.
\]

**Proof.** According to [1;(2.21)] we have

(10) \[
\text{Hl}\left(\frac{1}{2}, (k - n)(2k + 1); 2(k - n), 2k + 1; 2k + 2, 2k + 2; x\right) =
\]

\[
= \frac{4^k}{n+1} \binom{n+k+1}{n}^{-1} \binom{n}{k}^{-1} \times
\]

\[
\sum_{j=0}^{n-k} \binom{2k+2j+2}{2j} \binom{n+1}{k+j+1} r_{nj}(1-2x)^{2n-2k-2j}.
\]

By using (1) and (10) it follows that

\[
\text{Hl}\left(\frac{1}{2}, (2k + 3)(k - n + 1); 2(k - n + 1), 2k + 3; 2k + 3, 2k + 3; x\right) =
\]

\[
= \frac{2k + 2}{2(k-n)(2k + 1)} (1-2x)^{-1} \times
\]
\[ \frac{d}{dx} H_l(\frac{1}{2}, (2k + 1)(k - n); 2(k - n), 2k + 1; 2k + 2, 2k + 2; x) = \]
\[ = \frac{2k + 2}{2(k - n)(2k + 1)} 4^k 2k + 1 \left(\frac{n + k + 1}{n}\right)^{-1} \left(\frac{n}{k}\right)^{-1} \times \]
\[ \times \sum_{j=0}^{n-k} \left(\frac{2k + 2j + 2}{2j}\right) \left(\frac{n + 1}{k + j + 1}\right) r_{nj}(-4)(n - k - j)(1 - 2x)^{2n - 2k - 2j - 2} = \]
\[ = \frac{4^{k+1}(k + 1)}{(n - k)(n + 1)} \left(\frac{n + k + 1}{n}\right)^{-1} \left(\frac{n}{k}\right)^{-1} \times \]
\[ \times \sum_{j=0}^{n-k-1} \left(\frac{2k + 2j + 2}{2j}\right) \left(\frac{n + 1}{k + j + 1}\right) r_{nj}(n - k - j)(1 - 2x)^{2n - 2k - 2j - 2}. \]

This leads immediately to the first equality in (9). The second equality in (9) is a consequence of (3) with \( \gamma = 2k + 3, \theta = k + \frac{3}{2} \) and \( n \) replaced by \( n-k-1 \).

**Remark 3** Theorem 5 shows that
\( H_l(\frac{1}{2}, (2k+3)(k-n+1); 2(k-n+1), 2k+3, 2k+3, 2k+3; x) \) is a polynomial of degree \( 2(n-k-1) \). For example, \( H_l(\frac{1}{2}, 1-2n; -2, 2n-1; 2n-1, 2n-1, x) = 2x^2 - 2x + 1 \).

### 3 Combinatorial identities

By using the second equality in (5) we get
\[ \frac{(m + 1)2^{2m+1}}{(n - m)(2m + 1)} \left(\frac{n}{m}\right)^{-1} \left(\frac{2m}{m}\right)^{-1} \times \]
\[ \times \sum_{k=0}^{n-m-1} (k + 1) \left(\frac{m + k + 1}{m}\right) a_{n,m-k+1}(1 - 2x)^{2k} = \]
\[ = \frac{n-m-1}{\sum_{j=0}^{n-m-1} 4^j \left(\frac{n - m - 1}{j}\right) \left(\frac{m + \frac{3}{2}}{m + 2}\right)_j (x^2 - x)^j} = \]
\[ = \sum_{j=0}^{n-m-1} 4^j \left(\frac{n - m - 1}{j}\right) \left(\frac{m + \frac{3}{2}}{m + 2}\right)_j 4^{-j} \sum_{k=0}^{j} \left(\frac{j}{k}\right) (-1)^{j-k}(1 - 2x)^{2k} = \]
\[ = \sum_{k=0}^{n-m-1} \sum_{j=k}^{n-m-1} \left(\frac{n - m - 1}{j}\right) \left(\frac{m + \frac{3}{2}}{m + 2}\right)_j \left(\frac{j}{k}\right) (-1)^{j-k}(1 - 2x)^{2k}. \]

From this we derive
Corollary 1 Let $0 \leq m \leq n - 1$ and $0 \leq k \leq n - m - 1$. Then
\[
\sum_{j=k}^{n-m-1} \binom{n-m-1}{j} \frac{(m+\frac{3}{2})j}{(m+2)j} \binom{j}{k} (-1)^{j-k} = 
\]
\[
= \frac{2^{2m+1}}{2m+1} \binom{2m}{m}^{-1} \binom{n}{m+1}^{-1} (k+1) \binom{m+k+1}{m} a_{n,m+k+1}.
\]

Example 1 In the preceding corollary set $m = k = 0$ and replace $n$ by $n+1$; we get
\[
\sum_{j=0}^{n} (-4)^{-j} \binom{n}{j} \binom{2j+2}{j+1} = \frac{1}{(n+1)2^{2n-1}} \binom{2n}{n}
\]

Coming back to the second equality in (5), we can write
\[
\sum_{k=0}^{n-m-1} 4^k \binom{n-m-1}{k} \frac{(m+\frac{3}{2})k}{(m+2)k} (x^2 - x)^k = 
\]
\[
= \frac{2^{2m+1}}{2m+1} \binom{2m}{m}^{-1} \binom{n}{m+1}^{-1} \times
\]
\[
\times \sum_{j=0}^{n-m-1} (j+1) \binom{m+j+1}{m} a_{n,m+j+1} (1 - 2x)^{2j} = 
\]
\[
= \frac{2^{2m+1}}{2m+1} \binom{2m}{m}^{-1} \binom{n}{m+1}^{-1} \times
\]
\[
\times \sum_{j=0}^{n-m-1} (j+1) \binom{m+j+1}{m} a_{n,m+j+1} \sum_{k=0}^{j} \binom{j}{k} 4^k (x^2 - x)^k = 
\]
\[
= \sum_{k=0}^{n-m-1} \sum_{j=k}^{n-m-1} \frac{2^{2m+1}}{2m+1} \binom{2m}{m}^{-1} \binom{n}{m+1}^{-1} \times
\]
\[
\times (j+1) \binom{m+j+1}{m} \binom{j}{k} 4^k a_{n,m+j+1} (x^2 - x)^k.
\]

This leads to the following

Corollary 2 Let $0 \leq m \leq n - 1$ and $0 \leq k \leq n - m - 1$. Then
\[
\sum_{j=k}^{n-m-1} (j+1) \binom{m+j+1}{m} \binom{j}{k} a_{n,m+j+1} = \frac{2m+1}{2^{2m+1}} \binom{2m}{m} \times
\]
\[
\times \binom{n}{m+1} \binom{n-m-1}{k} \frac{(m+\frac{3}{2})k}{(m+2)k}. 
\]
Exemple 2 For $m=k=0$ and $n$ replaced by $n+1$, the previous identity reduces to

$$\sum_{j=0}^{n} (j+1) \binom{2n-2j}{n-j} \binom{2j+2}{j+1} = (n+1)2^{2n+1}.$$

See also [1;(2.19)].

References


Bărăr Adina - Elena
Technical University of Cluj-Napoca
Department of Mathematics
Str Memorandumului nr 28, 400114 Cluj-Napoca, Romania
e-mail: ellenasontica@yahoo.com
Voronovskaja-type formulas and applications

Fadel Nasaireh

Abstract
We continue previous investigations concerning Voronovskaja-type formulas for inverses of positive linear operators.

2010 Mathematics Subject Classification: 41A36.
Key words and phrases: positive operators; nonpositive operators; Voronovskaja type formulas.

1 Introduction
In this paper we continue previous investigations (see [5], [6], [8]) concerning Voronovskaja-type formulas for inverses of classical positive linear operators.

Section 2 is devoted to Voronovskaja-type formulas for $U^{-1}_n$, where $U_n$ is the genuine Bernstein-Durrmeyer operator. In Section 3 we present an application of the Voronovskaja-type formula for $B^{-1}_n$, the inverse of the Beta operator of Mühlbach and Lupas. Section 4 contains an application involving Bernstein operators.

Throughout the paper we consider the Banach space $C[0,1]$ of all continuous, real-valued functions, equipped with the supremum norm.

For a function $f \in C[0,1]$, $n \in \mathbb{N}$, $x \in [0,1]$ the Beta operators are given by

$$
\mathbb{B}_nf(x) = \begin{cases} 
  f(0), & x = 0, \\
  \frac{1}{B(x,n-nx)} \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t) dt, & 0 < x < 1, \\
  f(1), & x = 1,
\end{cases}
$$

with Euler’s Beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$, $x, y > 0$. The classical Bernstein operators $B_n : C[0,1] \to C[0,1]$ are defined by

$$
B_n(f;x) = \sum_{j=0}^{n} p_{n,j}(x) f \left( \frac{j}{n} \right), \quad x \in [0,1],
$$

received 1 June, 2017
accepted for publication (in revised form) 18 September, 2017

1

37
where
\[ p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}. \]

The genuine Bernstein-Durrmeyer operators \( U_n : C[0,1] \to \Pi_n \), are explicitly given by
\[
U_n(f;x) = f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1) \sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-2,k-1}(t)f(t)dt.
\]

where \( p_{-1,k} = 0 \) and \( p_{0,k} = 1 \).

By \( \Pi \) we denote the set of all polynomial functions defined on \([0,1]\). \( \Pi_m \) stands for the subset of \( \Pi \) consisting of the polynomial functions of degree \( \leq m \).

## 2 Voronovskaja-type formulas for \( U_n^{-1} \)

We recall a result from [8]; see also [6].

Let \( X \) be a Banach space and \( W \subset Z \subset Y \) linear subspaces of \( X \).

Let \( A, B : Y \to X; U, V : Z \to X; S, T : W \to X \) be linear operators.

Consider also two sequences of linear operators \( P_n : X \to X, Q_n : Y \to X, n \geq 1 \), and suppose that each \( P_n \) is bounded.

**Theorem 1** ([8; Th.2.1])

(i) Suppose that

1. \( \lim_{n \to \infty} P_n x = x, \ x \in X \),
2. \( \lim_{n \to \infty} n(P_n y - y) = Ay ; \lim_{n \to \infty} n(Q_n y - y) = By, \ y \in Y \).

Then

3. \( \lim_{n \to \infty} n(P_n Q_n y - y) = Ay + By, \ y \in Y \).

(ii) In addition to (1) and (2), suppose that

4. \( Bz \in Y, \ z \in Z \),
5. \( \lim_{n \to \infty} n(P_n z - z - Az) = Uz ; \lim_{n \to \infty} n(Q_n z - z - Bz) = Vz, \ z \in Z \).

Then

6. \( \lim_{n \to \infty} n(P_n Q_n z - z) - Az - Bz = Uz + Vz + ABz, \ z \in Z \).
(iii) Let (1), (2), (4), (5) be satisfied. Moreover, suppose that for each \( w \in W \) we have \( Vw \in Y, Bw \in Z, \) and

\[
\lim_{n \to \infty} n \left\{ n \left( P_n w - w \right) - Aw \right\} = Sw,
\]

\[
\lim_{n \to \infty} n \left\{ n \left( Q_n w - w \right) - Bw \right\} = Tw.
\]

Then, for all \( w \in W, \)

\[
\lim_{n \to \infty} n \left\{ n \left( P_n Q_n w - w \right) - Aw - Bw \right\} = Sw + Tw + AVw + UBw.
\]

Now we are in a position to prove

**Theorem 2** Let \( m \geq 0 \) and \( p_n \in \Pi_m, n \geq 1. \) Suppose that the sequence \( (p_n) \) is uniformly convergent on \([0, 1]\) to \( p \in \Pi_m. \) Then

\[
\lim_{n \to \infty} n \left( U_n^{-1} p_n(t) - p_n(t) \right) = -t (1 - t) p^{(2)}(t),
\]

\[
\lim_{n \to \infty} n \left( U_n^{-1} p_n(t) - p_n(t) \right) + t (1 - t) p^{(2)}(t) =\]

\[
\frac{t (1 - t)}{2} \left[ t (1 - t) p^{(4)}(t) + 2 (1 - 2t) p^{(3)}(t) - 2 p^{(2)}(t) \right],
\]

uniformly on \([0, 1].\)

**Proof.** It is well-known (see, e.g., [1], [2], [4], [7], [9], [10]) that \( U_n \) has eigenpolynomials \( q_0(t) = 1, q_1(t) = t, \) with eigenvalue 1, and

\[
q_k(t) = \frac{d^{k-2}}{dt^{k-2}} \left[ t^{k-1} (1 - t)^{k-1} \right], \quad k = 2, ..., n,
\]

with eigenvalues \( \lambda_{nk} = \frac{(n-1)!n!}{(n-k)!k!(n-k)!}. \) Moreover,

\[
t (1 - t) q_k^{(2)}(t) + k (k - 1) q_k(t) = 0, \quad k \geq 0.
\]

(7) Let \( n \geq k \geq 0. \) Then \( U_n q_k = \lambda_{nk} q_k, \) and therefore \( U_n^{-1} q_k = \frac{1}{\lambda_{nk}} q_k. \) For \( t \in [0, 1] \) we have

\[
\lim_{n \to \infty} n \left( U_n^{-1} q_k(t) - q_k(t) \right) = \lim_{n \to \infty} n \left( \frac{1}{\lambda_{nk}} - 1 \right) q_k(t) =
\]

\[
\lim_{n \to \infty} \left( \frac{n(n + 1) \ldots (n + k - 1)}{n(n - 1) \ldots (n - k + 1)} - 1 \right) q_k(t) = k (k - 1) q_k(t).
\]

Thus, according to (9), we have

\[
\lim_{n \to \infty} n \left( U_n^{-1} q_k(t) - q_k(t) \right) = -t (1 - t) q_k^{(2)}(t), \quad k \geq 0,
\]
uniformly on \([0, 1]\). This leads immediately to

\[
\lim_{n \to \infty} n \left( U_n^{-1} p(t) - p(t) \right) = -t \left( 1 - t \right) p^{(2)}(t), \quad p \in \Pi,
\]

uniformly on \([0, 1]\). Now let \(p_n = \sum_{j=0}^{m} a_{nj} e_j \in \Pi_m\) be uniformly convergent on \([0, 1]\) to \(p = \sum_{j=0}^{m} a_j e_j \in \Pi_m\). Then \(\lim_{n \to \infty} a_{nj} = a_j\), and (10) leads to

\[
\lim_{n \to \infty} n \left( U_n^{-1} p_n(t) - p_n(t) \right) = \lim_{n \to \infty} \sum_{j=0}^{m} a_{nj} \left( U_n^{-1} e_j(t) - e_j(t) \right) =
\]

\[
= -t \left( 1 - t \right) \left( \sum_{j=0}^{m} a_j e_j \right)^{(2)}(t) = -t \left( 1 - t \right) p^{(2)}(t).
\]

This proves (7).

(8) Let again \(n \geq k \geq 0\) and \(t \in [0, 1]\). Then

\[
n \left[ n \left( U_n^{-1} q_k(t) - q_k(t) \right) + t \left( 1 - t \right) q_k^{(2)}(t) \right] =
\]

\[
= n \left[ \frac{n(n+1) \ldots (n+k-1)}{n(n-1) \ldots (n-k+1)} - 1 \right] q_k(t) + t \left( 1 - t \right) q_k^{(2)}(t) =
\]

\[
= n \left[ \frac{n^{k-1}k(k-1) + \text{terms of degree } k-1}{(n-1)(n-2) \ldots (n-k+1)} - k(k-1) q_k(t) \right].
\]

Consider the operators \(P_n := U_n, Q_n := U_n^{-1},\)

\[
Ap(t) := t \left( 1 - t \right) p^{(2)}(t), \quad Bp(t) := -t \left( 1 - t \right) p^{(2)}(t),
\]

\[
U p(t) = \frac{t(1-t)}{2} \left[ t \left( 1 - t \right) p^{(4)}(t) + 2 \left( 1 - 2t \right) p^{(3)}(t) - 2p^{(2)}(t) \right].
\]

According to [8; Example 2.2], we have

\[
U p = \lim_{n \to \infty} n \left[ n (P_n p - p) - Ap \right], \quad p \in \Pi.
\]

The above computation shows that there exists the operator

\[
V p(t) := \lim_{n \to \infty} n \left( Q_n p(t) - p(t) - Bp(t) \right), \quad p \in \Pi.
\]

According to (6),

\[
U p + V p + ABp = 0, \quad p \in \Pi.
\]

By a straightforward calculation, (12) yields

\[
V p(t) = \frac{t(1-t)}{2} \left[ t \left( 1 - t \right) p^{(4)}(t) + 2 \left( 1 - 2t \right) p^{(3)}(t) - 2p^{(2)}(t) \right], \quad p \in \Pi.
\]
Now (11) and (13) imply
\[
\lim_{n \to \infty} n \left[ n U_n^{-1} p(t) - p(t) \right] + t (1 - t) p''(t) = \frac{t (1 - t)}{2} \left[ t (1 - t) p^{(4)}(t) + 2 (1 - 2t) p^{(3)}(t) - 2 p''(t) \right], \quad p \in \Pi.
\]
In order to get (8) it suffices to apply the argument which was used in the final part of the proof of (7). \qed

3 Voronovskaja-type formula for \( B_n \)

According [8, (3.4)],
\[
B_n^{-1} p(t) = \sum_{k=0}^{m} \frac{n (n + 1) \ldots (n + k - 1)}{n^k} \left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n}; p \right] t^k,
\]
where \([0, -\frac{1}{n}, \ldots, -\frac{k}{n}; p]\) is the divided difference of \( p \) on the knots \( 0, -\frac{1}{n}, \ldots, -\frac{k}{n} \). Moreover, it was proved in [5], [8] that
\[
\lim_{n \to \infty} n \left( B_n^{-1} p(t) - p(t) \right) = -\frac{t (1 - t)}{2} p''(t), \quad p \in \Pi
\]
uniformly on \([0, 1]\). By considering the difference of order \( k \), we have
\[
\left[ -\frac{k}{n}, \ldots, -\frac{1}{n}, 0; p \right] = \frac{n^k}{k!} \Delta^k \frac{p}{n} \left( -\frac{k}{n} \right),
\]
and so (14) can be written as
\[
B_n^{-1} p(t) = \sum_{k=0}^{m} \frac{n (n + 1) \ldots (n + k - 1) \Delta^k \frac{p}{n} \left( -\frac{k}{n} \right)}{k!} x^k,
\]
where \( p \in \Pi_m, \quad n \geq 1 \).
Now (15) and (16) imply
\[
\lim_{n \to \infty} \sum_{k=1}^{m} \frac{n (n + 1) \ldots (n + k - 1) \Delta^k \frac{p}{n} \left( -\frac{k}{n} \right) - p^{(k)}(0)}{k!} x^k = \frac{1}{2} \sum_{k=1}^{m} \frac{(k - 1) p^{(k)}(0) - p^{(k+1)}(0)}{(k - 1)!} x^k, \quad p \in \Pi_m.
\]
It follows immediately:
Corollary 1 For each $p \in \Pi$ and $k \geq 0$, we have

$$\lim_{n \to \infty} n \left[ \frac{(n+k-1)!}{(n-1)!} \Delta^k \frac{p}{n} \left( -\frac{k}{n} \right) - p^{(k)}(0) \right] = \frac{k}{2} \left[ (k-1) p^{(k)}(0) - p^{(k+1)}(0) \right].$$

It would be interesting to give a direct proof of (17).

4 An application to Bernstein polynomials

In this section we consider the operators $F_n : C[0,1] \to \Pi_n$, $F_n := \overline{E}_n B_n$. It was proved in [3, (19)] that

$$F_n f = \sum_{j=0}^{n} \left[ 0, \frac{1}{n}, \ldots, \frac{j}{n}; f \right] \frac{1}{(n-j)! n^{2j-1}} \left\{ \sum_{k=0}^{j} (-1)^{j-k} (n+k-1)! S(j,k) e_k, \right.$$  

where $S(j,k)$ are the Stirling numbers of second kind. The equation (18) can be written also as

$$F_n f = \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-1)! n^k} \left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n}; B_n f \right] e_k.$$

On the other hand, (14) implies, for $m = n$ and $f \in C[0,1]$,

$$\overline{E}_n B_n f = \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-1)! n^k} \left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n}; B_n f \right] e_k.$$

From (20) we infer that

$$F_n f = \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-1)! n^k} \left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n}; B_n f \right] e_k.$$

Now (19) and (21) lead to

$$\frac{(n+k-1)!}{(n-1)! n^k} \left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n}; B_n f \right] = (n+k-1)! \sum_{j=k}^{\infty} \left[ 0, \frac{1}{n}, \ldots, \frac{j}{n}; f \right] \frac{(-1)^{j-k}}{(n-j)! n^{2j-1}} S(j,k).$$

This implies

$$\left[ 0, -\frac{1}{n}, \ldots, -\frac{k}{n}; B_n f \right] = \sum_{j=k}^{\infty} (-1)^{j-k} S(j,k) \binom{n}{j} n^{k-j} \frac{j!}{n} \left[ 0, \frac{1}{n}, \ldots, \frac{j}{n}; f \right].$$

Thus we have proved
Theorem 3 Let $0 \leq k \leq n$ and $f \in C[0,1]$. Then

\begin{equation}
\left[0, -\frac{1}{n}, \ldots, -\frac{k}{n}; B_n f\right] = \sum_{j=k}^{n} (-1)^{j-k} S(j,k) \binom{n}{j} n^{k-j} \Delta_{\frac{j}{n}} f(0).
\end{equation}

It would be interesting to prove (22) directly.

References


Fadel Nasaireh
Technical University Cluj-Napoca, Romania
Faculty of Automation and Computer Science
Department of Mathematics
Memorandumului Street 28, 400114, Cluj-Napoca, Romania
e-mail: fadelnasierh@gmail.com
On nonexistence of solutions to a nonlinear Cauchy problem for a higher order hyperbolic equation

Djamal Foukrach

Abstract

In this paper we present a result on non-existence of global solutions to the nonlinear Cauchy problem for the higher-order hyperbolic equation

\[
\begin{align*}
    u_{tt} - \Delta |u|^m + \Delta^2 u - \Delta u_t &= |u|^p, & x \in \mathbb{R}^N, & t \in (0, T), & 0 < T < +\infty,
    
\end{align*}
\]

with the initial conditions \( u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ x \in \mathbb{R}^N, \) where \( m \geq 1, \ p > 1. \) The method used is the test function method which is based on a duality argument with an appropriate choice of the test function and a scaling argument.

2010 Mathematics Subject Classification: 35L75, 35A01.

Key words and phrases: Higher order hyperbolic equation; Nonexistence.

1 Introduction

The aim of this paper is to establish some conditions that ensure the nonexistence of global weak solutions to the nonlinear Cauchy problem

\[
\begin{align*}
    (P) \quad \left\{ \begin{array}{ll}
    u_{tt} - \Delta |u|^m + \Delta^2 u - \Delta u_t &= |u|^p, & x \in \mathbb{R}^N, & t \in (0, T), \\
    u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in \mathbb{R}^N,
    \end{array} \right.
    \end{align*}
\]

where \( 0 < T < +\infty, \ m \geq 1, \ p > 1, \ \Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_N^2} \) is the usual Laplacian in the space variable \( x \) and \( u_t \) is the time derivative of \( u. \)

The literature on this subject is very extensive, we refer the interested reader to [2], [4], [5] and the references therein. A sizeable number of articles on hyperbolic equations appeared the least years; we shall mention below only some results which motivate the main results of this paper.

1Received 5 August, 2017

Accepted for publication (in revised form) 5 October, 2017
In [7], G. F. Webb studied the existence and large time behavior of solutions to the damped wave equation

\[ w_{tt} - \alpha \Delta w_t - \Delta w = f(w), \]

for some nonlinearity \( f \).

Li and Zhou [3] studied the blowing-up solutions for the Cauchy problem

\[
\begin{cases}
  u_{tt} - \Delta u + u_t = |u|^p, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
  u(0, x) = u_0(x), & u_t(0, x) = u_1(x),
\end{cases}
\]

for \( 1 < p < 1 + 2/N \) and \( N \leq 2 \). In [6], Todorova et al. studied the same problem further. They showed that the critical exponent is \( p_c(N) = 1 + 2/N \), that is, if \( p > p_c(N) \) then all small initial data solutions of the Cauchy problem are global.

In the present paper we study the more general equation (P) which sheds light on the role played by the nonlinear terms. We consider problem (P), by employing the test function method we present some sufficient conditions guaranteeing the nonexistence of weak solutions. The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. The main result and proof will be given in Section 3. The method we will use is based on a duality argument with an appropriate choice of the test function and a scaling argument.

## 2 Preliminaries

Our basic definition of the solution to problem (P) is the following.

**Definition 1** (Weak solution). A function \( u \in L^p_0(\Omega) \cap L^m_0(\Omega) \) is a weak solution to problem (P) defined on \( \Omega := \mathbb{R}^N \times (0, T), \) if it satisfies

\[
\begin{align*}
  \int_{\Omega} u \varphi_t \, dxdt & - \int_{\Omega} |u|^m \Delta \varphi \, dxdt + \int_{\Omega} u \Delta^2 \varphi \, dxdt + \int_{\Omega} u \Delta \varphi_t \, dxdt \\
  &= \int_{\Omega} |u|^p \varphi \, dxdt + \int_{\mathbb{R}^N} u_1(x) \varphi(x, 0) \, dx - \int_{\mathbb{R}^N} \Delta u_0(x) \varphi(x, 0) \, dx,
\end{align*}
\]

for any function \( \varphi \in C^\infty_0(\Omega), \varphi \geq 0, \varphi(x, T) = 0, \varphi_t(x, T) = 0 \) and \( \varphi_t(x, 0) = 0 \), whenever \( u_0 \in W^{2,2}(\mathbb{R}^N) \) and \( u_1 \in L^1_0(\mathbb{R}^N) \).

We also need the following Lemma

**Lemma 1** (\( \varepsilon \)-Young’s inequality ) [1]. For all \( a \) and \( b \) nonnegative real numbers, we have

\[
a b \leq \varepsilon a^p + C_\varepsilon b^{\tilde{p}},
\]

where \( p, \tilde{p} > 1, \) \( p + \tilde{p} = p\tilde{p}, \varepsilon > 0; \) \( C_\varepsilon \) is a positive constant depending only on \( \varepsilon \).
3 Main result

In this section, we state and prove our main result.

**Theorem 1** Let $1 \leq m < p$, $u_0 \in W^{2,2}(\mathbb{R}^N)$ and $u_1 \in L^1_{\text{loc}}(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N} u_1(x) \, dx - \int_{\mathbb{R}^N} \Delta u_0(x) \, dx > 0. \quad (1)$$

If

$$\frac{N}{2} \leq p^* := \min\left\{\frac{p+1}{p-1}, \frac{m}{p-m}\right\}, \quad (2)$$

then no global weak solution to problem $(P)$ does exist for all time.

**Proof.** We argue by contradiction and assume that $u$ is a global solution. We have

$$\int_{\Omega} u \varphi_t \, dxdt - \int_{\Omega} |u|^m \Delta \varphi \, dxdt + \int_{\Omega} u \Delta^2 \varphi \, dxdt + \int_{\Omega} u \Delta \varphi_t \, dxdt$$

$$= \int_{\Omega} |u|^p \varphi \, dxdt + \int_{\mathbb{R}^N} u_1(x) \varphi(x,0) \, dx - \int_{\mathbb{R}^N} \Delta u_0(x) \varphi(x,0) \, dx. \quad (3)$$

Using Lemma 1, we estimate

$$\int_{\Omega} u \varphi_t \, dxdt \leq \int_{\Omega} |u| \varphi \, dxdt = \int_{\Omega} |u| \varphi^{\frac{1}{p}} |\varphi_t| \, dxdt$$

$$\leq \varepsilon \int_{\Omega} |u|^p \varphi \, dxdt + C_{\varepsilon} \int_{\Omega} |\varphi|^{\frac{p}{p-\tilde{\sigma}}} \, dxdt, \quad (4)$$

where $p + \tilde{p} = p\tilde{p}$, $\varepsilon > 0$; $C_{\varepsilon}$ is a constant depending only on $\varepsilon$.

Similarly, we have for

$$\sigma = \frac{p}{m} > 1$$

$$\int_{\Omega} |u|^m \Delta \varphi \, dxdt \leq \int_{\Omega} |u|^m |\Delta \varphi| \, dxdt = \int_{\Omega} |u|^m \varphi^{\frac{1}{\sigma}} |\Delta \varphi| \, dxdt$$

$$\leq \varepsilon \int_{\Omega} |u|^p \varphi \, dxdt + C_{\varepsilon} \int_{\Omega} |\Delta \varphi|^{\tilde{\sigma}} \varphi^{-\tilde{\sigma}} \, dxdt,$$

with $\sigma + \tilde{\sigma} = \sigma \tilde{\sigma}$; we remark that

$$\tilde{\sigma} = \frac{p}{p-m}.$$ 

So

$$\int_{\Omega} |u|^m \Delta \varphi \, dxdt \leq \varepsilon \int_{\Omega} |u|^p \varphi \, dxdt + C_{\varepsilon} \int_{\Omega} |\Delta \varphi|^{\frac{p}{p-m}} \varphi^{-\frac{m}{p-m}} \, dxdt. \quad (5)$$
Also by Lemma 1, we can get the estimates

\[ \int_{\Omega} u \Delta^2 \varphi \, dx \, dt \leq \varepsilon \int_{\Omega} |u|^p \varphi \, dx \, dt + C \int_{\Omega} |\Delta^2 \varphi|^\frac{p}{p-m} \varphi^{-\frac{m}{p-m}} \, dx \, dt, \]  
and

\[ \int_{\Omega} u \Delta \varphi_t \, dx \, dt \leq \varepsilon \int_{\Omega} |u|^p \varphi \, dx \, dt + C \int_{\Omega} |\Delta \varphi_t|^\frac{p}{p-m} \varphi^{-\frac{m}{p-m}} \, dx \, dt. \]

(6)

(7)

Now, we choose \( 4\varepsilon < 1 \). Then it follows from (3) via (4), (5), (6) and (7), that

\[ \int_{\Omega} |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_1(x) \varphi(x,0) \, dx - \int_{\mathbb{R}^N} \Delta u_0(x) \varphi(x,0) \, dx \leq C_0(I + J + K + L), \]

for some positive constant \( C_0 \), where

\[ I := \int_{\text{supp } \varphi_{tt}} |\varphi_{tt}|^\frac{p}{p-m} \varphi^{-\frac{m}{p-m}} \, dx \, dt, \]

\[ J := \int_{\text{supp } \Delta \varphi} |\Delta \varphi|^\frac{p}{p-m} \varphi^{-\frac{m}{p-m}} \, dx \, dt, \]

\[ K := \int_{\text{supp } \Delta^2 \varphi} |\Delta^2 \varphi|^\frac{p}{p-m} \varphi^{-\frac{m}{p-m}} \, dx \, dt, \]

\[ L := \int_{\text{supp } \Delta \varphi_t} |\Delta \varphi_t|^\frac{p}{p-m} \varphi^{-\frac{m}{p-m}} \, dx \, dt. \]

Let us choose

\[ \varphi(x,t) := \Phi\left(\frac{t^2 + |x|^4}{R^4}\right), \]

\[ R \in \mathbb{R}^+, \]

where \( \Phi \) is the standard cutoff function, \( \Phi \) decreasing,

\[ \Phi(r) := \begin{cases} 1, & 0 \leq r \leq 1, \\ 0, & r \geq 2, \end{cases} \]

that verifies

\[ 0 \leq \Phi \leq 1, \quad \text{supp } \Phi \subset \{ x \in \mathbb{R}^N / |x| \leq 2 \}. \]

From our choice of \( \varphi \), it is clear that

\[ \varphi(x,T) = \varphi_t(x,T) = \varphi_t(x,0) = 0. \]

Now, using the following scaled variables

\[ y = \frac{x}{R}, \quad \tau = \frac{t}{R^2}, \]

we can estimate \( I, J, K \) and \( L \) in terms of \( R \)

\[ I \leq C_1 R^{-4} \tilde{p} + N + 2, \]

\[ J \leq C_2 R^{-2} \left( \frac{p}{p-m} \right)^{N+2}, \]

\[ K \leq C_3 R^{-4} \tilde{p} + N + 2, \]

\[ L \leq C_4 R^{-4} \tilde{p} + N + 2, \]
with

\[ C_1 := \int_{\text{supp } \varphi_{\tau\tau}} |\varphi_{\tau\tau}|^{\tilde{p}} \varphi^{-\frac{\tilde{p}}{p}} \, dy \, d\tau < \infty, \]

\[ C_2 := \int_{\text{supp } \Delta_y \varphi} |\Delta_y \varphi|^{\tilde{p}} \varphi^{-\frac{\tilde{p}}{p-m}} \, dy \, d\tau < \infty, \]

\[ C_3 := \int_{\text{supp } \Delta^2_y \varphi} |\Delta^2_y \varphi|^{\tilde{p}} \varphi^{-\frac{\tilde{p}}{p}} \, dy \, d\tau < \infty, \]

\[ C_4 := \int_{\text{supp } \Delta_y \varphi_{\tau}} |\Delta_y \varphi_{\tau}|^{\tilde{p}} \varphi^{-\frac{\tilde{p}}{p}} \, dy \, d\tau < \infty. \]

Whereupon

\[ \int_\Omega |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_1(x) \varphi(x,0) \, dx - \int_{\mathbb{R}^N} \Delta u_0(x) \varphi(x,0) \, dx \leq C_5 \{ R^\alpha + R^\beta \} \]

for some positive constant \( C_5 \), where

\[ \alpha = -4 \tilde{p} + N + 2 = -4 \left( 1 + \frac{1}{p-1} \right) + N + 2, \quad \beta = -2 \left( 1 + \frac{m}{p-m} \right) + N + 2. \]

Now, we impose the condition in the estimate (2)

\[ \frac{N}{2} \leq p^*, \]

where \( p^* \) is defined in (2).

We have to distinguish two cases:

**Case 1: \( \alpha < 0, \beta < 0 \):** In this case, passing to the limit in inequality (8) as \( R \to +\infty \), using Lebesgue’s theorem of dominated convergence, and taking account of (2), we obtain

\[ 0 < \int_\Omega |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_1(x) \varphi(x,0) \, dx - \int_{\mathbb{R}^N} \Delta u_0(x) \varphi(x,0) \, dx \leq 0. \]

This is contradiction with (1).

**Case 2: \( \alpha = \beta = 0 \):** In this case, we obtain from (8)

\[ \int_\Omega |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_1(x) \varphi(x,0) \, dx - \int_{\mathbb{R}^N} \Delta u_0(x) \varphi(x,0) \, dx \leq C, \]

for some positive constant \( C \), and therefore

\[ \int_\Omega |u|^p \varphi \, dx \, dt \leq C. \]
Whereupon

\[ \lim_{R \to +\infty} \int_{\Omega_R} |u|^p \varphi \, dx \, dt = 0, \]

where

\[ \Omega_R := \{(x,t) : R^4 \leq t^2 + |x|^4 \leq 2R^4\}. \]

We can rely on the same argument as in Case 1. By using Hölder’s inequality we can easily get the estimates

\[ \int_{\Omega} u \varphi_{tt} \, dx \, dt \leq \int_{\Omega} |u| |\varphi_{tt}| \, dx \, dt \]
\[ \leq \int_{\Omega} |u| \varphi^{\frac{1}{p}} \varphi^{-\frac{1}{p}} |\varphi_{tt}| \, dx \, dt \]
\[ \leq \left( \int_{\text{supp } \varphi_{tt}} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{\text{supp } \varphi_{tt}} |\varphi_{tt}|^q \varphi^{-\frac{q}{p}} \, dx \, dt \right)^{\frac{1}{q}}, \]

where \( p + q = p \, q \). Also, we have for \( \sigma = p/m > 1 \)

\[ \int_{\Omega} |u|^m \Delta \varphi \, dx \, dt \leq \int_{\Omega} |u|^m |\Delta \varphi| \, dx \, dt \]
\[ \leq \int_{\Omega} |u|^m \varphi^{\frac{1}{p}} \varphi^{-\frac{1}{p}} |\Delta \varphi| \, dx \, dt \]
\[ \leq \left( \int_{\text{supp } \Delta \varphi} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{\text{supp } \Delta \varphi} |\Delta \varphi|^q \varphi^{-\frac{q}{p}} \, dx \, dt \right)^{\frac{1}{q}}, \]

where \( \gamma + \sigma = \gamma \, \sigma \).

In a similar way, we obtain

\[ \int_{\Omega} u \Delta^2 \varphi \, dx \, dt \leq \left( \int_{\text{supp } \Delta^2 \varphi} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{\text{supp } \Delta^2 \varphi} |\Delta^2 \varphi|^q \varphi^{-\frac{q}{p}} \, dx \, dt \right)^{\frac{1}{q}} \]

and

\[ \int_{\Omega} u \Delta \varphi_t \, dx \, dt \leq \left( \int_{\text{supp } \Delta \varphi_t} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{\text{supp } \Delta \varphi_t} |\Delta \varphi_t|^q \varphi^{-\frac{q}{p}} \, dx \, dt \right)^{\frac{1}{q}}. \]
Then, it follow from (3) via the estimates (11), (12), (13) and (14) that

\[
\int_\Omega |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_1(x) \varphi(x, 0) \, dx - \int_{\mathbb{R}^N} u_0(x) \Delta \varphi(x, 0) \, dx \\
\leq \left( \int_{\Omega_R} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{\text{supp} \varphi(t)} |\varphi(t)|^q \varphi^{-\frac{q}{p}} \, dx \, dt \right)^{\frac{1}{q}} \\
+ \left( \int_{\text{supp} \Delta \varphi} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{\text{supp} \varphi(t)} |\varphi(t)|^q \varphi^{-\frac{q}{p}} \, dx \, dt \right)^{\frac{1}{q}} \\
+ \left( \int_{\Omega_R} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{\text{supp} \Delta \varphi} |\Delta \varphi|^q \varphi^{-\frac{q}{p}} \, dx \, dt \right)^{\frac{1}{q}} \\
+ \left( \int_{\Omega_R} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left( \int_{\text{supp} \Delta \varphi(t)} |\Delta \varphi(t)|^q \varphi^{-\frac{q}{p}} \, dx \, dt \right)^{\frac{1}{q}}.
\]

Passing to the limit as \( R \to +\infty \) in the last inequality and taking into account (10), we obtain:

\[
\lim_{R \to +\infty} \left\{ \int_\Omega |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_1(x) \varphi(x, 0) \, dx - \int_{\mathbb{R}^N} \Delta u_0(x) \varphi(x, 0) \, dx \right\} = 0.
\]

This contradicts (1). We conclude that \( u \) can not be global. This completes the proof.

References


**Djamal Foukrach**
Hassiba Benbouali University of Chlef, Algeria,
Faculty of Exact Sciences and Computer Science,
Department of Mathematics.
and
E.N.S., Kouba, Algiers, Algeria,
Laboratory of Fixed Point Theory and Applications,
Department of Mathematics.
E-mail: d.foukrach@univ-chlef.dz
Extremal sets in a topological space

Poonam Agarwal, Chander Kant Goel

Abstract

In the present paper a new class of sets has been defined on a topological space. These sets have been named as extremal set. Properties of these extremal sets have been studied in terms of subsets, supersets, intersection and union of sets. It has been found that extremal sets and open sets are in general independent to each other. However in the presence of both the properties i.e. a set being both open and extremal, new characterizations of such sets arise. These characteristics have also been studied here. In support of these characteristics various examples have also been given.

2010 Mathematics Subject Classification: 54A05, 54B10, 54C10, 54D30.

Key words and phrases: extremal sets, extremal points.

1 Introduction

The concept of topology has always been studied in terms of its sets equipped with certain properties. The most common form of these sets has been a variety of open sets. A plethora of mathematicians have defined various new classes of open sets. Velicko [5] first gave the concept of \(\theta\)-closure. The concept of \(\theta\)-open sets was further defined in terms of \(\theta\)-closure [2]. Norman Levine [4] defined a subset A of a space X to be a semi-open if \(A = (\text{Cl} (\text{Int} A))\), or equivalently, a set A of a space X will be termed semi-open if and only if there exists an open set \(U \subseteq A \subseteq \text{cl}(U)\). Mashhour et al. [1] defined a subset A of a space X to be preopen if \(A \subseteq (\text{Int} (\text{Cl} A))\). Njastad [6] defined a subset A of a space X to be an \(\alpha\)-open if \(A \subseteq (\text{Int} (\text{Cl} (\text{Int} A)))\). Nakaoka and Oda [3] introduced and studied the concept of minimal open in a topological space.

Owing to the fact that such sets have many features in common, it is quite natural to reach to the conclusion that the concept of open sets can be applied widely to each
element of a set $X$ lying inside and outside of a subset of $X$ and a considerable part of the properties of set theory and topological spaces can be featured on them. All these events led to the development of a new class of sets defined on a topological space which is denoted as class of extremal sets. Let $X$ be a topological space equipped with some topology denoted by $\mathcal{I}$. Let $P(X)$ denote its power set. Let $A \subseteq X$ then by $\text{cl}(A)$ we mean closure of $A$ in $X$. Similarly, by $\text{int}(A)$ we mean the interior of $A$ in $X$. We note that for a topological space $(X, \mathcal{I})$ the collection of all open sets in $X$ with respect to the topology is denoted by $O(X)$. In this paper we introduce the concept of extremal sets and extremal points and study their properties.

## 2 Extremal Sets

In a topological space $X$, an extremal set is defined as follows:

**Definition 1** A subset $A \subseteq X$ is said to be an extremal set of $X$ if $a \in A$ and $x \in X-A$ there exists a proper open set in $X$ containing both $a$ and $x$.

**Example 1** In real line with usual topology, any subset of $X$ is an extremal set.

**Example 2** In a topological space $X$ with discrete topology every subset of $X$ is an extremal set.

**Example 3** In a topological space $X$ with indiscrete topology no subset of $X$ is an extremal set.

**Example 4** In point inclusion topology, every subset $A$ of $X$ is an extremal set as $a \in A$ and $x \in X-A$ there exists a proper open set $\{p, a, x\}$ in $X$ containing both $a$ and $x$ where $p$ is the point of inclusion.

**Example 5** In point exclusion topology, no subset $A$ of $X$ is an extremal set as if $p \in A$ then for any $x \in X-A$ there exists no proper open set in $X$ containing both $p$ and $x$. Similarly, if $p$ is not in $A$ then $p \in X-A$. Again there is no proper open set in $X$ containing $p$.

**Example 6** Let $X=\{a, b, c, d\}$ and let a topology on $X$ be defined as

1. $\mathcal{I}=\{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ then no set is extremal set as no open set contains $d$.

2. $\mathcal{I}=\{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. If $A_1 =\{a, b\}$, $A_2 =\{a\}$, $A_3 =\{b\}$, $A_4 =\{c\}$, $A_5 =\{d\}$, $A_6 =\{c, d\}$ and $A_7 =\{a, c\}$ then, $A_1$, $A_2$, $A_3$ are extremal sets while $A_4$, $A_5$ and $A_7$ are not. Further $A_6$ is an extremal set.

**Definition 2** A one point extremal set is called an extremal point.

**Example 7** In real line with usual topology every point of $X$ is an extremal point.
Extremal sets in a topological space

Example 8 In example 2.3 every point of $X$ is an extremal point.

Example 9 In example 2.4 no point of $X$ is an extremal point.

Example 10 In point inclusion topology, every point $a \in X$ is an extremal point as $\forall x \in X$ there exists a proper open set $\{p, a, x\}$ in $X$ containing both $a$ and $x$ where $p$ is the point of inclusion.

Example 11 In point exclusion topology, no point of $X$ is an extremal point.

Example 12 In example 2.7(ii) the points $a$ and $b$ are extremal points while $c$ and $d$ are not.

Definition 3 We define an extremal set $A$ to be maximal extremal set if no superset of $A$ is an extremal set.

Example 13 In example 2.7(ii) the sets $\{a, b\}$, $\{a, c, d\}$ and $\{b, c, d\}$ are maximal extremal sets.

Theorem 1 If $A$ is a maximal extremal set and $B$ is an extremal set in $X$ then, either $B \subseteq A$ or $B \cup A = X$.

Proof. If both of the statements do not hold then, $B \cup A \neq A$ and $B \cup A \neq X$. Hence $B \cup A$ is a proper open set containing $A$ properly, contradicting the maximality of $A$.

Proposition 1 Every subset of an extremal set need not be an extremal set.

Example 14 In Example 6(ii) $\{c, d\}$ is an extremal set while $\{c\}$ is not an extremal set.

Proposition 2 Every superset of an extremal set need not be an extremal set.

Example 15 In Example 6(ii) $\{a\}$ is an extremal set while $\{a, c\}$ is not an extremal set.

Proposition 3 An extremal set may or may not be open. Similarly an open set may or may not be an extremal set.

Proof In Example 6(ii) $\{a\}$ is both open and extremal, $\{a, b, c\}$ is open but not extremal, $\{c, d\}$ is extremal but not open and $\{c\}$ in neither open nor extremal.

Theorem 2 The intersection of two extremal sets is an extremal set.
Proof  Let A and B be two extremal sets of a topological space X. Let \( p \in A \cap B \) then \( p \in A \) and \( p \in B \). Let \( x \in X-(A \cap B) \) then \( x \in (X-A) \cup (X-B) \).

(i) If \( x \in X-(A \cup B) \) then \( x \in (X-A) \cap (X-B) \). Now \( p \in A \) and \( x \in (X-A) \) then there exists an open set \( U \) containing \( p \) and \( x \). Similarly, \( p \in B \) and \( x \in (X-B) \) then there exists an open set \( V \) containing \( p \) and \( x \). All of \( U, V \) and \( U \cap V \) serve the purpose.

(ii) If \( x \in A-B \) then \( p \in B \) and \( x \in X-B \) then there exists an open set \( U \) containing \( p \) and \( x \). (iii) If \( x \in B-A \) then \( p \in A \) and \( x \in X-A \) then there exists an open set \( U \) containing \( p \) and \( x \).

Theorem 3  The union of two extremal sets is an extremal set.

Proof  Let A and B be two extremal sets of a topological space X. Let \( p \in A \cup B \).

Let \( x \in X-(A \cup B) \) then \( x \in (X-A) \cap (X-B) \). If (i) \( x \in X-(A \cup B) \) then \( x \in (X-A) \cap (X-B) \). If \( p \in A \) then since \( x \in (X-A) \) there exists an open set \( U_A \) containing both \( p \) and \( x \). If \( p \in B \) then since \( x \in (X-B) \) there exists an open set \( U_B \) containing both \( p \) and \( x \).

Definition 4  We define a subset \( A \) of a topological space \( X \) as an open extremal set if \( A \) is both open and extremal in \( X \).

Theorem 4  Every subset of an open extremal set in a topological space \( X \) is an extremal set in \( X \).

Proof  Let \( A \) be an open extremal set. Let \( B \subseteq A \). Let \( p \in B \) and \( x \in X-B \).

If \( x \in X-A \) then there exists an open set in \( X \) containing both \( p \) and \( x \) as \( A \) is extremal while if \( x \in A-B \) then \( A \) is the open set in \( X \) containing both \( p \) and \( x \). Hence \( B \) is an extremal set.

Theorem 5  If \( A \) is an extremal set then \( A \) intersects with at least one neighborhood of each element of \( X \).

Proof  Let \( x \in X-A \). Since \( A \) is an extremal set, for every \( p \) in \( A \) there exists an open set containing both \( x \) and \( p \). Each such neighborhood of \( x \) intersects \( A \). Further let \( x \in A \) then every neighborhood of \( x \) contains an element of \( A \) namely \( x \) and therefore intersects \( A \).

Theorem 6  If \( A \) is an open extremal set then \( A \) intersects with at least one neighborhood of each element of \( X \).

Proof  Let \( x \in X-A \). Since \( A \) is extremal set, for every \( p \) in \( A \) there exists an open set containing both \( x \) and \( p \). Each such neighborhood of \( x \) intersects \( A \). Then since \( A \) is open it contains a neighborhood of \( x \) which is contained in \( A \).

Theorem 7  If \( A \) is an extremal set and \( x \in X-A \) then there exists a neighborhood of \( x \) containing \( A \).
Extremal sets in a topological space

Proof Let \( A = \{ a_i \mid i \in \Delta \} \) be an extremal set in \( X \). Let \( x \in X - A \) then \( \forall i \in \Delta \) there exists an open set \( U_i \) containing both \( x \) and \( a_i \). Then the set \( \Delta = \bigcup_{i \in \Delta} U_i \) is an open set containing \( x \) and whole of \( A \).

Theorem 8 If \( A \) is an open extremal set and \( x \in X \) then there exists a neighborhood of \( x \) containing \( A \).

Proof Let \( A = \{ a_i \mid i \in \Delta \} \) be an extremal set in \( X \). If \( x \in A \) then \( A \) serves the purpose. Let \( x \in X - A \) then as before there is an open set containing \( x \) and whole of \( A \).

Proposition 4 A superset of an open extremal set may or may not be an extremal set.

Proof In example 6(ii) \( \{ a \} \) is an extremal open set, \( \{ a, b \} \) is an extremal set but \( \{ a, c \} \) is not an extremal set.

Proposition 5 An open superset of an open extremal set may or may not be an extremal set.

Proof In example 2.7(ii) \( \{ a \} \) is an extremal open set, \( \{ a, b \} \) is an extremal set but \( \{ a, b, c \} \) is not an extremal set although it is open.

Theorem 9 Let \( a, b \) be two elements of \( X \) such that there exists no open set containing both \( a \) and \( b \) then, every extremal set contains either both or none.

Proof If \( A \) is an extremal set containing \( a \) but not \( b \) then, there must exist an open set containing both \( a \) and \( b \) which is contrary to the assumption. Hence \( A \) must contain either both or none.

Theorem 10 Let \( U = \{ U_i \mid U_i \cap A \neq \emptyset \} \) be the collection of all open sets which intersect \( A \) where \( A \) is an extremal set then, \( A \cup \bigcup_{U_i \in U} U_i = X \).

Proof Let \( x \in X - (A \cup \bigcup_{U_i \in U} U_i) \) then \( x \in X - A \). Then there exists an open set in \( X \) containing \( x \) and intersecting \( U \). Then \( x \in (\bigcup_{U_i \in U} U_i) \) which is a contradiction. Hence we must have \( A \cup \bigcup_{U_i \in U} U_i = X \).

Theorem 11 An open subset of an open extremal set is an open extremal set.

Proof Obvious.

Proposition 6 If \( U \) is an open set which does not intersect with any of the other open sets then \( U \) is not an extremal set.

Proposition 7 If \( U \) is the open set containing all the other open sets then \( U \) is not an extremal set.
Theorem 12 Let $U = \{U_i \mid U_i \cap A \neq \emptyset\}$ be the collection of all open sets which intersect $A$ where $A$ is an extremal set then, $\bigcup_{U_i \in U} U_i = X$.

Proposition 8 Let $x \in A$ and $x \in X-A$ then there exists an open set $U_i$ containing $x$ and $A$. Then $U_i \in U$. Then as before $A \cup \bigcup_{U_i \in U} U_i = X$. But $A \subseteq U_i$ therefore $\bigcup_{U_i \in U} U_i = X$.

Proposition 9 If $X$ has an element that does not belong to any proper open set then $X$ has no extremal set.

Proposition 10 If a topological space $X$ has a topology $\mathfrak{T}$ defined over it as $\mathfrak{T} = \{\varnothing, X, U, X-U\}$ then $X$ has no extremal set.

Theorem 13 If $A$ is an extremal set then so is $X-A$.

Proof Let $B = X-A$. Let $p \in B = X-A$ and $x \in X-B = A$. Then since $A$ is extremal there exists an open set containing both $p$ and $x$. Since $p$ and $x$ are arbitrary therefore $B = X-A$ is also extremal.

Acknowledgement The authors are grateful to Prof. M. K. Gupta for his valuable suggestions towards the improvement of the paper.

References


Poonam Agarwal  
Inderprastha Engineering College  
Applied Sciences and Humanities  
Plot No. 63, Surya Nagar Flyover Road, Sahibabad, Ghaziabad, U.P., India  
e-mail: poonam_r5@yahoo.com

Chander Kant Goel  
Amity University  
Department of Mathematics  
Sector 125, Noida, U.P., India  
e-mail: ckgoel@amity.edu
A Young’s inequality for the Sugeno integrals

Ali Hassanzadeh

Abstract

In this paper we study the Young’s inequality for the Sugeno integral. Indeed we investigate the integral inequality

\[ \int_{[0,a]} f \, dx + \int_{[0,b]} f^{-1} \, dx \geq ab \]

in fuzzy version, where \( f \) is a continuous and increasing function on \([0, c]\) \((c > 0)\), \( a \in [0, c] \), \( b \in [0, f(c)] \) and \( f^{-1} \) stands for the inverse function of \( f \). Some relevant counterexamples are indicated. Finally we derive a new inequality for the classical Young’s inequality in fuzzy version.

2010 Mathematics Subject Classification: 28E10, 35A23.

Key words and phrases: Sugeno integral, fuzzy measure, Young’s inequality.

1 Introduction

The concept of a Sugeno integral (fuzzy integral) for a measurable function on a normalized monotone measure space was introduced by Sugeno [12], who also discussed some elementary properties of this integral. Further investigations of the integral were also pursued by Batle and Trillas [3], Wierzchon [15], Dubois and Prade [4], Grabisch et al. [6], and other researchers.

These concepts were envisioned by Sugeno in his efforts to compare membership grade functions of fuzzy sets with probabilities. Since no direct comparison is possible, Sugeno conceived of the generalization of classical measures into fuzzy measures as an analogy of the generalization of classical (crisp) sets into fuzzy sets. Using this analogy he coined for the nonclassical (nonadditive) measures the term fuzzy measures. Fuzzy measures, according to Sugeno, are obtained by replacing the additivity

\[1\text{Received 1 August, 2017} \]

\[ \text{Accepted for publication (in revised form) 6 November, 2017} \]
requirement of classical measures with the weaker requirements of increasing monotonicity (with respect to set inclusion) and continuity. The requirement of continuity was later found to be too restrictive and was replaced with a weaker requirement of semicontinuity. The term fuzzy measure in the sense Sugeno introduced it has been accepted by most researchers working in the area of generalized measures [13, 14].

It is well known that integral inequalities are instrumental in studying the qualitative analysis of solutions to differential and integral equations [9].

The study of inequalities for Sugeno integral was initiated by Román-Flores and Chalco-Cano [10]. Since then, the fuzzy integral counterparts of several classical inequalities, including Chebyshevs, Jensens, Minkowskis and Hölders inequalities, are given by Flores-Franulić and Román-Flores [5], Agahi et al. [2, 1] Mesiar and Ouyang [7], Román-Flores et al. [11], and other researchers. In this paper, we study the classical Young’s inequality for the fuzzy integrals.

2 Preliminaries

For convenience of the reader we give a survey of the relevant materials from [14] and [13], without proofs, thus making our exposition self-contained.

Throughout this paper, $X$ is a nonempty set, $\Sigma$ is a $\sigma$-algebra of subsets of $X$ and all considered subsets belong to $\Sigma$.

**Definition 1** A set function $\mu : \Sigma \rightarrow [0, \infty]$ is called a fuzzy measure if the following properties are satisfied:

(i) $\mu(\emptyset) = 0$;

(ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$;

(iii) $A_1 \subseteq A_2 \subseteq \ldots$ implies $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$;

(iv) $A_1 \supseteq A_2 \supseteq \ldots$ and $\mu(A_1) < \infty$ imply $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$.

If $\mu(X) = 1$, then $\mu$ is called a normalized fuzzy measure. When $\mu$ is a fuzzy measure, the triple $(X, \Sigma, \mu)$ is called a fuzzy measure space. If $\mu$ is a fuzzy measure on $X$ and $D \in \Sigma$, we denote by $\mathcal{F}^\mu(D)$ the set of all non-negative $\mu$-measurable functions $f : D \to [0, \infty]$.

For any given $f \in \mathcal{F}^\mu(X)$, we write $F_\alpha = \{x : f(x) \geq \alpha\}$, where $\alpha \in [0, \infty]$.

**Definition 2** Let $A \in \Sigma$, $f \in \mathcal{F}^\mu(X)$. The fuzzy integral of $f$ on $A$ with respect to $\mu$, which is denoted by $\int_A f d\mu$, is defined by

$$\int_A f d\mu = \sup_{\alpha \in [0, \infty]} [\alpha \land \mu(A \cap F_\alpha)].$$
If $A = X$, then
\[ \int_X f d\mu = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \mu(F_{\alpha})]. \]

When $A = X$, the fuzzy integral may also be denoted by $\int f d\mu$.

Sometimes, the fuzzy integral is also called Sugeno’s integral in the literature.

From now on, we make the convention that the appearance of a symbol $\int_A f d\mu$ implies that $A \in \Sigma$ and $f \in F^\mu(X)$.

The following theorem gives the most elementary properties of the fuzzy integral.

**Theorem 1** ([13])

1. $\int_A f d\mu \leq \mu(A)$;
2. If $\int_A f d\mu = 0$, then $\mu(A \cap \{x : f(x) > 0\}) = 0$;
3. If $f \leq g$, then $\int_A f d\mu \leq \int_A g d\mu$;
4. If $\mu(A) = 0$, then $\int_A f d\mu = 0$ for any $f \in F^\mu(X)$;
5. $\int_A k d\mu = k \wedge \mu(A)$, $k$ is a nonnegative constant;
6. $\mu(A \cap f \geq \alpha) \geq \alpha \Rightarrow \int_A f d\mu \geq \alpha$;
7. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow \int_A f d\mu \leq \alpha$;
8. $\int_A f d\mu < \alpha \iff \exists \gamma < \alpha; \mu(A \cap \{f \geq \gamma\}) < \alpha$;
9. $\int_A f d\mu > \alpha \iff \exists \gamma > \alpha; \mu(A \cap \{f \geq \gamma\}) > \alpha$.

### 3 Main results

The following theorem is called “Young’s inequality” [8]. Let $f$ be a continuous and increasing function on $[0, c]$ where $c > 0$. If $f(0) = 0$, $a \in [0, c]$ and $b \in [0, f(c)]$, then

\[ \int_a^0 f(x)dx + \int_0^b f^{-1}(x)dx \geq ab, \]

where $f^{-1}$ stands for the inverse function of $f$.

In fact, W. H. Young [16] has proved (1) for differentiable functions.

The following examples show that, in general, the Young’s inequality is not true for Sugeno integrals.

**Example 1** Let $\mu$ be the usual Lebesgue measure on $\mathbb{R}$. Take the function $f(x) = x^2$ for all $x \in [0, c]$. First let $c > 1$. Take $c = \frac{3}{2}$. The simple calculation shows that

\[ \int_0^{\frac{3}{2}} f d\mu = \int_0^{\frac{3}{2}} x^2 d\mu = \int_{\alpha \in [0, \frac{3}{2}]} [\alpha \wedge \mu(x^2 \geq \alpha)] \]

\[ = \int_{\alpha \in [0, \frac{3}{2}]} [\alpha \wedge (\frac{3}{2} - \sqrt{\alpha})] \approx 0.677, \]
and

\[
\int_{0}^{\frac{3}{4}} f^{-1} \, d\mu = \int_{0}^{\frac{3}{4}} \sqrt{x} \, d\mu = \bigvee_{\alpha \in [0, \frac{3}{4}]} [\alpha \wedge \mu(\sqrt{x} \geq \alpha)]
\]

\[
= \bigvee_{\alpha \in [0, \frac{3}{4}]} [\alpha \wedge \left(\frac{9}{4} - \alpha^2\right)] \approx 1.081.
\]

So

\[
\int_{0}^{\frac{3}{4}} f \, d\mu + \int_{0}^{\frac{3}{4}} f^{-1} \, d\mu \approx 1.758 \neq cf(c) = 3.375.
\]

Now let \( c \leq 1 \). By taking \( c = \frac{1}{2} \), we have

\[
\int_{0}^{\frac{1}{2}} f \, d\mu = \int_{0}^{\frac{1}{2}} x^2 \, d\mu = \bigvee_{\alpha \in [0, \frac{1}{2}]} [\alpha \wedge \mu(x^2 \geq \alpha)]
\]

\[
= \bigvee_{\alpha \in [0, \frac{1}{2}]} [\alpha \wedge (\frac{1}{2} - \sqrt{\alpha})] \approx 0.134,
\]

and

\[
\int_{0}^{\frac{1}{2}} f^{-1} \, d\mu = \int_{0}^{\frac{1}{2}} \sqrt{x} \, d\mu = \bigvee_{\alpha \in [0, \frac{1}{2}]} [\alpha \wedge \mu(\sqrt{x} \geq \alpha)]
\]

\[
= \bigvee_{\alpha \in [0, \frac{1}{2}]} [\alpha \wedge \left(\frac{1}{4} - \alpha^2\right)] \approx 0.207.
\]

So

\[
\int_{0}^{\frac{1}{2}} f \, d\mu + \int_{0}^{\frac{1}{2}} f^{-1} \, d\mu \approx 0.341 > cf(c) = 0.125.
\]

The following example shows that in the case of \( c < 1 \), the Young’s inequality necessarily is not true.

**Example 2** Let \( \mu \) be the usual Lebesgue measure on \( \mathbb{R} \). If take the function \( f(x) = 10x \) for all \( x \in [0, \frac{1}{2}] \), then

\[
\int_{0}^{\frac{1}{2}} f \, d\mu = \int_{0}^{\frac{1}{2}} 10x \, d\mu = \bigvee_{\alpha \in [0, \frac{1}{2}]} [\alpha \wedge \mu(10x \geq \alpha)]
\]

\[
= \bigvee_{\alpha \in [0, \frac{1}{2}]} [\alpha \wedge (\frac{1}{2} - \frac{\alpha}{10})] \approx 0.454,
\]
A Young’s inequality for the Sugeno integrals\[\int_0^5 f^{-1}d\mu = \int_0^5 \frac{x}{10}d\mu = \bigvee_{\alpha \in [0,5]} [\alpha \land \mu(\frac{x}{10} \geq \alpha)] = \bigvee_{\alpha \in [0,5]} [\alpha \land (\frac{1}{4} - 10\alpha)] \approx 0.454.\]

Therefore \[\int_0^2 f^2d\mu + \int_0^5 f^{-1}d\mu \approx 0.908 \nless cf(c) = 2.5.\]

The following result presents a new inequality for the Young’s inequality in the fuzzy version.

**Theorem 2** Let \( f \) be a continuous, increasing and concave function on [0,1] and \( f(1) \geq 1 \). If \( f(0) = 0 \), then

\[1 + \frac{f(1)f^{-1}(1)}{1 + f^{-1}(1)} \geq \int_0^1 f(x)d\mu + \int_0^{f(1)} f^{-1}(x)d\mu \geq \frac{f(1)}{1 + f(1)},\]

where \( f^{-1} \) stands for the inverse function of \( f \) and \( \mu \) is the usual Lebesgue measure on \( \mathbb{R} \).

**Proof.** Since \( f \) is concave, we have

\( f(x) = f(x1 + (1 - x)0) \geq xf(1) + (1 - x)f(0) = xf(1). \)

From Theorem 2.3 (3), it follows that

\[\int_0^1 fd\mu \geq \int_0^1 xf(1)d\mu.\]

Therefore

\[\int_0^1 fd\mu + \int_0^{f(1)} f^{-1}(x)d\mu \geq \bigvee_{\alpha \in [0,1]} (\alpha \land \mu([0,1] \cap \{x : xf(1) \geq \alpha\})) = \bigvee_{\alpha \in [0,1]} (\alpha \land \mu([0,1] \cap \{x : x \geq \frac{\alpha}{f(1)}\})) = \bigvee_{\alpha \in [0,1]} (\alpha \land \mu[0,1] \land 1 - \frac{\alpha}{f(1)}) = \bigvee_{\alpha \in [0,1]} (\alpha \land (1 - \frac{\alpha}{f(1)})).\]

The last one is equal to the root of the following equation:

\[\alpha = 1 - \frac{\alpha}{f(1)}.\]
Therefore
\[ \int_{0}^{1} f d\mu + \int_{0}^{f(1)} f^{-1}(x) d\mu \geq \frac{f(1)}{1 + f(1)}. \]

Since \( f \) is concave, so \( f^{-1} \) is a convex function. Hence
\[ f^{-1}(x) = f^{-1}(x1 + (1 - x)0) \leq xf^{-1}(1) + (1 - x)f^{-1}(0) = xf^{-1}(1). \]

Similarly
\[ \int_{0}^{1} f(x) d\mu + \int_{0}^{f(1)} f^{-1}(x) d\mu \leq 1 + \frac{f(1)f^{-1}(1)}{1 + f^{-1}(1)}, \]

which complete the proof.

Finally, we can prove the following theorem similarly, in general.

**Theorem 3** Let \( f \) be a continuous, increasing and concave function on \([0, c]\) where \( c > 1 \). If \( f(0) = 0 \), \( 1 \leq a \in [0, c] \) and \( 1 \leq b \in [0, f(c)] \), then
\[ a + b \frac{f^{-1}(1)}{1 + f^{-1}(1)} \geq \int_{0}^{a} f(x) d\mu + \int_{0}^{b} f^{-1}(x) d\mu \geq a \frac{f(1)}{1 + f(1)}, \]

where \( f^{-1} \) stands for the inverse function of \( f \) and \( \mu \) is the usual Lebesgue measure on \( \mathbb{R} \).

**References**


A Young’s inequality for the Sugeno integrals


**Ali Hassanzadeh**
Sahand University of Technology
Faculty Of Sciences
Department of Mathematics
Tabriz, Iran
e-mail: ali.hassanzadeh@guest.unimi.it, a_hassanzadeh@sut.ac.ir
Blending type approximation by summation-integral operators based on Polya distribution

Arun Kajla

Abstract
In this note we construct the new kind a Durrmeyer type operators based on Polya distribution. We present a Voronovskaja type theorem, local approximation and global approximation theorem by means of first order Ditzian-Totik modulus of smoothness. Also, we derived the rate of convergence for absolutely continuous functions having a derivative equivalent with a function of bounded variation of these operators. Furthermore, we prove the rate of convergence for these operators to some functions by illustrative graphics with the help of Mathematica algorithms.

2010 Mathematics Subject Classification: 41A25, 26A15.
Key words and phrases: Polya distribution, Mathematica, rate of convergence, Steklov mean.

1 Introduction
Lupaş and Lupaş [25] presented and established the rate of convergence of the following operators:

\[ P_{n}^{(1/n)}(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(1/n)}(x), \]

where \( f \in C(J) \), with \( J = [0, 1] \), \( p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!} \binom{n}{k} (nx)_{k} (n-nx)_{n-k}, \) \((n)_{k}\) being the rising factorial given by \((n)_{k} = n(n+1)...(n+k-1), (n)_{0} = 1.\)

Miclaus [26] estimated some direct results of the Bernstein-Stancu type operators involving Polya distribution. In [11], to approximate integrable functions,
Kantorovich type operators given by (1) was introduce and the properties of local and global approximation were investigated in univariate and bivariate cases. In [21], Gupta and Rassias introduced a Durrmeyer type operators involving Polya and Bernstein basis functions as follows;

\[ D_n^{(1/n)}(f; x) = (n + 1) \sum_{k=0}^{n} P_{n,k}^{(1/n)}(x) \int_0^1 p_{n,k}(t) f(t) dt, \]

where \( f \in L_1(J) \) (the space of all Lebesgue integrable functions on \( J \)), \( p_{n,k}(t) = \binom{n}{k} t^k (1 - t)^{n-k} \) and \( P_{n,k}^{(1/n)}(x) \) is defined as above. In [21], Voronoskaja type asymptotic results, local and global approximations properties were obtained for the operators (2). Agrawal et al. [12] considered a Bezier variant of the operators (2) and studied direct theorems and the rate of convergence of these operators. In the recent years, several mathematicians have made significant contributions in this direction. We refer the reader to some of the related papers (cf. [1, 2, 3, 4, 5, 6, 7, 9, 8, 10, 14, 15, 19, 20, 22, 23, 24, 27, 28, 18, 30] etc.).

For \( f \in C(J) \), we construct the following linear positive operators with the help of Polya distribution:

\[ \mathcal{P}_{n,\rho}^{(1/n)}(f, t) = \sum_{k=1}^{n-1} P_{n,k}^{(1/n)}(x) \frac{1}{B(nk, n(n-k))} \int_0^1 t^{nk-1} (1 - t)^{n(n-k)-1} \cdot f \left( \rho t + (1 - \rho) \frac{k}{n} \right) dt + p_{n,0}^{(1/n)}(x) f(0) + p_{n,n}^{(1/n)}(x) f(1), \]

where \( \rho \in [0, 1] \), \( B(nk, n(n-k)) \) is the beta function and \( P_{n,k}^{(1/n)}(x) \) is defined as above.

The goal of this note is to present a new kind Durrmeyer type operators involving Polya distribution and study some approximation properties of these operators.

## 2 Basic Results

**Lemma 1.** [26] For \( e_i = t^i, i = 0, 4 \) we have

(i) \( P_{n}^{(1/n)}(e_0; x) = 1; \)

(ii) \( P_{n}^{(1/n)}(e_1; x) = x; \)

(iii) \( P_{n}^{(1/n)}(e_2; x) = \frac{nx^2 + 2x - x^2}{n + 1}; \)
Lemma 2. For the operators $P_{n,\rho}^{(1/n)}$, we have

(i) $P_{n,\rho}^{(1/n)}(e_0; x) = 1$;

(ii) $P_{n,\rho}^{(1/n)}(e_1; x) = x$;

(iii) $P_{n,\rho}^{(1/n)}(e_2; x) = \frac{(n - 1)(n^2 - \rho^2 + 1)}{(n + 1)(n^2 + 1)} x^2 + \left(\frac{2}{n + 1} + \frac{(n - 1)\rho^2}{(n + 1)(n^2 + 1)}\right) x$;

(iv) $P_{n,\rho}^{(1/n)}(e_3; x) = \frac{(n - 1)(n - 2)(n^2 + 2)(n^2 + 1) - 3(n^2 + 2)\rho^2 + 4\rho^3}{(n + 1)(n + 2)(n^2 + 1)(n^2 + 2)} x^3$

\(\frac{3(n - 1)(2(n^2 + 2)(n^2 + 1) + (n - 4)(n^2 + 2)\rho^2 - 2(n - 2)\rho^3}{(n + 1)(n + 2)(n^2 + 1)(n^2 + 2)} x^2
\)

\(\frac{(6(n^2 + 2)(n^2 + 1) + 6(n - 1)(n^2 + 2)\rho^2 + 2(n - 2)(n - 1)\rho^3)}{(n + 1)(n + 2)(n^2 + 1)(n^2 + 2)} x\)

(v) $P_{n,\rho}^{(1/n)}(e_4; x) = \frac{1}{(n + 1)(n + 2)(n + 3)(n^2 + 1)(n^2 + 2)(n^2 + 3)}$

\(\left[\frac{(n - 1)(n - 2)(n - 3)(n^6 - 6(\rho - 1)^3(3\rho + 1) - 6n^4(\rho^2 - 1)}{(n + 1)(n + 2)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \right] x^4
\)

\(\frac{6(n - 2)(n - 1)}{(n + 1)(n + 2)(n + 3)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \left[\frac{(2(n^2 + 1)(n^2 + 2)(n^2 + 3) + (n - 9)(n^2 + 2)(n^2 + 3)\rho^2 - 4(n - 5)(n^2 + 3)\rho^3 - (n - 3)(n^2 - 6)\rho^4}{(n + 1)(n + 2)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \right] x^3
\)

\(\frac{1}{(n + 1)(n + 2)(n + 3)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \left[\frac{(n - 1)(2(18n - 1) + 2(n + 2)(n^2 + 3))/(n^2 + 2) + (2 + n(n - 11)(n^2 + 3))\rho^3 + 3(12 + n(-48 + n(30 + n(-2 + n(n - 7))))(n^2 + 3)\rho^4}{(n + 1)(n + 2)(n + 3)(n^2 + 1)(n^2 + 2)(n^2 + 3)\rho^2 + 8(n - 2)\rho^3 + 3(12 + n(-48 + n(30 + n(-2 + n(n - 7))))\rho^4)} \right] x^2
\)

\(\frac{1}{(n + 1)(n + 2)(n + 3)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \left[\frac{(n - 1)(2(-6 + n(78 + n(13n - 1)) + 6n^2 + n^4)) + 36\rho^2 + 6n(-24 + n(23 + n(-20 + n(16 + n(3n - 4))))\rho^3 + 8(n - 2)(n^2 + 3)\rho^4 + 3(n - 1)(-6 + 6n + n^3 + 4)\rho^4}{(n + 1)(n + 2)(n + 3)(n^2 + 1)(n^2 + 2)(n^2 + 3)\rho^2 + 8(n - 2)(n - 1)^2(n^2 + 3)\rho^3 + 3(n - 1)(-6 + 6n + n^3 + 4)\rho^4)} \right]
\).
Lemma 3. For \( m = 1, 2, 4 \) the \( m \)th order central moments of \( P_{n,\rho}^{(1/n)} \) defined as 
\[
\mu_n^{m,\rho}(x) = P_{n,\rho}^{(1/n)}((t-x)^m; x)
\]
we have

(i) \( \mu_n^{0,\rho}(x) = 0; \)

(ii) \( \mu_n^{2,\rho}(x) = \frac{(\rho^2 - n(2n + \rho^2) - 2)}{(n+1)(n^2 + 1)} x^2 + \left( \frac{2}{n+1} + \frac{(n-1)\rho^2}{(n+1)(n^2 + 1)} \right) x; \)

(iii) \( \mu_n^{4,\rho}(x) = \frac{1}{(n+1)(n+2)(n+3)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \left[ 3(4(n-7)(n^2 + 1)) \right. \)

\( (n^2 + 2)(n^2 + 3) + 4(n-12)(n-1)(n^2 + 2)(n^2 + 3)\rho^2 - 32(n-2)(n-1) \)

\( (n^2 + 3)\rho^3 + (n-3)(n-2)(n-1)(n^2 - 6)\rho^4 \right] x^4 \)

\[
+ \frac{1}{(n+1)(n+2)(n+3)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \left[ 6(-4(n-7)(n^2 + 1)\right. \)

\( (n^2 + 2)(n^2 + 3) - 4(n-12)(n-1)(n^2 + 2)(n^2 + 3)\rho^2 + 32(n-2)(n-1)(n^2 + 3)\rho^3 \)

\( - (n-3)(n-2)(n-1)(n^2 - 6)\rho^4 \right] x^3 \)

\[
+ \frac{1}{n(n+1)(n+2)(n+3)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \left[ (2(n^2 + 1)(n^2 + 2)(n^2 + 3) \right. \)

\( (1 + n(-55 + 6n)) + 12(n-1)(1 + n(n-15))(n^2 + 2)(n^2 + 3)\rho^2 - 16(n-2) \)

\( (n-1)(7n-1)(n^2 + 3)\rho^3 + 3(n-1)(12+n(-48+n(30+n(-2+(-7+n)n))))\rho^4 \right] \)

\( x^2 \)

\[
+ \frac{1}{n(n+1)(n+2)(n+3)(n^2 + 1)(n^2 + 2)(n^2 + 3)} \left[ 2(-6 + n(78 + n(13n-1) \right. \)

\( (11 + 6n^2 + n^4)) + 36\rho^2 + 6n(-24 + n(23 + n(-20 + n(16 + n(3n-4)))) \right) \rho^2 \)

\( + 8(n-2)(n-1)(n^2 + 3)\rho^3 + 3(n-1)(-6 + 6n + n^3 + n^4)\rho^4 \right] x. \)

Lemma 4. For \( n \in \mathbb{N} \), we have

\( P_{n,\rho}^{(1/n)}((t-x)^2; x) \leq \frac{M_{\rho}}{n+1} \phi^2(x), \quad n \geq 1, \quad x \in J, \)

where \( M_{\rho} \) is a positive constant depending on \( \rho \) and \( \phi^2(x) = x(1-x). \)

Remark 1. For the operators \( P_{n,\rho}^{(1/n)} \), we get

\[
\lim_{n \to \infty} n \mu_n^{0,\rho}(x) = 0,
\]

\[
\lim_{n \to \infty} n \mu_n^{2,\rho}(x) = 2x(1-x),
\]

\[
\lim_{n \to \infty} n^2 \mu_n^{4,\rho}(x) = 12x^2(x-1)^2.
\]

3 Direct Estimates

Theorem 1. Let \( f \in C(J) \). Then \( \lim_{n \to \infty} P_{n,\rho}^{(1/n)}(f; x) = f(x), \) uniformly in \( J. \).
**Proof.** By Lemma 2, \( P_{n,\rho}^{(1/n)}(e_0; x) = 1, \) \( P_{n,\rho}^{(1/n)}(e_1; x) \to x, \) \( P_{n,\rho}^{(1/n)}(e_2; x) \to x^2 \) as \( n \to \infty, \) uniformly in \( J. \) By Korovkin Theorem, it follows that \( P_{n,\rho}^{(1/n)}(f; x) \to f(x) \) as \( n \to \infty, \) uniformly in \( J. \)

### 3.1 Voronovskaja type theorem

In this section we establish Voronovskaja type result for the operators \( P_{n,\rho}^{(1/n)}. \)

**Theorem 2.** Let \( f \in C(J). \) If \( f'' \) exists at a point \( x \in J, \) then we have

\[
\lim_{n \to \infty} n \left[ P_{n,\rho}^{(1/n)}(f; x) - f(x) \right] = x(1-x)f''(x).
\]

**Proof.** Applying Taylor’s expansion of \( f, \) we have

\[
f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varpi(t, x)(t-x)^2,
\]

where \( \lim_{t \to x} \varpi(t, x) = 0. \) By using the linearity of the operator \( P_{n,\rho}^{(1/n)} \), we get

\[
P_{n,\rho}^{(1/n)}(f; x) - f(x) = P_{n,\rho}^{(1/n)}((t-x); x)f'(x) + \frac{1}{2}P_{n,\rho}^{(1/n)}((t-x)^2; x)f''(x)
+ P_{n,\rho}^{(1/n)}(\varpi(t, x)(t-x)^2; x).
\]

Now, using the Cauchy-Schwarz property, we have

\[
nP_{n,\rho}^{(1/n)}(\varpi(t, x)(t-x)^2; x) \leq \sqrt{nP_{n,\rho}^{(1/n)}(\varpi^2(t, x); x)} \sqrt{n^2P_{n,\rho}^{(1/n)}((t-x)^4; x)}.
\]

In view of Theorem 1, we obtain \( \lim_{n \to \infty} P_{n,\rho}^{(1/n)}(\varpi^2(t, x); x) = \varpi^2(x, x) = 0, \) since \( \varpi(t, x) \to 0 \) as \( t \to x, \) and using Remark 1 for every \( x \in J, \) we get

\[
\lim_{n \to \infty} n^2P_{n,\rho}^{(1/n)}((t-x)^4; x) = 12x^2(x-1)^2.
\]

Hence,

\[
\lim_{n \to \infty} nP_{n,\rho}^{(1/n)}(\varpi(t, x)(t-x)^2; x) = 0.
\]

From Remark 1, we have

\[
\lim_{n \to \infty} nP_{n,\rho}^{(1/n)}(t-x; x) = 0,
\]

\[
\lim_{n \to \infty} nP_{n,\rho}^{(1/n)}((t-x)^2; x) = 2x(1-x).
\]

Combining the results from above the theorem is proved.
3.2 Local approximation

We begin by recalling the following K-functional:

\[ K_2(f, \delta) = \inf \{ ||f - g|| + \delta ||g''|| : g \in W^2 \} \ (\delta > 0), \]

where \( W^2 = \{ g : g'' \in C(J) \} \) and \( ||.|| \) is the uniform norm on \( C(J) \). By [16] there exists a positive constant \( M > 0 \) such that

\[ K_2(f, \delta) \leq M \omega_2(f, \sqrt{\delta}), \tag{8} \]

where the modulus of smoothness of second order for \( f \in C(J) \) is defined as

\[ \omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in J} |f(x + 2h) - 2f(x + h) + f(x)|. \]

The modulus of continuity for \( f \in C(J) \) is defined by

\[ \omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in J} |f(x + h) - f(x)|. \]

The Steklov mean (see [13]) is defined as

\[ f_h(x) = \frac{4}{h^2} \int_0^h \int_0^h [2f(x + u + v) - f(x + 2(u + v))] \, du \, dv. \tag{9} \]

By simple computation, it is observed that

a) \( ||f_h - f||_{C(J)} \leq \omega_2(f, h). \)

b) \( f'_h, f''_h \in C(J) \) and \( ||f'_h||_{C(J)} \leq \frac{5}{h} \omega(f, h), \ |f''_h||_{C(J)} \leq \frac{9}{h^2} \omega_2(f, h). \)

Theorem 3. Let \( f \in C(J) \). Then for every \( x \in J \), the following inequality holds

\[ |P_n^{(1/n)}(f; x) - f(x)| \leq 5 \omega \left( f, \sqrt{\mu_n^{(2)}(x)} \right) + \frac{13}{2} \omega_2 \left( f, \sqrt{\mu_n^{(2)}(x)} \right). \]

Proof. For \( x \in J \), and using the Steklov mean \( f_h \) that is given by (9), we may write

\[ |P_n^{(1/n)}(f; x) - f(x)| \leq |P_n^{(1/n)}(f - f_h; x)| + |P_n^{(1/n)}(f_h - f_h(x); x)| + |f_h(x) - f(x)|. \]

From (3), for every \( f \in C(J) \) we get

\[ |P_n^{(1/n)}(f; x)| \leq ||f||. \tag{11} \]

Using assumption (a) of Steklov mean and (11), we have

\[ P_n^{(1/n)}(||f - f_h||; x) \leq ||P_n^{(1/n)}(f - f_h)|| \leq ||f - f_h|| \leq \omega_2(f, h). \]
By Taylor’s expansion and Cauchy-Schwarz property, we obtain
\[
\left| P_{n,\rho}^{(1/n)}(f_h - f_h(x); x) \right| \leq \| f_h' \| \sqrt{P_{n,\rho}^{(1/n)}}((t - x)^2; x) + \frac{1}{2} f_h'' \| P_{n,\rho}^{(1/n)}((t - x)^2; x). 
\]

From Lemma 3 and inequality (b) of Steklov mean, we have
\[
\left| P_{n,\rho}^{(1/n)}(f_h - f_h(x); x) \right| \leq \frac{5}{h} \omega(f, h) \sqrt{\mu_0^2(x)} + \frac{9}{2h^2} \omega_2(f, h) \mu_0^2(x).
\]

Choosing \( h = \sqrt{\mu_0^2(x)} \), and substituting the values of the above estimates in (10), we get the desired result.

Remark 2. [29] Let \( L : C(I) \to \mathcal{F}(I) \) be a positive linear operator, where \( I \) is an arbitrary interval and \( \mathcal{F}(I) \) is the space of real functions defined on \( I \). Then, for any \( f \in C(I), x \in I, 0 < h < \frac{1}{2} \text{length}(I) \) we have
\[
L(f; x) \leq |f(x)||L(e_0; x) - 1| + \frac{1}{h} |L(e_1; x) - x| \omega_1(f; h) + \left[ L(e_0; x) + \frac{1}{2h^2} L((e_1 - e_0 x)^2; x) \right] \omega_2(f, h).
\]

From this for \( L = P_{n,\rho}^{(1/n)} \) and \( h = \sqrt{\mu_0^2(x)} \) the upper bound becomes, \( \frac{3}{2} \omega_2 \left( f, \sqrt{\mu_0^2} \right) \).

### 3.3 Global Approximation

In this section, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the \( K \)-functional [17]. Let \( \phi(x) = \sqrt{x(1-x)} \) and \( f \in C(J) \). The first order modulus of smoothness is defined by
\[
\omega_{\phi}(f, t) = \sup_{0 < h \leq t} \left\{ \left| f \left( x + \frac{h\phi(x)}{2} \right) - f \left( x - \frac{h\phi(x)}{2} \right) \right|, x \pm \frac{h\phi(x)}{2} \in J \right\},
\]
and the Petree’s \( K \)-functional is given by
\[
K_{\phi}(f, t) = \inf_{g \in W_{\phi}} \{ ||f - g|| + t||\phi g'|| + t^2 ||g'|| \} (t > 0),
\]
where \( W_{\phi} = \{ g : g \in AC_{loc}, ||\phi g'|| < \infty, ||g'|| < \infty \} \) and ||.|| is the uniform norm on \( C(J) \). It is well known that (Theorem 3.1.2, [17]) \( K_{\phi}(f, t) \sim \omega_{\phi}(f, t) \) which means that there exists a constant \( M > 0 \) such that
\[
M^{-1} \omega_{\phi}(f, t) \leq K_{\phi}(f, t) \leq M \omega_{\phi}(f, t).
\]

Now we study a global approximation theorem for the operators \( P_{n,\rho}^{(1/n)} \).
A. Kajla

Theorem 4. Let $f$ be in $C(J)$ and $\phi(x) = \sqrt{x(1-x)}$, then for every $x \in [0,1)$, we have

$$|P_{n,\rho}^{(1/n)}(f; x) - f(x)| \leq C\omega_\phi \left( f, \sqrt{\frac{M_\rho}{n+1}} \right),$$

where $M_\rho$ is defined in Lemma 4 and $C > 0$ is a constant.

Proof. Applying the relation $g(t) = g(x) + \int_x^t g'(u)du$, we may write

$$|P_{n,\rho}^{(1/n)}(g; x) - g(x)| = \left| \frac{1}{n} \int_f^{f(t)} g'(u)du; x \right|. \tag{13}$$

For any $x, t \in (0,1)$, we have

$$\left| \int_x^t g'(u)du \right| \leq ||\phi g'|| \left| \int_x^t \frac{1}{\phi(u)} du \right|. \tag{14}$$

Therefore,

$$\left| \int_x^t \frac{1}{\phi(u)} du \right| = \left| \int_x^t \frac{1}{\sqrt{u(1-u)}} du \right| \leq \left| \int_x^t \left( \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}} \right) du \right| \leq 2 \left( |\sqrt{t} - \sqrt{x}| + |\sqrt{1-t} - \sqrt{1-x}| \right) = 2|t-x| \left( \frac{1}{\sqrt{t} + \sqrt{x}} + \frac{1}{\sqrt{1-t} + \sqrt{1-x}} \right) < 2|t-x| \left( \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}} \right) \leq \frac{2\sqrt{2}}{\phi(x)} \tag{15}.$$}

Collecting (13)-(15) and using Cauchy-Schwarz property, we may write

$$|P_{n,\rho}^{(1/n)}(g; x) - g(x)| < 2\sqrt{2}||\phi g'|| \phi^{-1}(x)P_{n,\rho}^{(1/n)}(|t-x|; x) \leq 2\sqrt{2}||\phi g'|| \phi^{-1}(x) \left( P_{n,\rho}^{(1/n)}((t-x)^2; x) \right)^{1/2}.$$}

Now applying Lemma 4, we obtain

$$|P_{n,\rho}^{(1/n)}(g; x) - g(x)| < C \sqrt{\frac{M_\rho}{n+1}} ||\phi g'||. \tag{16}$$
Using Lemma 2 and (16), we get
\[
|\mathcal{P}_{n,\rho}^{(1/n)}(f) - f| \leq |\mathcal{P}_{n,\rho}^{(1/n)}(f - g; x)| + |f - g| + |\mathcal{P}_{n,\rho}^{(1/n)}(g; x) - g(x)|
\]
(17)
\[
\leq C \left( \|f - g\| + \sqrt{\frac{M_{\rho}}{(n+1)}} \|\phi g'\| \right).
\]
Taking infimum on the right hand side of the above inequality over all \(g \in W_{\phi}\), we have
\[
|\mathcal{P}_{n,\rho}^{(1/n)}(f; x) - f(x)| \leq C \overline{\mathcal{K}}_{\phi} \left( f; \sqrt{\frac{M_{\rho}}{(n+1)}} \right).
\]
Using \(\overline{\mathcal{K}}_{\phi}(f,t) \sim \omega_{\phi}(f,t)\), we get the desired relation.

3.4 Rate of convergence

DBV\((J)\) denotes the class of all absolutely continuous functions \(f\) defined on \(J\), having on \(J\) a derivative \(f'\) equivalent with a function of bounded variation on \(J\).

We notice that the functions \(f \in DBV(J)\) possess a representation
\[
f(x) = \int_0^x g(t)dt + f(0)
\]
where \(g \in BV(J)\), i.e., \(g\) is a function of bounded variation on \(J\).

The operators \(\mathcal{P}_{n,\rho}^{(1/n)}(f; x)\) also admit the integral representation
\[
\mathcal{P}_{n,\rho}^{(1/n)}(f; x) = \int_0^1 \mathcal{H}_{n,\rho}(x,t) f \left( \frac{\rho t + (1 - \rho) \frac{k}{n}}{n} \right) dt,
\]
where the kernel \(\mathcal{H}_{n,\rho}(x,t)\) is given by
\[
\mathcal{H}_{n,\rho}(x,t) = \sum_{k=1}^{n-1} p_{n,k}^{(1/n)}(x) \frac{1}{B(nk, n(n-k))} t^{nk-1}(1-t)^{n(n-k)-1}
\]
\[
+ p_{n,0}^{(1/n)}(x) \delta(t) + p_{n,n}^{(1/n)}(x) \delta(1-t),
\]
\(\delta(u)\) being the Dirac-delta function.

Lemma 5. For a fixed \(x \in (0,1)\) and sufficiently large \(n\), we have
\[
(i) \quad \vartheta_{n,\rho}(x,y) = \int_0^y \mathcal{H}_{n,\rho}(x,t) dt \leq \frac{M_{\rho}}{(n+1)} \frac{\phi^2(x)}{(x-y)^2}, \quad 0 \leq y < x,
\]
\[
(ii) \quad 1 - \vartheta_{n,\rho}(x,z) = \int_z^1 \mathcal{H}_{n,\rho}(x,t) dt \leq \frac{M_{\rho}}{(n+1)} \frac{\phi^2(x)}{(z-x)^2}, \quad x < z < 1.
\]
Proof. (i) Using Lemma 4 we get
\[ \vartheta_{n,\epsilon}(x,y) = \int_0^y H_{n,\epsilon}(x,t)dt \leq \int_0^y \left( \frac{x-t}{x-y} \right)^2 H_{n,\epsilon}(x,t)dt \]
\[ = \mathcal{P}_{n,\epsilon}^{(1/n)}((t-x)^2;x)(x-y)^{-2} \leq \frac{M_\epsilon}{n+1} \phi^2(x) (x-y)^2. \]

The proof of (ii) is similar hence the details are omitted.

Theorem 5. Let \( f \in DBV(J) \). Then for every \( x \in (0,1) \) and sufficiently large \( n \), we have
\[ |\mathcal{P}_{n,\epsilon}^{(1/n)}(f;x) - f(x)| \leq \sqrt{\frac{M_\epsilon}{n+1}} \phi(x) \left| f'(x+) - f'(x-) \right| \]
\[ + \frac{M_\epsilon \phi^2(x)}{(n+1)} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} x^{-1} \left( f'_x \right) \sum_{k=1}^{\lfloor (1-x)/k \rfloor} x^{(1-x)/k} \left( f'_x \right) \]
\[ + \frac{M_\epsilon \phi^2(x)}{(1-x)(n+1)} \sum_{k=1}^{\lfloor (1-x)/k \rfloor} x^{(1-x)/k} \left( f'_x \right) \]
\[ + \frac{(1-x)}{\sqrt{n}} \sum_{x} \left( f'_x \right), \]
where \( \mathcal{V}_a^b(f'_x) \) denotes the total variation of \( f'_x \) on \([a,b]\) and \( f'_x \) is defined by
\[
\begin{cases}
  f'_x(t) = \begin{cases}
    f'(t) - f'(x-), & 0 \leq t < x \\
    0, & t = x \\
    f'(t) - f'(x+) & x < t < 1.
  \end{cases}
\end{cases}
\]
(20)

Proof. Since \( \mathcal{P}_{n,\epsilon}^{(1/n)}(1;x) = 1 \), by using (19), for every \( x \in (0,1) \) we get
\[ \mathcal{P}_{n,\epsilon}^{(1/n)}(f;x) - f(x) = \int_0^1 H_{n,\epsilon}(x,t)(f(t) - f(x))dt \]
\[ = \int_0^1 H_{n,\epsilon}(x,t) \int_x^t f'(u)du dt. \]
(21)

For any \( f \in DBV(J) \), by (20) we can write
\[ f'(u) = f'_x(u) + \frac{1}{2}(f'(x+) + f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-))\text{sgn}(u-x) \]
\[ + \delta_x(u)[f'(u) - \frac{1}{2}(f'(x+) + f'(x-))], \]
(22)
where
\[ \delta_x(u) = \begin{cases}
  1, & u = x \\
  0, & u \neq x.
\end{cases} \]
Obviously,

\[
\int_0^1 \left( \int_x^t \left( f'(u) - \frac{1}{2} \left( f'(x^+) + f'(x^-) \right) \delta_x(u) \right) \mathcal{H}_{n,\rho}(x, t) \right) dt = 0.
\]

By (19) and simple calculations we have

\[
\int_0^1 \left( \int_x^t \frac{1}{2} (f'(x^+) + f'(x^-)) du \right) \mathcal{H}_{n,\rho}(x, t) dt
\]

\[
= \frac{1}{2} (f'(x^+) + f'(x^-)) \int_0^1 (t - x) \mathcal{H}_{n,\rho}(x, t) dt
\]

\[
= \frac{1}{2} (f'(x^+) + f'(x^-)) \mathcal{P}^{(1/n)}((t - x); x)
\]

and

\[
\left| \int_0^1 \mathcal{H}_{n,\rho}(x, t) \left( \int_x^t \frac{1}{2} (f'(x^+) - f'(x^-)) \text{sgn}(u - x) du \right) dt \right|
\]

\[
\leq \frac{1}{2} | f'(x^+) - f'(x^-) | \int_0^1 |t - x| \mathcal{H}_{n,\rho}(x, t) dt
\]

\[
\leq \frac{1}{2} | f'(x^+) - f'(x^-) | \mathcal{P}^{(1/n)}(|t - x|; x)
\]

\[
\leq \frac{1}{2} | f'(x^+) - f'(x^-) | \left( \mathcal{P}^{(1/n)}((t - x)^2; x) \right)^{1/2}.
\]

Considering Lemmas 3 and 4 and using (21)-(22) we obtain the following estimate

\[
|\mathcal{P}^{(1/n)}(f; x) - f(x)| \leq \frac{1}{2} | f'(x^+) - f'(x^-) | \sqrt{\frac{M_{\rho}}{n+1}} \phi(x)
\]

\[
+ \left| \int_0^x \left( \int_x^t f_x'(u) du \right) \mathcal{H}_{n,\rho}(x, t) dt \right|
\]

\[
+ \left| \int_x^1 \left( \int_x^t f_x'(u) du \right) \mathcal{H}_{n,\rho}(x, t) dt \right|.
\]

(23)

Let

\[
\mathcal{T}_{n,\rho}(f_x', x) = \int_0^x \left( \int_x^t f_x'(u) du \right) \mathcal{H}_{n,\rho}(x, t) dt,
\]

\[
\mathcal{S}_{n,\rho}(f_x', x) = \int_x^1 \left( \int_x^t f_x'(u) du \right) \mathcal{H}_{n,\rho}(x, t) dt.
\]

To complete the proof, it is sufficient to estimate the terms \( \mathcal{T}_{n,\rho}(f_x', x) \) and \( \mathcal{S}_{n,\rho}(f_x', x) \).

Since \( \int_a^b dt \mathcal{H}_{n,\rho}(x, t) \leq 1 \) for all \( [a, b] \subseteq [0, 1] \), using integration by parts and applying
Lemma 5 with \( y = x - (x/\sqrt{n}) \), we have

\[
|T_{n,\rho}(f'_x, x)| = \left| \int_0^x \left( \int_x^t f'_x(u)du \right) dt \varphi_{n,\rho}(x, t) \right|
\]
\[
= \left| \int_0^x \varphi_{n,\rho}(x, t)f'_x(t)dt \right|
\]
\[
\leq \left( \int_y^x + \int_x^y \right) |f'_x(t)| \varphi_{n,\rho}(x, t)dt
\]
\[
\leq \frac{M\rho^2(x)}{(n+1)} \int_0^y x (f'_x)(x - t)^{-2}dt + \int_x^y x (f'_x)dt
\]
\[
\leq \frac{M\rho^2(x)}{(n+1)} \int_0^y x (f'_x)(x - t)^{-2}dt + \frac{x}{\sqrt{n}} \int_{x-(x/\sqrt{n})}^x (f'_x).
\]

By the substitution of \( u = x/(x - t) \), we obtain

\[
\frac{M\rho^2(x)}{(n+1)} \int_0^{x-(x/\sqrt{n})} (x - t)^{-2} \frac{x}{t} (f'_x)dt = \frac{M\rho^2(x)}{(n+1)} x^{-1} \int_1^{\sqrt{n}} \int_{x/u}^x (f'_x)du
\]
\[
\leq \frac{M\rho^2(x)}{(n+1)} x^{-1} \sum_{k=1}^{\sqrt{n}} \int_k^{k+1} \int_{x-(x/k)}^{x} (f'_x)du
\]
\[
\leq \frac{M\rho^2(x)}{(n+1)} x^{-1} \sum_{k=1}^{\sqrt{n}} \int_{x-(x/k)}^x (f'_x).
\]

Thus,

\[
(24) \quad |T_{n,\rho}(f'_x, x)| \leq \frac{M\rho^2(x)}{(n+1)} x^{-1} \sum_{k=1}^{\sqrt{n}} \int_{x-(x/k)}^{x} (f'_x) + \frac{x}{\sqrt{n}} \int_{x-(x/\sqrt{n})}^x (f'_x).
\]

Using integration by parts and applying Lemma 5 with \( z = x + ((1 - x)/\sqrt{n}) \), we have
Blending type approximation

\[ |S_{n,\rho}(f'_x, x)| \]

\[
= \left| \int_x^1 \left( \int_x^t f'_x(u)du \right) \mathcal{H}_{n,\rho}(x, t)dt \right| \\
= \left| \int_x^z \left( \int_x^t f'_x(u)du \right) dt(1 - \vartheta_{n,\rho}(x, t)) \right. \\
\left. + \int_x^1 \left( \int_x^t f'_x(u)du \right) dt(1 - \vartheta_{n,\rho}(x, t)) \right| \\
= \left| \left[ \int_x^t f'_x(u)(1 - \vartheta_{n,\rho}(x, t))du \right]_x^z - \int_x^z f'_x(t)(1 - \vartheta_{n,\rho}(x, t))dt \\
\left. + \int_x^1 \left( \int_x^t f'_x(u)du \right) dt(1 - \vartheta_{n,\rho}(x, t)) \right| \\
= \left| \int_x^z f'_x(t)(1 - \vartheta_{n,\rho}(x, t))dt + \int_x^1 f'_x(t)(1 - \vartheta_{n,\rho}(x, t))dt \right| \\
\leq \frac{M_\rho \phi^2(x)}{(n+1)} \int_x^1 \sqrt{x} (f'_x)(t-x)^{-2}dt + \int_x^z \sqrt{x} (f'_x)dt \\
= \frac{M_\rho \phi^2(x)}{(n+1)} \int_x^1 \sqrt{x} (f'_x)(t-x)^{-2}dt + \frac{(1-x)^{x+(1-x)/\sqrt{n}}}{\sqrt{n}} \int_x^z (f'_x). \\
\]

By the substitution of \( v = (1-x)/(t-x) \), we get

\[
|S_{n,\rho}(f'_x, x)| \leq \frac{M_\rho \phi^2(x)}{(n+1)} \int_1^{\sqrt{n} x+(1-x)/\sqrt{n}} \sqrt{x} (f'_x)(1-x)^{-1}dv + \frac{(1-x)^{x+(1-x)/\sqrt{n}}}{\sqrt{n}} \int_x^z (f'_x) \\
\leq \frac{M_\rho \phi^2(x)}{(1-x)(n+1)} \sum_{k=1}^{\sqrt{n}} \sqrt{x} (f'_x)dv + \frac{(1-x)^{x+(1-x)/\sqrt{n}}}{\sqrt{n}} \int_x^z (f'_x) \\
= \frac{M_\rho \phi^2(x)}{(1-x)(n+1)} \sum_{k=1}^{\sqrt{n}} \sqrt{x} (f'_x) + \frac{(1-x)^{x+(1-x)/\sqrt{n}}}{\sqrt{n}} \int_x^z (f'_x). \\
\]

Collecting the estimates (23)-(25), we get the required result. This completes the proof of theorem.

3.5 Numerical examples

Example 1. In Figure 1, for \( n = 25 \), the collation of convergence of \( P_{25,\rho}(f; x) \) (orange) and the Bernstein-Durrmeyer (green) operators to
\( f(x) = x^4 - 12x^3 + 4x^2 - 10x \) (blue) is illustrated. It is seen that the Bernstein-Durrmeyer operators gives a better approximation to \( f(x) \) than \( P_{25,k}^{0.4}(f;x) \) for \( n = 25 \).

\[ \text{Figure 1.} \]

**Example 2.** In Figure 2, for \( n = 50 \), the collation of convergence of \( P_{50,k}^{0.6}(f;x) \) (orange) and the Bernstein-Durrmeyer (green) operators to \( f(x) = x^4 - 12x^3 + 4x^2 - 10x \) (blue) is illustrated. It is note that the Bernstein-Durrmeyer operators gives a better approximation to \( f(x) \) than \( P_{50,k}^{0.6}(f;x) \) for \( n = 50 \).

\[ \text{Figure 2.} \]

**References**


Arun Kajla
Central University of Haryana, India
Department of Mathematics
Central University of Haryana, Haryana-123031, India
e-mail:rachitkajla47@gmail.com
On some subclasses of bi-univalent functions associating
pseudo-starlike functions with Sakaguchi
type functions\footnote{Received 1 August, 2017
Accepted for publication (in revised form) 6 October, 2017}

Emeka P. Mazi, Timothy O. Opoola

Abstract

In this paper, we define two new subclasses of bi-univalent functions associating
\( \lambda \)-pseudo-starlike function with Sakaguchi type functions
\( \angle^\lambda_2(s,t,\alpha) \) and
\( \angle^\lambda_3(s,t,\beta) \). The Coefficients estimates for
\( |a_2| \) and \( |a_3| \) are obtained. Results obtained generalized some known results.

\textbf{2010 Mathematics Subject Classification:} 30C45, 30C50.

\textbf{Key words and phrases:} bi-univalent function, coefficient bounds,
pseudo-starlike function, Sakaguchi type function, Taylor- Maclaurin coefficients.

1 Introduction

Let \( A \) denotes the class of functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

which are analytic in the open unit disk \( U = \{ z \in C : |z| < 1 \} \). Let \( S \) be the subclass of \( A \) consisting of functions which are analytic and univalent in \( U \). Some of the important and well known subclasses of univalent function class \( S \) includes (for example) the class \( S^*(\beta) \) of starlike functions of order \( \beta \) in \( U \) and the class \( K(\beta) \) of convex function of order \( \beta \) in \( U \). By definition we have

\[
S^*(\beta) := \left\{ f : f \in S \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta; z \in U; 0 \leq \beta < 1 \right\}
\]
and

\[ K(\beta) := \left\{ f : f \in S \quad \text{and} \quad \Re \left( 1 + z \frac{f''(z)}{f'(z)} \right) > \beta ; z \in U ; 0 \leq \beta < 1 \right\}. \]

It readily follows from the definition (2) and (3) that \( f(z) \in K(\beta) \iff zf'(z) \in S^*(\beta). \)

It is well known by Keobe One-Quarter Theorem [5] that the range of every function of the class \( S \) contains the disk \( \{ w : |w| < 1/4 \} \). Therefore, every \( f \in S \) has an inverse function \( f^{-1} \) satisfying \( f^{-1}(f(z)) = z \) \( z \in U \) and

\[ f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4). \]

The inverse of \( f(z) \) has a series expansion in some disc about the origin of the form

\[ (4) \quad f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots. \]

A function \( f(z) \) univalent in a neighbourhood of the origin and its inverse satisfy the condition \( f(f^{-1}(w)) = w \)

using (4) yields

\[ (5) \quad w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots. \]

and now using (5) we get the following results

\[ (6) \quad g(w) = f^{-1}(w) = w - a_2 w^2 + (2 a_2^2 - a_3) w^3 - (5 a_3^2 - 5 a_2 a_3 + a_4) w^4 + \cdots. \]

An analytic function \( f(z) \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \). The class of analytic bi-univalent function in \( U \) is denoted by \( \Sigma \).

Example of functions in the class \( \Sigma \) are

\[ \frac{z}{1 - z}; \quad -\log(1 - z); \quad \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right). \]

However the familiar Keobe function \( \frac{z}{(1 - z)^2} \) is not a member of the class \( \Sigma \)

(see[12]).

Other example of function that does not belong to \( \Sigma \) are

\[ z - \frac{z^2}{2}; \quad \frac{z}{1 - z^2} \]

(see[12]).

Lewin [7] investigated the bi-univalent class of functions in \( \Sigma \) and showed that from

(1) the \( |a_2| < 1.51 \). Netanyahu [9] on the other hand showed that from (1) the \( \max |a_2| < 4/3 \). Brannan and Clunie [3] conjectured that \( |a_2| \leq \sqrt{2} \. Brannan and Taha [4] introduced these subclasses \( S^*_\Sigma, S^*_\Sigma(\rho), K_\Sigma(\alpha) \) and \( K_\Sigma(\beta) \) of bi-univalent class of functions \( \Sigma \). Similar to the two subclasses \( S^*(\alpha) \) and \( K(\alpha) \) of the univalent functions in class \( S \) (see[3]). Brannan and Taha [4] found non sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) (see also [15]).

Babalola [2] defined the class \( L_\lambda(\beta) \) of \( \lambda \)-pseudo-starlike functions of order \( \beta \) as below:
Definition 1 Let $f \in A$, suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real. Then $f(z) \in L_\lambda(\beta)$ of $\lambda$-pseudo-starlike functions of order $\beta$ in the unit disk if and only if
\[
(\text{7}) \quad \text{Re} \left( \frac{zf'(z)}{f(z)} \right)^\lambda > \beta.
\]

Babalola [2] proved that, all pseudo-starlike functions are Bazilevic of type $(1 - \frac{1}{\lambda})$ order $\beta \lambda$ and univalent in open unit disk $\mathbb{U}$.

Sakaguchi [11] investigated class of starlike functions with respect to symmetric points denoted by $S_s$ consisting of functions $f \in A$ satisfy the condition
\[
\text{Re} \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0, \quad (z \in \mathbb{U})
\]

Joshi et al. [6] studied some subclasses of bi-univalent functions associated with pseudo-starlike functions and obtained bounds of initial coefficient $|a_2|$ and $|a_3|$.

Altinkaya and Yalcin [1] also investigated a new subclass of bi-univalent functions of Sakaguchi type satisfying subordinate conditions and obtained bounds of initial coefficients $|a_2|$ and $|a_3|$. Many author investigated bounds for various subclasses of bi-univalent functions ([8], [10], [13], [14] and [16]). But the coefficients estimates problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N}\setminus\{1, 2\}$; $N = 1, 2, 3$ is presumably still an open problem.

The purpose of these paper is to introduce two new subclasses of bi-univalent functions associating pseudo-starlike function with Sakaguchi type functions and to obtain the coefficient estimate $|a_2|$ and $|a_3|$ for functions of these subclasses. Let $P$ be the class of Carathéodory functions i.e $P$ is the family of functions $\varphi$ analytic in $U$ for which
\[
\text{Re} \{\varphi(z)\} > 0 \quad \varphi(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad \text{for} \quad z \in U.
\]

Lemma 1 ([5]) If $\varphi \in P$ then
\[
\varphi(z) = 1 + \sum_{k=1}^{\infty} c_k z^k
\]
and $|c_k| \leq 2 \quad (k \in \mathbb{N} = (1, 2, \cdots))$. 

2 Coefficient bounds for the function class $\angle_\Sigma^\lambda(s, t, \alpha)$

**Definition 2** A function $f(z)$ given by (1) is said to be in the class $\angle_\Sigma^\lambda(s, t, \alpha)$ if the following conditions are satisfied

$$f \in \Sigma,$$

$$\left| \arg \left\{ \frac{(s - t)z[f'(z)]^\lambda}{f(sz) - f(tz)} \right\} \right| < \frac{\alpha \pi}{2},$$

$$\{ \lambda \geq 1 \in \mathbb{R}, s, t \in \mathbb{C}, s \neq t, |t| < 1, 0 < \alpha \leq 1; z \in U \}$$

and

$$\left| \arg \left\{ \frac{(s - t)w[g'(w)]^\lambda}{g(sw) - g(tw)} \right\} \right| < \frac{\alpha \pi}{2},$$

$$\{ \lambda \geq 1 \in \mathbb{R}, s, t \in \mathbb{C}, s \neq t, |t| < 1, 0 < \alpha \leq 1; w \in U \}$$

where $g(w) = f^{-1} = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$.

**Theorem 1** Let $f(z)$ given by (1) be in the class $\angle_\Sigma^\lambda(s, t, \alpha)$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(6\lambda - 4\lambda(s + t - \lambda + 1) + 2st)\alpha - (\alpha - 1)(2\lambda - s - t)^2}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(2\lambda - s - t)^2} + \frac{2\alpha}{(3\lambda - s^2 - t^2 - st)}.$$

**Proof:** Let $f \in \angle_\Sigma^\lambda(s, t, \alpha)$, then its follow from (8) and (9) that

$$\frac{(s - t)z[f'(z)]^\lambda}{f(sz) - f(tz)} = [p(z)]^\alpha$$

and

$$\frac{(s - t)w[g'(w)]^\lambda}{g(sw) - g(tw)} = [q(w)]^\alpha,$$

where $p(z)$ and $q(w)$ are in the class $P$ and we have the forms

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \cdots$$
clearly,
\[
[p(z)]^\alpha = 1 + \alpha p_1 z + \left( \alpha p_2 + \frac{\alpha(\alpha - 1)p_1^2}{2!} \right) z^2 + \cdots ,
\]
\[
[q(w)]^\alpha = 1 + \alpha q_1 w + \left( \alpha q_2 + \frac{\alpha(\alpha - 1)q_1^2}{2!} \right) w^2 + \cdots .
\]

Now, equating the coefficient in (13) and (14) we get
\[
(2\lambda - s - t)a_2 = \alpha p_1 , \tag{17}
\]
\[
(3\lambda - s^2 - t^2 - st)a_3 - (2\lambda s + t - \lambda + 1) - s^2 - t^2 - 2st)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 , \tag{18}
\]
\[
-(2\lambda - s - t)a_2 = \alpha q_1 , \tag{19}
\]
and
\[
(6\lambda - s^2 - t - 2\lambda s + t - \lambda + 1)a_2^2 - (3\lambda - s^2 - t^2 - st)a_3 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 . \tag{20}
\]

From (17) and (19) we obtain
\[
p_1 = -q_1 \tag{21}
\]
and
\[
2(2\lambda - s - t)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2) . \tag{22}
\]

From (18) and (20) we obtain
\[
(6\lambda - s^2 - t^2 - 2\lambda s + t - \lambda + 1) - (2\lambda s + t - \lambda + 1) - s^2 - t^2 - 2st)a_2^2 = \\
= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2) .
\]

A rearrangement together with the identity in (22) yields
\[
(6\lambda - 4\lambda(s + t - \lambda + 1) + 2st)a_2^2 = \alpha(p_2 + q_2) + (\alpha - 1)\frac{(2\lambda - s - t)^2}{\alpha} .
\]

Thus, we have
\[
a_2^2 = \frac{\alpha^2(p_2 + q_2)}{(6\lambda - 4\lambda(s + t - \lambda + 1) + 2st)\alpha - (\alpha - 1)(2\lambda - s - t)^2} . \tag{23}
\]
Applying Lemma 1 for the coefficients $p_2$ and $q_2$ we have

$$|a_2| \leq \frac{2\alpha}{\sqrt{(6\lambda - 4\alpha(s + t - \lambda + 1) + 2st)\alpha - (\alpha - 1)(2\lambda - s - t)^2}}.$$ 

This gives the bounds on $|a_2|$ as asserted in (11).

Next, in order to find the bounds on $|a_3|$, subtracting (20) from (18) we get

$$(24) \quad 2(3\lambda - s^2 - t^2 - st)a_3 - 2(3\lambda - s^2 - t^2 - st)a_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2).$$

It follow from (21),(22) and (24) that

$$(25) \quad a_3 = \frac{\alpha(p_2 - q_2)}{2(3\lambda - s^2 - t^2 - st)} + \frac{\alpha(p_1^2 + q_1^2)}{2(2\lambda - s - t)^2}.$$ 

Applying Lemma 1 once again for the coefficients $p_1, p_2, q_1$ and $q_2$

$$|a_3| \leq \frac{2\alpha}{(3\lambda - s^2 - t^2 - st)} + \frac{4\alpha^2}{(2\lambda - s - t)^2}.$$ 

This complete the proof of Theorem 1.

Taking $s=1$ and $t=0$ in Theorem 1 we obtain the corollary.

**Corollary 1** let $f(z)$ given by (1) be in the class $LB_{\lambda}^\alpha(\alpha)$ and $0 < \alpha \leq 1$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(2\lambda - 1)(2\lambda - 1 + \alpha)}}$$

and

$$|a_3| = \frac{4\alpha^2}{(2\lambda - 1)^2} + \frac{2\alpha}{(3\lambda - 1)^2}.$$ 

which are the results obtained by Joshi et al. [6].

Taking $s=1$ and $t=0$ and $\lambda = 1$ in Theorem 1 we obtain the corollary.

**Corollary 2** Let $f(z)$ given by (1) be in the class $S_{\lambda}^{\alpha}$ and $0 < \alpha \leq 1$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{1 + \alpha}}$$

and

$$|a_3| \leq 4\alpha^2 + \alpha.$$
3 Coefficient bounds for the function class $\angle_\Sigma \lambda(s, t, \beta)$

**Definition 3** A function $f(z)$ given by (1) is said to be in the class $\angle_\Sigma \lambda(s, t, \beta)$ if the following conditions are satisfied

$$f \in \Sigma, \quad \text{Re} \left\{ \frac{(s - t)z[f'(z)]^\lambda}{f(sz) - f(tz)} \right\} > \beta \quad \{ \lambda \geq 1 \in \mathbb{R}, s, t \in \mathbb{C}, s \neq t, |t| < 1, 0 \leq \beta < 1; z \in U \}$$

and

$$\text{Re} \left\{ \frac{(s - t)w[g'(w)]^\lambda}{g(sw) - g(tw)} \right\} > \beta \quad \{ \lambda \geq 1 \in \mathbb{R}, s, t \in \mathbb{C}, s \neq t, |t| < 1, 0 \leq \beta < 1; w \in U \}$$

(28) $g(w) = f^{-1} = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_3^2 - 5a_2a_3 + a_4)w^4 + \cdots$.

**Theorem 2** Let $f(z)$ given by (1) be in the class $\angle_\Sigma \lambda(s, t, \beta)$. Then

(29) $|a_2| \leq \sqrt{\frac{2(1 - \beta)}{3\lambda - 2\lambda(s + t - \lambda + 1) + st}}$

and

(30) $|a_3| \leq \frac{4(1 - \beta)^2}{(2\lambda - s - t)^2} + \frac{2(1 - \beta)}{(3\lambda - s^2 - t^2 - st)}$

**Proof:** It follows from (15) and (16) that there exist $p(z) \in P$ and $q(w) \in P$ such that

(31) $\frac{(s - t)z[f'(z)]^\lambda}{f(sz) - f(tz)} = \beta + (1 - \beta)p(z)$

and

(32) $\frac{(s - t)w[g'(w)]^\lambda}{g(sw) - g(tw)} = \beta + (1 - \beta)q(w)$.

where

$$\beta + (1 - \beta)p(z) = \beta + (1 - \beta) \left( 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \right)$$

$$= \beta + 1 + p_1z + p_2z^2 + p_3z^3 - \beta p_1z + \beta p_2z^2 + \beta p_3z^3 + \cdots$$

$$= 1 + (1 - \beta)p_1z + (1 - \beta)p_2z^2 + \cdots$$
\[ \beta + (1 - \beta)q(w) = 1 + (1 - \beta)q_1w + (1 - \beta)q_2w^2 + \cdots. \]

Equating the coefficients of (31) and (32) we have
\[ (2\lambda - s - t)a_2 = (1 - \beta)p_1, \tag{33} \]
\[ (3\lambda - s^2 - t^2 - st)a_3 - (2\lambda(s + t - \lambda + 1) - s^2 - t^2 - 2st)a_2^2 = (1 - \beta)p_2, \tag{34} \]
\[ -(2\lambda - s - t)a_2 = (1 - \beta)q_1, \tag{35} \]
and
\[ (6\lambda - s^2 - t^2 - 2\lambda(s + t - \lambda + 1))a_2^2 - (3\lambda - s^2 - t^2 - st)a_3 = (1 - \beta)q_2. \tag{36} \]

From (33) and (35) we obtain
\[ p_1 = -q_1 \tag{37} \]
and
\[ 2(2\lambda - s - t)^2a_2^2 = (1 - \beta)^2(p_1^2 + q_1^2). \tag{38} \]

From (34) and (36) we obtain
\[ (6\lambda - s^2 - t^2 - 2\lambda(s + t - \lambda + 1) - (2\lambda(s + t - \lambda + 1) - s^2 - t^2 - 2st))a_2^2 = (1 - \beta)(p_2 + q_2). \]

Upon simplification and rearrangement we get
\[ (6\lambda - 4\lambda(s + t - \lambda + 1) + 2st)a_2^2 = (1 - \beta)(p_2 + q_2). \]

Therefore, we have
\[ a_2^2 = \frac{(1 - \beta)(p_2 + q_2)}{(6\lambda - 4\lambda(s + t - \lambda + 1) + 2st)}. \]

Applying Lemma 1 for the coefficient of \( p_2 \) and \( q_2 \) lead to
\[ |a_2| \leq \sqrt{\frac{2(1 - \beta)}{(3\lambda - 2\lambda(s + t - \lambda + 1) + st)}}. \]

This gives the bound on \( |a_2| \) as asserted in (29).

Next, in other to find \( |a_3| \), we subtract (36) from (34) and we get
\[ 2(3\lambda - s^2 - t^2 - st)a_3 - 2(3\lambda - s^2 - t^2 - st)a_2^2 = (1 - \beta)(p_2 - q_2) \tag{39} \]
or equivalent

\[ a_3 = a_2^2 + \frac{(1 - \beta)(p_2 - q_2)}{2(3\lambda - s^2 - t^2 - st)}. \]

Upon substituting the values of \( a_2^2 \) from (38) we get

\[ a_3 = \frac{(1 - \beta)^2(p_1^2 + q_1^2)}{2(2\lambda - s - t)^2} + \frac{(1 - \beta)(p_2 - q_2)}{2(3\lambda - s^2 - t^2 - st)}. \]

Applying Lemma 1 for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \) we get

\[ |a_3| \leq \frac{4(1 - \beta)^2}{(2\lambda - 1)^2} + \frac{2(1 - \beta)}{(3\lambda - 1)}. \]

This complete the proof of Theorem 2. Taking \( s=1 \) and \( t=0 \) in Theorem 2 we obtain the corollary.

**Corollary 3** let \( f(z) \) given by (1) be in the class. Then \( LB_\lambda^S(\beta) \) and \( 0 \leq \beta < 1 \). Then

\[ |a_2| \leq \sqrt{\frac{2(1 - \beta)}{\lambda(2\lambda - 1)}} \]

and

\[ |a_3| \leq \frac{4(1 - \beta)^2}{(2\lambda - 1)^2} + \frac{2(1 - \beta)}{(3\lambda - 1)}. \]

which are the results obtained by Joshi et al. [6].

Taking \( s=1 \) and \( t=0 \) and \( \lambda = 1 \) in Theorem 2 we obtain the Corollary.

**Corollary 4** Let \( f(z) \) given by (1) be in the class. Then \( S_{\lambda}^S(\beta) \) and \( 0 \leq \beta < 1 \). Then

\[ |a_2| \leq \sqrt{2(1 - \beta)} \]

and

\[ |a_3| \leq 4(1 - \beta)^2 + (1 - \beta). \]

**References**


Emeka P. Mazi
University of Ilorin, Nigeria
Faculty of Science
Department of Mathematics
Department of Mathematics, University of Ilorin, Nigeria
e-mail: emekmazi21@gmail.com

Timothy O. Opoola
University of Ilorin, Nigeria
Faculty of Science
Department of Mathematics
Department of Mathematics, University of Ilorin, Nigeria
e-mail: opoola_stc@yahoo.com
A note on the Pompeiu-Stamate mean-value theorem

Vicuta Neagos

Abstract

We provide a simple proof of the Stamate mean-value theorem.

2010 Mathematics Subject Classification: 26A24.
Key words and phrases: mean-value theorems, divided differences, interpolation.

1 Preliminaries and auxiliary results

Throughout the paper \( n \geq 1 \) denotes an integer. Let \([a, b]\) be a real interval and \( a \leq x_0 < \cdots < x_n \leq b \).

1.1 The divided difference

In most books on numerical analysis, the divided difference \([x_0, \ldots, x_n; f]\) of a function \( f: \{x_0, \ldots, x_n\} \to \mathbb{R} \) is defined recursively. We prefer in this note to use the following definition in terms of determinants,

\[
[x_0, \ldots, x_n; f] = \frac{\begin{vmatrix}
1 & x_0 & \cdots & x_0^{n-1} & f(x_0) \\
1 & x_1 & \cdots & x_1^{n-1} & f(x_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^{n-1} & f(x_n)
\end{vmatrix}}{\begin{vmatrix}
1 & x_0 & \cdots & x_0^n \\
1 & x_1 & \cdots & x_1^n \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \cdots & x_n^n
\end{vmatrix}}.
\]

Received 17 July, 2017
Accepted for publication (in revised form) 8 October, 2017
The interpolation polynomial \( L[x_0, \ldots, x_n; f] \) of degree at most \( n \) satisfying the interpolation conditions:

\[
\begin{align*}
\{ & L[x_0, \ldots, x_n; f](x_i) = f(x_i), \quad i = 0, \ldots, n, \\
& \text{is uniquely defined by the equation} \\
& \begin{vmatrix}
1 & x_0 & \cdots & x_0^n & f(x_0) \\
1 & x_1 & \cdots & x_1^n & f(x_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^n & f(x_n) \\
1 & x & \cdots & x^n & L[x_0, \ldots, x_n; f](x) \\
\end{vmatrix} = 0.
\end{align*}
\]

The interpolation polynomial can also be written into the form

\[
L(x_0, \ldots, x_n; f)(x) = \sum_{i=0}^{n} f(x_i) \prod_{j=0 \atop j \neq i}^{n} \frac{x-x_j}{x_i-x_j}.
\]

From (2) and (3) we deduce that the number \( L[x_0, \ldots, x_n; f](0) \) satisfies the equalities:

\[
\begin{vmatrix}
1 & x_0 & \cdots & x_0^n & f(x_0) \\
1 & x_1 & \cdots & x_1^n & f(x_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^n & f(x_n) \\
1 & 0 & \cdots & 0 & L[x_0, \ldots, x_n; f](0) \\
\end{vmatrix} = 0,
\]

and

\[
L(x_0, \ldots, x_n; f)(0) = \sum_{i=0}^{n} f(x_i) \prod_{j=0 \atop j \neq i}^{n} \frac{x_j}{x_j-x_i}.
\]

If \( f \) has a derivative of order \( n \) at \( \alpha \), we denote by \( T_n(f; \alpha) \) the Taylor polynomial of degree \( n \) associated to \( f \) at \( \alpha \),

\[
T_n(f; \alpha)(x) := \sum_{i=0}^{n} \frac{f^{(i)}(\alpha)}{i!} (x-\alpha)^i.
\]

It is important to note that the use of the Leibniz rule for higher derivatives of the product of two functions gives

\[
\binom{f(t)}{t}^{(n)} = \frac{(-1)^{n}n!}{t^{n+1}} T_n[f; t](0).
\]

One connection between the divided difference and the \( n \)-th derivative of a function is given by a classical result of Cauchy.
Proposition 1 (Cauchy [8, p. 36]) If \( \varphi : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and \( n \) times differentiable on \((a, b)\), then there exists \( c \in (a, b) \) such that

\[
[t_0, \ldots, t_n; \varphi] = \frac{\varphi^{(n)}(c)}{n!}.
\]

Pompeiu gave the following variant of the Lagrange mean-value theorem:

Theorem 1 ([7, Pompeiu (1946)]) Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), differentiable on \((a, b)\) and \( 0 \notin [a, b] \). Then there exists a point \( c \in (a, b) \) such that

\[
a f(b) - b f(a) = f(c) - c f'(c).
\]

He also gave the following geometric interpretation:

The tangent line to the graph of \( f \) at the point \((c, f(c))\), the line joining the points \((a, f(a))\) and \((b, f(b))\) and the y-axis intersect in the same point.

A Pompeiu-type mean-value theorem was obtained by Mircea Ivan in 1970.

Theorem 2 ([4, Ivan (1970)], [5, Ivan (1973)], [6, Ivan (2002)]) Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f \) has no roots in \([a, b]\) and \( f(a) \neq f(b) \) then there exists a point \( c \in (a, b) \) such that:

\[
(a) \frac{af(b) - bf(a)}{a - b} = f(c) - c f'(c).
\]

Geometrically this means that the graph of the Taylor polynomial \( T_1(f; c) \) and the graph of the Lagrange interpolation polynomial \( L_1(a, b; f) \) intersect the Ox axis in the same point.

In 1948, T. Boggio obtained the following generalization of Pompeiu’s mean-value theorem:

Theorem 3 ([2, T. Boggio (1948)]) Let \( f, g : [a, b] \to \mathbb{R} \) be two functions satisfying the conditions:

(i) are continuous on \([a, b]\);

(ii) are differentiable on \((a, b)\);

(iii) \( g(x) \neq 0, \forall x \in [a, b] \);

(iv) \( g'(x) \neq 0, \forall x \in (a, b) \).

Then there exists a point \( c \in (a, b) \) such that

\[
(8) \quad \frac{g(a) f(b) - g(b) f(a)}{g(a) - g(b)} = f(c) - g(c) \frac{f'(c)}{g'(c)}.
\]
A proof of this theorem was given by Mircea Ivan [6]. He showed that the assertion of Boggio’s theorem results by applying Pompeiu’s theorem to the function $F = f \circ g^{-1}$.

Among the many other extensions of Pompeiu’s theorem we focus on that of I. Stamate:

**Theorem 4 ([9, Stamate (1958)])** Assume that $0 \notin [a, b]$ and let $f$ be continuous on $[a, b]$ and $n$ times differentiable on $(a, b)$. Then there exists a point $c \in (a, b)$ such that

$$
\sum_{i=0}^{n} f(x_i) \prod_{j=0}^{n} \frac{x_j}{x_i} = \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!} (-c)^i.
$$

We note that, using (3) and (6) we can write (9) in the form

$$
L[x_0, \ldots, x_n; f](0) = T_n[f; c](0).
$$

Eq. (10) gives a geometric interpretation of Stamate’s theorem:

*The graph of the Taylor polynomial $T_n(f; c)$, the graph of the Lagrange interpolating polynomial $L_n = L_n[x_0, \ldots, x_n; f]$ and the $y$-axis intersect in the same point.*

In the special case $n = 1$, we obtain Pompeiu’s theorem 1.

The proof of Stamate is based on the well-known extension of Lagrange’s mean-value theorem to the case of divided differences. Stamate obtained the relation (9) by applying Proposition1 for $t_i = \frac{1}{x_i}$, $i = 0, \ldots, n$ and $\varphi(t) = t^n f\left(\frac{1}{t}\right)$.

**Remark 1** A proof of Stamate’s mean-value theorem was given by U. Abel and M. Ivan in [1]. They showed that the relation (10) results by using some properties of divided differences.

## 2 Main results

The following is a very simple mean-value result but has a lot of applications.

**Proposition 2 ([3, Gavrea, Ivan, Neagos (2017)])** If $F, G: [a, b] \to \mathbb{R}$ are continuous and $n$ times differentiable on $(a, b)$, then there exists a point $c \in (a, b)$ such that

$$
\begin{vmatrix}
1 & x_0 & \cdots & x_0^{n-1} & F(x_0) & G(x_0)
1 & x_1 & \cdots & x_1^{n-1} & F(x_1) & G(x_1)
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^{n-1} & F(x_n) & G(x_n)
0 & 0 & \cdots & 0 & F^{(n)}(c) & G^{(n)}(c)
\end{vmatrix} = 0.
$$
A note on the Pompeiu-Stamate mean-value theorem

Proof. The function
\[
\begin{vmatrix}
1 & x_0 & \cdots & x_0^{n-1} & F(x_0) & G(x_0) \\
1 & x_1 & \cdots & x_1^{n-1} & F(x_1) & G(x_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^{n-1} & F(x_n) & G(x_n) \\
1 & x & \cdots & x^{n-1} & F(x) & G(x)
\end{vmatrix},
\]
has \( n \) roots, namely \( x_i, i = 0, 1, \ldots, n \). By using the Generalized Rolle theorem, we deduce that there exists a point \( c \in (x_0, x_n) \) such that \( h^{(n)}(c) = 0 \), and the proof is complete.

\[\square\]

2.1 A short proof of Boggio’s mean-value theorem

Proof. In (11) we take \( n = 1 \), \( F(x) := \frac{f(x)}{g(x)} \) and \( G(x) := \frac{1}{g(x)} \), and we get:
\[
\begin{vmatrix}
1 & F(a) & G(a) \\
1 & F(b) & G(b) \\
0 & F'(c) & G'(c)
\end{vmatrix} = 0,
\]
that is,
\[
\begin{vmatrix}
1 & \frac{f(a)}{g(a)} & \frac{1}{g(a)} \\
1 & \frac{f(b)}{g(b)} & \frac{1}{g(b)} \\
0 & \frac{f'(c)}{g^2(c)} g(c) - f(c) \frac{g'(c)}{g^2(c)} & -\frac{g'(c)}{g^2(c)}
\end{vmatrix} = 0.
\]
We expand the determinant in terms of the third row and we have:
\[
-(f'(c) g(c) - f(c) g'(c)) \left( \frac{1}{g(b)} - \frac{1}{g(a)} \right) - g'(c) \left( \frac{f(b)}{g(b)} - \frac{f(a)}{g(a)} \right) = 0,
\]
hence
\[
g'(c) \left( f(b) g(a) - f(a) g(b) \right) = \left( f'(c) g(c) - f(c) g'(c) \right) \left( g(a) - g(b) \right).
\]
Since \( g'(c) \neq 0 \) and \( g(a) \neq g(b) \), we obtain the relation (8).

\[\square\]

Remark 2 For using Rolle’s theorem we consider the function
\[
h : [a, b] \to \mathbb{R}, h(x) = \begin{vmatrix}
1 & f(x) & 1 \\
1 & g(x) & g(x) \\
1 & f(a) & 1 \\
1 & g(a) & g(a) \\
1 & f(b) & 1 \\
1 & g(b) & g(b)
\end{vmatrix}.
\]
h are continuous on \([a, b]\), differentiable on \((a, b)\) and \( h(a) = h(b) = 0 \). It results that there exists \( c \in (a, b) \) such that \( h'(c) = 0 \), from where we deduce the relation (8).
Remark 3 A proof of Pompeiu’s Theorem results by applying the Rolle’s theorem

for \( g : [a, b] \to \mathbb{R} \),
\[
\begin{vmatrix}
1 & f(x) & 1 \\
1 & \frac{x}{a} & 1 \\
1 & \frac{g}{b} & 1 \\
\end{vmatrix}.
\]

Remark 4 We can also prove the Theorem 2 using Rolle’s theorem if we consider the auxiliary function

\[
h : [a, b] \to \mathbb{R},
\begin{vmatrix}
1 & x & 1 \\
1 & \frac{a}{f(a)} & 1 \\
1 & \frac{b}{f(b)} & 1 \\
\end{vmatrix}.
\]

2.2 A simple proof of Stamate’s mean-value formula

Proof. For \( f(t) := f(t)/t \) and \( g(t) := 1/t \), using (6), (11) becomes

\[
\begin{vmatrix}
1 & x_0 & \cdots & x_0^{n-1} & \frac{f(x_0)}{x_0} & 1 & \frac{1}{x_0} \\
1 & x_1 & \cdots & x_1^{n-1} & \frac{f(x_1)}{x_1} & 1 & \frac{1}{x_1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^{n-1} & \frac{f(x_n)}{x_n} & 1 & \frac{1}{x_n} \\
0 & 0 & \cdots & 0 & \frac{(-1)^n n!}{c^{n+1}} T_n[f; c](0) & 0 & \frac{(-1)^n n!}{c^{n+1}} \\
\end{vmatrix} = 0,
\]

i.e.,

\[
\begin{vmatrix}
1 & x_0 & \cdots & x_0^n & f(x_0) \\
1 & x_1 & \cdots & x_1^n & f(x_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_n & \cdots & x_n^n & f(x_n) \\
1 & 0 & \cdots & 0 & T_n[f; c](0) \\
\end{vmatrix} = 0.
\]

From (4) and (13) we deduce (10) and the proof is complete. □

We note that (13) is a new form of Stamate’s mean-value theorem.
A note on the Pompeiu-Stamate mean-value theorem

References


Vicuta Neagos
Technical University of Cluj-Napoca
Department of Mathematics
Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania
e-mail: vicuta.neagos@math.utcluj.ro
Lattice $g$-2-normed spaces and 2-best approximation properties of their downward subsets

Mahdi Iranmanesh, Ali Ganjbakhsh Sanatee

Abstract

We developed a theory of downward subsets of a $g$-2-Banach lattice $X$ with a strong unite $1$. For an application of this concept, we study 2-best approximation of $X$ by elements of downward sets and give necessary and sufficient conditions for elements of 2-best approximation from closed downward subsets of $X$.

2010 Mathematics Subject Classification: 41A50, 41A65.

Key words and phrases: $g$-2-normed space, 2-Best approximation, Donward set, $g$-2-Banach lattis space.

1 Introduction

The concept of linear 2-normed spaces has been introduced by S.Gahler [8]. Z. Lewandowska in [11] introduced a generalization of Gahler 2-normed space as follows; Let $X$ and $Y$ be two real linear spaces. A function $\| \cdot, \cdot \| : X \times Y \rightarrow [0,\infty)$ is called a generalized 2-norm on $X \times Y$ if satisfies the following conditions:

1. $\| \alpha x, y \| = |\alpha| \| x, y \| = \| \alpha x, y \|$ for any real number $\alpha$ and all $(x, y) \in X \times Y$,
2. $\| x, y + z \| \leq \| x, y \| + \| x, z \|$ for $x \in X$ and $y, z \in Y$ with $(x, y), (x, z) \in X \times Y$,
3. $\| x + y, z \| \leq \| x, z \| + \| y, z \|$ for $x, y \in X$ and $z \in Y$ with $(x, z), (y, z) \in X \times Y$.

Then the function $\| \cdot, \cdot \|$ is said to be a generalized 2-norm on $X \times Y$ and the pair $(X \times Y, \| \cdot, \cdot \|)$ is called generalized 2-normed space. When $X = Y$, the generalized 2-normed space $(X \times X, \| \cdot, \cdot \|)$ is denoted by $(X, \| \cdot, \cdot \|)$.

---

1 Received 5 September, 2017

   Accepted for publication (in revised form) 10 October, 2017
Recall that in Gahler definition of a 2-norm, $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent, and this is a crucial difference between Gahlers approach and Lewandowskas one.

A real vector space $X$ which is ordered by some order relation $\leq$, is called vector lattice if any two elements $x, y \in X$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$ and the following properties are satisfied:

1. $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$
2. $0 \leq x$ implies $0 \leq tx$ for all $x \in X$ and $t \in \mathbb{R}^+$.

For $x \in X$, positive part, negative part and absolute value of $x$ define as $x^+ := x \vee 0$, $x^- := (-x) \vee 0$, $|x| := x \vee (-x)$.

**Definition 1** A norm $\| \cdot \|$ on a vector lattice $X$ is called a lattice norm if

1. $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in X$.

A Banach lattice is a real Banach space $X$ endowed with an ordering $\leq$ such that $(X, \leq)$ is a vector lattice and the norm on $X$ is a lattice norm. An element $1 \in X$ is called a strong unit if for each $x \in X$ there exist $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda 1$.

Using a strong unit $1$, we can define a norm on $X$ by

$$
\|x\| = \inf\{\lambda \geq 0; |x| \leq \lambda 1\} \quad \forall x \in X.
$$

Then

$$
B(x, r) = \{y \in X; \|x - y\| \leq r\} = \{y \in X; x - r1 \leq y \leq x + r1\}.
$$

We have also

$$
|x| \leq \|x\|1 \quad \text{for all } x \in X.
$$

A subset $W$ of an ordered set $X$ is said to be downward, if $w \in W, x \leq w$, it implies that $x \in W$.

In this paper we define a new form of generalized 2-normed space and call it $g$-2-normed space. In the next step we define $g$-2-Banach lattice space and develope a theory of 2-best approximation by elements of closed downward sets in a $g$-2-Banach lattice $X$ which its $g$-2-norm is induced by the strong unit. We show that a closed downward set is 2-proximinal. This means that there exist a 2-best approximation from this set for each $x \in X$. We show that the least element of the set of 2-best approximations exists. Finally we investigate 2-best approximation in $g$-2-normed space $(X \times Y, \|\cdot, \cdot\|)$ by elements of downward sets and show that the least element of the set of 2-best approximations exists.
2 Notations and preliminary result

Now we define a new form of generalized 2-normed space as follows.

Definition 2 Let $X$ and $Y$ be two real linear spaces. A function
$$\|\cdot,\cdot\| : X \times Y \rightarrow [0, \infty)$$
called a $g$-2-norm on $X \times Y$ if it satisfies the following conditions:

1. For every $x \neq 0$ in $X$ there exists $y \neq 0$ in $Y$ such that $\|x, y\| \neq 0$,
2. $\|x, \alpha y\| = |\alpha| \|x, y\|$ for any real number $\alpha$ and all $(x, y) \in X \times Y$,
3. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for $x \in X$ and $y, z \in Y$ with $(x, y), (x, z) \in X \times Y$,
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for $x, y \in X$ and $z \in Y$ with $(x, z), (y, z) \in X \times Y$.

Example 1 Let $X$ and $Y$ be two real linear spaces having seminorms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then $(X \times Y, \|\cdot,\cdot\|)$ is a $g$-2-normed space with the $2$-norm defined by

$$\|x, y\| = \|x\|_1 \|y\|_2.$$

Then the function $\|\cdot,\cdot\|$ is said to be a $g$-2-norm on $X \times Y$ and the pair $(X \times Y, \|\cdot,\cdot\|)$ is called $g$-2-normed space. When $X = Y$, the $g$-2-normed space $(X \times X, \|\cdot,\cdot\|)$ is denoted by $(X, \|\cdot,\cdot\|)$.

Definition 3 A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a $g$-2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent if there exists $x \in X$ such that $\|x_n - x, y\|_{n \in \mathbb{N}}$ tends to zero for all $y \in X$. In this case, we write $\lim_{n \to \infty} x_n = x$ and call $x$ the limit of $\{x_n\}_{n \in \mathbb{N}}$.

The uniqueness of the limit of a convergent can be shown as follows: To show this, suppose $\{x_n\}_{n \in \mathbb{N}}$ is convergent to two distinct limits $x$ and $y$ in $X$. For this choose $z \in X$ such that $\|x - y, z\| \neq 0$ and taking $n_0 \in \mathbb{N}$ sufficiently large such that $\|x_{n_0} - x, z\| < \frac{1}{2}\|x - y, z\|$ and $\|x_{n_0} - y, z\| < \frac{1}{2}\|x - y, z\|$ simultaneously. Then we get

$$\|x - y, z\| \leq \|x - x_{n_0}, z\| + \|x_{n_0} - y, z\| < \frac{1}{2}\|x - y, z\| + \frac{1}{2}\|x - y, z\| = \|x - y, z\|$$

which is a contradiction. Hence, whenever limit exists, it must be unique.

Definition 4 A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a $g$-2-normed space $(X, \|\cdot,\cdot\|)$ is called a Cauchy sequence if there exist two linearly independent elements $y$ and $z$ in $X$ such that $\|x_n, y\|_{n \in \mathbb{N}}$ and $\|x_n, z\|_{n \in \mathbb{N}}$ are real Cauchy sequences.

Definition 5 A $g$-2-normed space $(X, \|\cdot,\cdot\|)$ is called $g$-2-Banach space if every Cauchy sequence is convergent.
If $X$, $Y$ be two vector lattices with strong units $1_X$ and $1_Y$ respectively, then strong units induced a $g$-2-norm on $X \times Y$.

Let $\| \cdot \|_1, \| \cdot \|_2$ be two norms which is induced by strong units $1_X, 1_Y$ on vector lattices $X, Y$. Then this norms induce a $g$-2-norm on $X \times Y$ by

\[
\| \cdot, \cdot \| : X \times Y \rightarrow \mathbb{R}^+
\]
\[
\| (x, y) \| = \| x \|_1 \| y \|_2.
\]

Let $b \in Y, x \in X$ we define the set $V^b_\epsilon(x)$ as follows

\[
V^b_\epsilon(x) = \{ y \in X; \| x - y, b \| \leq \epsilon \} = \{ y \in X; x - \frac{\epsilon 1}{\| b \|} \leq y \leq x + \frac{\epsilon 1}{\| b \|} \}.
\]

Then the family $\{V^b_\epsilon(x)\}_{\epsilon \geq 0, b \in Y, x \in X}$ form a subbasis of a locally convex topology on $X$.

**Definition 6** Let $X, Y$ be two vector lattices and $(X \times Y, \| \cdot, \cdot \|)$ be a $g$-2-normed space. Then $(X \times Y, \| \cdot, \cdot \|)$ is called lattice $g$-2-normed space if

\[
(x_1 \leq x_2, \ y_1 \leq y_2) \implies \| x_1, y_1 \| \leq \| x_2, y_2 \|, \ x_1, x_2 \in X, \ y_1, y_2 \in Y.
\]

A $g$-2-Banach lattice is a real $g$-2-Banach space $X$ such that $(X, \leq)$ is vector lattice and the $g$-2-norm on $X$ is a lattice $g$-2-norm.

**Theorem 1** Let $W$ be a downward subset of a $g$-2-normed space $X$, $W^o$ be the interior of $W$ and $x \in X$. Then the following statements are true;

1. If $x \in X$ then $x - \epsilon 1 \in W^o$ for all $\epsilon \geq 0$
2. We have $W^o = \{ x \in X : x + \epsilon 1 \in W \ for some \ \epsilon \geq 0 \}$.

**Proof.** (1). Let $\epsilon \geq 0$ be arbitrary and $x \in W$. For a given $a \in X$ we have

\[
V^a_{\epsilon \| a \|}(x - \epsilon 1) = \{ y \in X : \| y - (x - \epsilon 1) \|, a \| \leq \epsilon \| a \| \} = \{ y \in X : x - 2\epsilon 1 < y < x \}.
\]

Therefore, since $W$ is downward and $x \in W$, we have

\[
V^a_{\epsilon \| a \|}(x - \epsilon 1) \subseteq W
\]

and so $x - \epsilon 1 \in W^o$.

(2)Let $x \in W^o$. Then there exists an $a \in X$ and $\epsilon > 0$ such that

\[
V^a_{\epsilon \| a \|}(x) \subseteq W
\]

so

\[
\{ y \in X : x - \epsilon 1 < y < x + \epsilon 1 \} \subseteq W
\]

and hence $x \in W^o$. 
Corollary 1 Let $\mathbb{W}$ be a closed downward subset of $g$-2-normed space $X$, $bd(\mathbb{W})$ be the boundary of $\mathbb{W}$ and $w \in \mathbb{W}$. Then the following statements are equivalent.

1. $w \in bd(\mathbb{W})$.
2. For any $\epsilon > 0$, $w + \epsilon 1 \notin \mathbb{W}$.

Proof. Suppose that (1) holds but (2) is not true. Then there exists $\epsilon > 0$ such that $w + \epsilon 1 \in \mathbb{W}$. So, by theorem 1, we get $w \in \mathbb{W}^\circ$. This make a contradiction with $w \in bd(\mathbb{W})$.

Conversly, let $w \in \mathbb{W}$, and (2) holds. So, by theorem 1, $w \notin \mathbb{W}^\circ$. Since $\mathbb{W}$ is closed, we have

$\mathbb{W} = \mathbb{W}$.

So $w \in bd(\mathbb{W})$.

Definition 7 Let $(\mathbb{X} \times \mathbb{Y}, \| \cdot, \|)$ be a $g$-2-normed space and $\mathbb{W}_1 \subset \mathbb{X}, \mathbb{W}_2 \subset \mathbb{Y}$. Then $\mathbb{W}_1 \times \mathbb{W}_2$ is said to be a $g$-2-proximinal if for each element $(x, y)$ of $\mathbb{X} \times \mathbb{Y}$ there exists an element $(w_0, g_0) \in \mathbb{W}_1 \times \mathbb{W}_2$ such that

$$\|x - w_0, y - g_0\| = \inf_{(w, g) \in \mathbb{W}_1 \times \mathbb{W}_2} \|x - w, y - g\|.$$ 

Then $(w_0, g_0)$ is called $g$-2-best approximation of $(x, y)$ in $\mathbb{W}_1 \times \mathbb{W}_2$. We shall denote the set of all $g$-2-best approximations from $(x, y)$ to $\mathbb{W}_1 \times \mathbb{W}_2$ by $P(2)_{\mathbb{W}_1 \times \mathbb{W}_2}(x, y)$ which is defined by

$$P(2)_{\mathbb{W}_1 \times \mathbb{W}_2}(x, y) = \{(w_0, g_0) \in \mathbb{W}_1 \times \mathbb{W}_2 ; \|x - w_0, y - g_0\| = \inf_{(w, g) \in \mathbb{W}_1 \times \mathbb{W}_2} \|x - w, y - g\| \}.$$ 

Definition 8 Let $(\mathbb{X}, \| \cdot, \|)$ be a $g$-2-normed space and $\mathbb{G} \subset \mathbb{X}$, $x \in \mathbb{X} \setminus \mathbb{G}$ and $g_0 \in \mathbb{G}$. Then $g_0$ is said to be a $2$-$a$ best approximation of $x$ in $\mathbb{G}$ if

$$\|x - g_0, a\| = \inf_{g \in \mathbb{G}} \|x - g, a\| \text{ for all } a \in \mathbb{X}.$$ 

We shall denote it by $P(a)_{\mathbb{G}}(x)$. In fact

$$P(a)_{\mathbb{G}}(x) := \{g_0 \in \mathbb{G} ; \|x - g_0, a\| = \inf_{g \in \mathbb{G}} \|x - g, a\| \}.$$ 

$\mathbb{G}$ is said to be $2$-$a$-proximinal if $P(a)_{\mathbb{G}}(x) \neq \emptyset$ for all $x \in \mathbb{X}$, and $\mathbb{G}$ is said to be $2$-proximinal if $P(a)_{\mathbb{G}}(x) \neq \emptyset$ for all $a, x \in \mathbb{X}$. 
3 Main results

Lemma 1 Let \( \mathcal{W} \) be a downward subset of a lattice 2-Banach space \( \mathcal{X} \). Then \( \mathcal{W} \) is 2-\( a \)-proximinal for each fixed element \( a \in \mathcal{X} \).

**Proof.** Let \( x \in \mathcal{X} \setminus \mathcal{W} \) and
\[
\epsilon_a = \inf \left\{ \frac{\|x_0 - w, a\|}{\|a\|} : w \in \mathcal{W} \right\} = \left\{ \|x_0 - w\| : w \in \mathcal{W} \right\}.
\]
Then for each \( \delta > 0 \), there exists \( w_\delta \in \mathcal{W} \) such that
\[
\frac{\|x_0 - w_\delta, a\|}{\|a\|} \leq \epsilon_a + \delta.
\]
Hence
\[
-(\epsilon_a + \delta)1 \leq w_\delta - x_0 \leq (\epsilon_a - \delta)1.
\]
Let \( w_0 = x_0 - \epsilon_a1 \). Then we have
\[
\frac{\|x_0 - w_0\|}{\|a\|} = \frac{\|\epsilon_a1\| - \epsilon_a = \inf_{w \in \mathcal{W}} \|w - x_0\|}{\|a\|},
\]
and so
\[
w_0 - \delta1 = x_0 - (\epsilon_a1 + \delta1) \leq w_\delta.
\]
Since \( \mathcal{W} \) is downward and \( w_\delta \in \mathcal{W} \), it follows that \( w_0 - \delta1 \in \mathcal{W} \) for all \( \delta > 0 \) and so \( w_0 \in \text{bd}(\mathcal{W}) \). Since \( \mathcal{W} \) is closed, we have \( w_0 \in \mathcal{W} \). We have
\[
\|x_0 - w_0\| = \inf_{w \in \mathcal{W}} \|w - x_0, a\|.
\]
Hence
\[
\|x_0 - w_0, a\| = \inf_{w \in \mathcal{W}} \|w - x_0, a\|
\]
and so
\[
w_0 \in \mathcal{P}_{\mathcal{W}}(x_0).
\]
Thus we get the result.

Theorem 2 Let \( \mathcal{W} \) be a closed downward subset of \( \mathcal{X} \) and \( x_0 \in \mathcal{X} \). Then there exists the element \( w_0 := \min \mathcal{P}_{\mathcal{W}}(x_0) \) of the set \( \mathcal{P}_{\mathcal{W}}(x_0) \), namely, \( w_0 = x_0 - \epsilon_a1 \), where \( \epsilon_a = \inf_{w \in \mathcal{W}} \|w - x_0, a\| \).

**Proof.** If \( x_0 \in \mathcal{W} \), then the result is trivial. Now consider that \( x_0 \notin \mathcal{W} \) and \( \epsilon_a = \inf_{w \in \mathcal{W}} \|w - x_0, a\| \) then \( w_0 = x_0 - \frac{\epsilon_a}{\|a\|}1 \in \mathcal{W} \).

On the other hand, if \( x \in V_{\epsilon_a}(x_0) \) then by the property (7), \( x \geq x_0 - \frac{\epsilon_a}{\|a\|}1 = w_0 \) and hence
\[
w_0 = \min V_{\epsilon_a}(x_0).
\]
Now, let \( w \in P^a_W(x_0) \) be arbitrary. Then
\[
\| x_0 - w, a \| = \epsilon_a
\]
and so
\[
w \in V^a_{\epsilon_a}(x_0).
\]
Since \( w_0 = x_0 - \frac{\epsilon_a}{\| a \|} 1 \) and by the property (7) we have
\[
w_0 = \min P^a_W(x_0)
\]
and hence the result is true.

**Corollary 2** Let \( W \) be a closed downward subset of lattice normed space \( X \), \( x_0 \in X \). Then we have
\[
w_0 := \min P^a_W(x_0) \leq x_0.
\]
**Proof.** Let \( x_0 < w_0 \). Since \( W \) is downward, \( x_0 \in W \). So
\[
x_0 \in P^a_W(x_0).
\]
Therefore
\[
w_0 \leq x_0.
\]
This is a contradiction.

The following theorem is proved by H. Mohebi and A. M. Rubinove in [15]

**Theorem 3** Let \( W \) be a closed downward subset of lattice normed space \( X \) and \( x \in X \) be arbitrary. Then
\[
\inf_{w \in W} \| x - w \| = \min \{ \lambda \geq 0 : x - \lambda 1 \in W \}.
\]

**Corollary 3** Let \( W \) be a closed downward subset of generalized lattice 2-normed space \( X \) and \( x \in X \) be arbitrary. Then
\[
\inf_{w \in W} \| x - w, a \| = \min \{ \lambda \geq 0 : x - \frac{\lambda}{\| a \|} 1 \in W \}.
\]

**Proof.** The result is a consequence of theorem 3.

**Theorem 4** Let \( D_1, D_2 \) be two downward subsets of \( X, Y \) respectively. Then for every \( (x, y) \in X \times Y \) and \( (a, b) \in X \times Y \setminus D_1 \times D_2 \), there are \( d_1 \in D_1, d_2 \in D_2 \) such that
\[
\| x - d_1, b \| = \inf_{\alpha \in D_1} \| x - \alpha, b \|
\]
\[
\| a, y - d_2 \| = \inf_{\beta \in D_2} \| a, y - \beta \|.
\]

**Proof.** The proof of this theorem and lemma 1 are the same.
Theorem 5 Let \((X \times Y, \|\cdot, \|)\) be a lattice g-2-normed space and \(D_1, D_2\) be a downward subsets of \(X, Y\) respectively. Then \(D = D_1 \times D_2\) is 2-proximinal.

Proof. Let \((x, y) \in X \times Y \setminus D\) be arbitrary. Then
\[
\inf_{(\alpha, \beta) \in D} \|x - \alpha, y - \beta\| = \inf_{\beta \in D_2} (\inf_{\alpha \in D_1} \|x - \alpha, y - \beta\|).
\]

By theorem 4, there is \(d_1 \in D_1\) such that
\[
\inf_{\beta \in D_2} (\inf_{\alpha \in D_1} \|x - \alpha, y - \beta\|) = \inf_{\beta \in D_2} (\|x - d_1, y - \beta\|).
\]

Again, by theorem 4, there is \(d_2 \in D_2\) such that
\[
\inf_{\beta \in D_2} (\|x - d_1, y - \beta\|) = \|x - d_1, y - d_2\|.
\]

Therefore
\[
\inf_{(\alpha, \beta) \in D} \|x - \alpha, y - \beta\| = \|x - d_1, y - d_2\|.
\]

Hence \(P_{D_1 \times D_2}^{(2)}(x, y) \neq \emptyset\) which completes the proof.

Corollary 4 Let \((x_0, y_0) \in X \times Y\) be arbitrary, \(\omega_0 = P_{W_b}(x_0), \gamma_0 = P_{W_a}(y_0)\) and \(W \times V \subset X \times Y\). If \(V, W\) are closed downward subset. Then for every \(0 \neq a \in X, 0 \neq b \in Y\)
\[
(x_0, y_0) \geq (\frac{\omega_0}{b}, \frac{\gamma_0}{a}).
\]

Proof. The result follows from corollary 2.

Definition 9 A downward subset \(W\) of \(X\) is called strictly downward if for each boundary point \(w_0\) of \(W\), the inequality \(w > w_0\) implies \(w \notin W\).

This definition was introduced in [18] for finite dimensional spaces. The following lemma gives an example of strictly downward set.

Lemma 2 Suppose that \(f : X \rightarrow \mathbb{R}\) be a continuous strictly increasing function and let
\[
S_c(f) = \{x \in X : f(x) \leq c\} \quad (c \in \mathbb{R}).
\]
Then \(S_c(f)\) is strictly downward.

Proof. Since \(f\) is continuous strictly increasing, it follows that
\[
bd S_c(f) = \{x \in X : f(x) = c\}.
\]

Let \(x \in bd S_c(f)\) be arbitrary and \(y \in X\) with \(y > x\). Since \(f\) is strictly increasing and \(f(x) = c\), then \(f(y) > f(x) = c\), and so \(y \notin S_c(f)\). Hence, \(S_c(f)\) is strictly downward.
Definition 10. Let \( \mathcal{W} \) be a downward set. We say that \( \mathcal{W} \) is strictly downward at a point \( w' \in \text{bd}(\mathcal{W}) \) if for all \( w_0 \in \text{bd}(\mathcal{W}) \) with \( w_0 \leq w' \), the inequality \( w > w_0 \) implies \( w \notin \mathcal{W} \).

The following theorem is proved by H. Mohebi and A. M. Rubinov in [15].

Theorem 6. Let \( \mathcal{W} \) be closed downward set. Then \( \mathcal{W} \) is strictly downward at \( w' \in \text{bd}(\mathcal{W}) \) if and only if

1. \( w > w' \Rightarrow w \in \mathcal{W} \)
2. \( (w_0 \leq w', w_0 \in \text{bd}(\mathcal{W})) \Rightarrow w_0 = w' \).

We recall that if \( A, B \) are two closed subsets of topological space \( X \), then

\[
\text{bd}(A \times B) = (\text{bd}(A) \times B) \cup (A \times \text{bd}(B)).
\]

Definition 11. Let \( (X_1 \times X_2, \| \cdot \|) \) be a \( g \)-2-normed space and let \( \mathcal{W} \) be a downward subset of \( X_1 \) and \( b \in X_2 \). Then \( w' \in \text{bd}(\mathcal{W}) \) is said to be a \( 2-b \)-Chebyshev point if for each \( w_0 \in \text{bd}(\mathcal{W}) \) with \( w_0 \leq w' \) and for each \( x_0 \notin \mathcal{W} \) such that \( w_0 \in P^b_W(x_0) \) it follows that \( P^b_W(x_0) = w_0 \).

Let \( (X_1, \leq_1) \) and \( (X_2, \leq_2) \) be two ordered set. We can define an order \( \leq \) on \( X_1 \times X_2 \) as

\[
(x_1, x_2) \leq (x'_1, x'_2) \iff (x_1 \leq_1 x'_1 \text{ and } x_2 \leq_2 x'_2)
\]

This order is called product order.

Definition 12. Let \( (X_1 \times X_2, \| \cdot \|) \) be a \( g \)-2-normed space, and let \( D \) be a downward subset of \( X_1 \times X_2 \), \( X_1 \) and \( X_2 \) be an ordered set and \( \leq \) be product order on \( X_1 \times X_2 \). A point \( (w'_1, w'_2) \in \text{bd}(D) \) is said to be a \( 2 \)-Chebyshev point if for each \( (w_1, w_2) \in \text{bd}(D) \) with \( (w_1, w_2) \leq (w'_1, w'_2) \) and for each \( (x_1, x_2) \notin D \) such that \( (w_1, w_2) \in P^2_D(x_1, x_2) \) it follows that \( P^2_D(x_1, x_2) = (w_1, w_2) \).

Theorem 7. Let \( (X_1 \times X_2, \| \cdot \|) \) be a \( g \)-2-normed space, \( b \in X_2 \), \( \mathcal{W} \) be a downward subset of \( X_1 \) and \( \leq \) be an order on \( X_1 \). Then the following statements are equivalent

1. \( w' \) is a \( 2 \)-Chebyshev point of \( \mathcal{W} \).
2. \( \mathcal{W} \) is a strictly downward set at \( w' \).

Proof. Suppose that (1) holds. Let \( \mathcal{W} \) is not strictly downward at \( w' \). Then we can find \( w \in \text{bd}(\mathcal{W}) \) such that \( w \leq w' \) and there exists \( a \in \mathcal{W}_1 \) with \( a > w \). Let

\[
r \geq \| a - w, b \| > 0
\]

It follows from (2.6) that

\[
a - w \leq |a - w| \leq \| a - w \| 1 \leq \frac{r^1}{\| b \|}.
\]
so

\[(11)\]
\[a \leq \frac{r1}{\|b\|} + w.\]

Let

\[(12)\]
\[x_0 = \frac{r1}{\|b\|} + w.\]

We claim that

\[(13)\]
\[d(x_0, W) = \frac{r}{\|b\|}.\]

Suppose equation (13) does not hold. Then there exists an element \(y \in W\) such that \(\|x_0 - y\| < \frac{r}{\|b\|}\). Then there exists \(r_0 \in (0, \frac{r}{\|b\|})\) such that \(\|x_0 - y\| < r_0\). Hence, by Equation (3) we have \(x_0 \leq y + r_01\). So, by Equation (12), we get

\[x_0 = \frac{r1}{\|b\|} + w \leq y + r_01.\]

Hence

\[w + \lambda_01 \leq y\quad (\lambda_0 = \frac{r}{\|b\|} - r_0)\]

and so by theorem 1 (2) \(w_0 \in W^o\). This is a contradiction with \(w \in bd(W)\). Therefore,

\[d(x_0, W) = \frac{r}{\|b\|} = \|x_0 - w\|.\]

Hence

\[d(x_0, W)\|b\| = \|x_0 - w\|\|b\|\]

Therefore

\[\inf_{\alpha \in W} \|x_0 - \alpha\|\|b\| = \|x_0 - w\|\|b\|\]

and so

\[\inf_{\alpha \in W} \|x_0 - \alpha, b\| = \|x_0 - w, b\|.\]

Therefore \(w \in p^b_W(x_0)\). On the other hand, by relation (11), we have

\[a \leq \frac{r1}{\|b\|} + w = x_0, \quad a > w\]

and so

\[0 \leq x_0 - a \leq x_0 - w = \frac{r1}{\|b\|}.\]

Hence

\[\|x_0 - a\| \leq \|x_0 - w\| = \frac{r}{\|b\|} = d(x_0, W) \leq \|x_0 - a\|\]
Therefore

\[ \inf_{\alpha \in \mathcal{W}} \| x_0 - \alpha, b \| = \| x_0 - a, b \| \]

and so

\[ \| x_0 - a \| = d(x_0, \mathcal{W}). \]

This is impossible since \( w' \) is a 2-best approximation point.

Conversely, suppose that \( \mathcal{W} \) is strictly downward at \( w' \in \text{bd}(\mathcal{W}) \). Then for each point \( w_0 \leq w' \), \( w_0 \in \text{bd}(\mathcal{W}) \) we have \( w_0 = w' \). So we need only to check that \( \mathbf{P}^b_{\mathcal{W}}(x_0) = w' \) for each \( x_0 \not\in \mathcal{W} \) such that \( w' \in \mathbf{P}^b_{\mathcal{W}}(x_0) \). Let \( x_0 \) be such an element. Applying Theorem 2, we conclude that the least element \( w_0 \) of the set \( \mathbf{P}^b_{\mathcal{W}}(x_0) \) exists. We have \( w_0 \in \text{bd}(\mathcal{W}) \), \( w_0 \leq w' \), hence \( w_0 = w' \). Since \( \mathcal{W} \) is strictly downward at \( w_0 = w' \), \( w \geq w' \) for all \( w \in \mathbf{P}^b_{\mathcal{W}}(x_0) \), \( w \not\in \mathcal{W} \). Let \( \gamma \) be such an element. Applying Theorem 2, we have \( \gamma = w' \), and so \( w' \) is a Chebyshev point of \( \mathcal{W} \).

**Theorem 8** Let \( (X_1 \times X_2, \| \cdot, \cdot \|) \) be a g-2-normed space, \( X_1 \) and \( X_2 \) be an ordered set, \( \leq \) be product order on \( X_1 \times X_2 \), \( X_1 \times X_2 \) be a downward subset of \( X_1 \times X_2 \) and \( \mathcal{W}_1 \times \mathcal{W}_2 \) be a strictly downward set at \((w'_1, w'_2)\). Then \((w'_1, w'_2)\) is a 2-Chebyshev point of \( \mathcal{W}_1 \times \mathcal{W}_2 \).

**Proof.** Suppose \( \mathcal{W}_1 \times \mathcal{W}_2 \) is strictly downward at \((w'_1, w'_2)\) and \((w'_1, w'_2) \in \text{bd}(\mathcal{W}_1 \times \mathcal{W}_2)\). Then if \((w_1, w_2) \leq (w'_1, w'_2)\), we have \((w_1, w_2) = (w'_1, w'_2)\).

We need only to check that \( \mathbf{P}_{\mathcal{W}_1 \times \mathcal{W}_2}^{(2)}(x_0, y_0) = (w'_1, w'_2) \) for each \((x_0, y_0) \not\in \mathcal{W}_1 \times \mathcal{W}_2\) such that \((w'_1, w'_2) \in \mathbf{P}_{\mathcal{W}_1 \times \mathcal{W}_2}^{(2)}(x_0, y_0)\). Let \((x_0, y_0)\) be such an element. Applying Theorem 2, we conclude that the least element of the set \( \mathbf{P}_{\mathcal{W}_1 \times \mathcal{W}_2}^{(2)}(x_0, y_0) \) exists. Let \((\gamma_0, \lambda_0) = \min \mathbf{P}_{\mathcal{W}_1 \times \mathcal{W}_2}^{(2)}(x_0, y_0)\). Then \((\gamma_0, \lambda_0) \in \text{bd}(\mathcal{W}_1 \times \mathcal{W}_2)\), \((\gamma_0, \lambda_0) \leq (w'_1, w'_2)\).

Hence \((\gamma_0, \lambda_0) = (w'_1, w'_2)\). Since \( \mathcal{W} \) is strictly downward at \((\gamma_0, \lambda_0) = (w'_1, w'_2)\), \((w_1, w_2) \geq (w'_1, w'_2)\) for all \((w_1, w_2) \in \mathbf{P}_{\mathcal{W}_1 \times \mathcal{W}_2}^{(2)}(x_0, y_0) \subseteq \mathcal{W}_1 \times \mathcal{W}_2\). In view of Theorem 6, we have \((w_1, w_2) = (w'_1, w'_2)\) for all \((w_1, w_2) \in \mathbf{P}_{\mathcal{W}_1 \times \mathcal{W}_2}^{(2)}(x_0, y_0)\). Hence \( \mathbf{P}_{\mathcal{W}_1 \times \mathcal{W}_2}^{(2)}(x_0, y_0) = \{ (w'_1, w'_2) \} \).

**References**


Lattice $g$-2-Normed spaces and 2-best approximation

Mahdi Iranmanesh
Shahrood university of thechnology
Department of pure mathematic
Shahrood, Iran
e-mail: m.iranmanesh2012@gmail.com

Ali Ganjbakhsh Sanatee
Shahrood university of thechnology
Department of pure mathematic
Shahrood, Iran
e-mail: alisanatee62@gmail.com
Differential subordination results obtained by using a new operator

Eszter Szatmari, Ágnes Orsolya Páll-Szabó

Abstract

In this paper, is defined the operator $D_{\lambda,\nu,n}^{\alpha,\beta}$: $A \to A$, given by

$$D_{\lambda,\nu,n}^{\alpha,\beta}f(z) = (1-\alpha-\beta)R_{\nu}D_{n}f(z) + \alpha R_{\nu}\Omega_{\lambda}z^{\alpha}f(z) + \beta D_{n}\Omega_{\lambda}z^{\alpha}f(z),$$

for $z \in U$, where $R_{\nu}$ is the Ruscheweyh derivative, $D_{n}$ is the Sălăgean operator, $\Omega_{\lambda}z^{\alpha}$ is a fractional differintegral operator introduced by S. Owa and H. M. Srivastava, $A = \{f \in H(U) : f(z) = z + a_{2}z^{2} + a_{3}z^{3} + ..., z \in U \}$, $\alpha, \beta \geq 0$, $\nu > -1$, $n \in \mathbb{N}_0 = \{0,1,2,3,...\}$, $-\infty < \lambda < 2$. A certain subclass of analytic functions in the open unit disk, $R_{\lambda,\nu,n}^{\alpha,\beta}(\delta)$, where $0 \leq \delta \leq 1$, is introduced using the new operator. Are obtained some properties of the class $R_{\lambda,\nu,n}^{\alpha,\beta}(\delta)$ and some differential subordinations using the operator $D_{\lambda,\nu,n}^{\alpha,\beta}$.

2010 Mathematics Subject Classification: 30C45.

Key words and phrases: differential subordination, convex function, Sălăgean operator, Ruscheweyh derivative, fractional differintegral operator.

1 Introduction

Let $H(U)$ denote the class of functions which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

For $a \in \mathbb{C}$ and $k \in \mathbb{N} = \{1,2,...\}$, let

$$H[a,k] = \{f \in H(U) : f(z) = a + a_{k}z^{k} + a_{k+1}z^{k+1} + ...\},$$

and

$$A = \{f \in H(U) : f(z) = z + a_{2}z^{2} + a_{3}z^{3} + ..., z \in U\}.$$

We shall use the following operators to define our new operator.

---

1 Received 20 August, 2017
Accepted for publication (in revised form) 3 October, 2017
In [7] is defined the Ruscheweyh operator $R^\nu : A \to A$, by

$$R^\nu f(z) = \frac{z}{(1 - z)^{1 + \nu}} * f(z), z \in U, \nu \geq -1.$$  

If $\nu \in \mathbb{N}_0$, then

$$R^\nu f(z) = \frac{z(z^\nu - f(z))^\nu}{\nu!}, z \in U,$$

whose series expression for $f \in A$ is given by

$$R^\nu f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(k + 1 + \nu)}{\Gamma(\nu + 1)\Gamma(k + 1)} a_{k+1} z^{k+1}, \nu > -1, z \in U.$$  

In [8] is defined the Sălăgean operator $D^n$ of order $n$, $n \in \mathbb{N}_0$, by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = D f(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)), n \in \mathbb{N}, f \in A.$$  

The series expression of the operator $D^n$ for the function $f \in A$ is given by

$$D^n f(z) = z + \sum_{k=1}^{\infty} (k + 1)^n a_{k+1} z^{k+1}, n \in \mathbb{N}_0.$$  

In [5] are defined the following operators:

the fractional integral operator $D^{-\mu}_z$ of order $\mu$, by

$$D^{-\mu}_z f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt, z \in U, f \in A, \mu > 0,$$

where the multiplicity of $(z - t)^{\mu - 1}$ is removed by requiring $\log(z - t)$ to be real when $z - t > 0$, and

the fractional derivative operator $D^{\lambda}_z$ of order $\lambda$, by

$$D^{\lambda}_z f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, & 0 \leq \lambda < 1 \\ \frac{d^n}{dz^n} D^{\lambda-n}_z f(z), & n \leq \lambda < n + 1 \end{cases}, n \in \mathbb{N}_0, f \in A, \lambda \geq 0,$$

where the multiplicity of $(z - t)^{-\lambda}$ is likewise understood.

In [6] is defined the fractional differintegral operator $\Omega^{\lambda}_z : A \to A$, by

$$\Omega^{\lambda}_z f(z) = \Gamma(2 - \lambda) z^\lambda D^{\lambda}_z f(z), z \in U, -\infty < \lambda < 2,$$

where $D^{\lambda}_z f(z)$ is the fractional integral of order $\lambda, -\infty < \lambda < 0$, and a fractional derivative of order $\lambda, 0 \leq \lambda < 2$.  

The series expression of the operator $\Omega^\lambda_z$ for the function $f \in A$ is given by

$$\Omega^\lambda_z f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(2 - \lambda) \Gamma(k + 2)}{\Gamma(k + 2 - \lambda)} a_{k+1} z^{k+1}, -\infty < \lambda < 2, z \in U.$$ 

In [9] is defined the fractional operator $D_{\alpha}^{\nu,n} : A \to A$ for $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$ as a composition of fractional differintegral operator, the Sălăgean operator and the Ruscheweyh operator:

$$D_{\alpha}^{\nu,n} f(z) = R_{\nu} \Omega^\lambda_z f(z).$$

The series expression of $D_{\alpha}^{\nu,n} f(z)$ for $f \in A$ is given by

$$D_{\alpha}^{\nu,n} f(z) = z + \sum_{k=1}^{\infty} \frac{(\nu + 1) k}{(2 - \lambda) k} (k + 1)^n a_{k+1} z^{k+1},$$

$-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, z \in U$, where the symbol $(\gamma)k$ denotes the usual Pochhammer symbol, for $\gamma \in \mathbb{C}$, defined by

$$(\gamma)k = \begin{cases} 1, k = 0 \\ (\gamma)(\gamma + 1)...(\gamma + k - 1), k \in \mathbb{N} = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)}; \gamma \in \mathbb{C} \setminus \mathbb{Z}^- \end{cases}.$$ 

Forwards, is defined the notion of differential subordination.

**Definition 1.** [3, p. 4] Let $f,F \in H(U)$. The function $f$ is said to be subordinate to $F$, written $f \prec F$, or $f(z) \prec F(z)$, if there exists a function $w \in H(U)$, with $w(0) = 0$ and $|w(z)| < 1, z \in U$, such that $f(z) = F[w(z)], z \in U$.

**Definition 2.** [3, p. 16] Let $\Psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination

$$(1) \quad \Psi(p(z), zp'(z), z^2 p''(z); z) < h(z),$$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all $p$ satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1) is said to be the best dominant of (1). (Note that the best dominant is unique up to a rotation of $U$).

We shall use the following lemmas to prove our results.

**Lemma 1.** [1] If $p(z)$ is analytic in $U, p(0) = 1$ and $\Re(p(z)) > \frac{1}{2}, z \in U$, then for any function $F$ analytic in $U$, the function $p \ast F$ takes its values in the convex hull of $F(U)$. 
Lemma 2. [4] Let \( q \) be a convex function in \( U \) and let
\[
h(z) = q(z) + n\alpha z^q'(z),
\]
where \( \alpha > 0 \) and \( n \) is a positive integer. If
\[
p(z) = q(0) + p_n z^n + \ldots \in \mathcal{H}[q(0), n]
\]
and
\[
p(z) + \alpha z p'(z) \prec h(z)
\]
then
\[
p(z) \prec q(z),
\]
and this result is sharp.

Lemma 3. [3, Theorem 3.1b, p.71] Let \( h \) be convex in \( U \), with \( h(0) = a \), \( \gamma \neq 0 \) and \( \Re \gamma \geq 0 \). If \( p \in \mathcal{H}[a, n] \) and
\[
p(z) + \gamma z p'(z) \prec h(z)
\]
then
\[
p(z) < q(z) < h(z),
\]
where
\[
q(z) = \frac{\gamma}{n z^\frac{\lambda}{\pi}} \int_0^z h(t)t^{\frac{\lambda}{\pi} - 1} dt.
\]
The function \( q \) is convex and is the best \((a, n)\)-dominant.

2 Main results

Definition 3. Let \(-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, \alpha, \beta \geq 0\). Denote by \( \mathcal{D}_{\gamma,n}^{\lambda,\nu} \) the operator given by
\[
\mathcal{D}_{\gamma,n}^{\lambda,\nu} : \mathcal{A} \rightarrow \mathcal{A},
\]
\[
\mathcal{D}_{\gamma,n}^{\lambda,\nu} f(z) = (1 - \alpha - \beta) \mathcal{R}^{\nu} \mathcal{D}^n f(z) + \alpha \mathcal{R}^{\nu} \Omega_{\lambda}^z f(z) + \beta \mathcal{D}^n \Omega_{\lambda}^z f(z),
\]
for \( z \in U \).

Remark 1. \( \mathcal{R}^{\nu} \mathcal{D}^n f(z) \) is the composition of the Sălăgean operator and the Ruscheweyh derivative, \( \mathcal{R}^{\nu} \Omega_{\lambda}^z f(z) \) is the composition of fractional differintegral operator and the Ruscheweyh derivative, and \( \mathcal{D}^n \Omega_{\lambda}^z f(z) \) is the composition of fractional differintegral operator and the Sălăgean operator.

Remark 2. If \( f \in \mathcal{A}, f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1} \), then
\[
\mathcal{D}_{\gamma,n}^{\lambda,\nu} f(z) = z + \sum_{k=1}^{\infty} \left( (1 - \alpha - \beta) \frac{(\nu + 1)_k}{(2)_k} (k+1)^{n+1} + \alpha \frac{(\nu + 1)_k}{(2 - \lambda)_k} (k+1) + \right.
\]

for $z \in U$.

**Remark 3.** Differential subordination results obtained by using a new operator $D_{\alpha,\beta}^{0,\nu,n} f(z) = (1 - \alpha - \beta)D_0^nf(z) + \alpha D_0^{\nu}f(z) + \beta D_0^{n}f(z)$, for $z \in U$.

**Remark 4.** For $\alpha = 0$ and $\beta = 0$, we obtain $D_{0,0}^{0,\nu,n} f(z) = R^\nu D^n f(z)$, where $z \in U$.

For $\alpha = 1$ and $\beta = 0$, we obtain $D_{1,0}^{0,\nu,n} f(z) = R^\nu \Omega_1 f(z)$, where $z \in U$.

For $\alpha = 0$ and $\beta = 1$, we obtain $D_{0,1}^{0,\nu,n} f(z) = D^n \Omega_1 f(z)$, where $z \in U$.

For $\beta = 0$ and $\nu = 0$, we obtain $D_{\alpha,0}^{0,\nu,n} f(z) = (1 - \alpha)D^n f(z) + \alpha \Omega_1 f(z)$, where $z \in U$.

For $\alpha = 0$ and $n = 0$, we obtain $D_{0,0}^{0,\nu,0} f(z) = (1 - \beta)R^\nu f(z) + \beta \Omega_1 f(z)$, where $z \in U$.

For $\alpha + \beta = 1$ and $\lambda = 0$, we obtain $D_{1-\beta,0}^{0,\nu,0} f(z) = (1 - \beta)R^\nu f(z) + \beta D^n f(z)$, where $z \in U$.

For $\alpha + \beta = 1$, $\lambda = 0$ and $\nu = n$, we obtain $D_{1-\beta,\lambda}^{0,\nu,0} f(z) = (1 - \beta)R^\nu f(z) + \beta D^n f(z)$, where $z \in U$.

For $\beta = \lambda = n = 0$, we obtain $D_{0,\nu}^{0,\nu,0} f(z) = R^\nu f(z)$, and for $\beta = \lambda = n = 0$, we obtain $D_{0,0}^{0,0} f(z) = D^n f(z)$, and for $\alpha = \lambda = \nu = 0$, we obtain $D_{0,\nu,n}^{0,0} f(z) = D^n f(z)$, where $z \in U$.

For $\alpha = 0$ and $\nu = 1$, we obtain $D_{0,\nu}^{0,0,0} f(z) = D^{n+1} f(z)$, where $z \in U$.

For $\alpha = 1$ and $\beta = 0$, we obtain $D_{0,1}^{\nu,0,0} f(z) = \Omega_1 f(z)$ and for $\alpha = n = 0$ and $\beta = 1$, we obtain $D_{1,1}^{\nu,0,0} f(z) = \Omega_1 f(z)$, where $z \in U$.

For $\lambda = \nu = 0$, we obtain $D_{\alpha,\beta}^{0,0,n} f(z) = (1 - \alpha)D^n f(z) + \alpha f(z)$, where $z \in U$.

For $\lambda = n = 0$, we obtain $D_{\alpha,\beta}^{0,\nu,0} f(z) = (1 - \beta)R^\nu f(z) + \beta f(z)$, where $z \in U$.

For $\nu = n = 0$, we obtain $D_{\alpha,\beta}^{0,\nu,0} f(z) = (1 - \alpha - \beta)f(z) + (\alpha + \beta)\Omega_1 f(z)$, where $z \in U$.

For $\lambda = 0$ and $\nu = 1$, we obtain $D_{\alpha,\beta}^{0,1,n} f(z) = (1 - \alpha - \beta)D^{n+1} f(z) + \alpha D^1 f(z) + \beta D^n f(z)$, where $z \in U$.

For $\lambda = 1$ and $\nu = 0$, we obtain $D_{\alpha,\beta}^{1,0,n} f(z) = (1 - \alpha - \beta)D^n f(z) + \alpha D^1 f(z) + \beta D^{n+1} f(z)$, where $z \in U$.

For $\lambda = \nu = 1$, we obtain $D_{\alpha,\beta}^{1,1,n} f(z) = (1 - \alpha)D^{n+1} f(z) + \alpha D^2 f(z)$, where $z \in U$.

For $\lambda = \nu = n = 0$, we obtain $D_{\alpha,\beta}^{0,0,0} f(z) = f(z)$, for $\alpha = \beta = \nu = n = 0$, we obtain $D_{0,0}^{0,0,0} f(z) = f(z)$, for $\alpha = 1$ and $\lambda = \nu = 0$, we obtain $D_{1,\beta}^{0,0,n} f(z) = f(z)$, and for $\beta = 1$ and $\lambda = n = 0$, we obtain $D_{\alpha,\nu,n}^{0,0} f(z) = f(z)$, for $z \in U$.

**Definition 4.** Let $f \in \mathcal{A}$. We say that the function $f$ is in the class $\mathcal{H}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$, where $0 \leq \delta \leq 1$, $\alpha, \beta \geq 0$, $-\infty < \lambda < 2$, $\nu > -1$, $n \in \mathbb{N}_0$, if $f$ satisfies the condition

$$
\Re(D_{\alpha,\beta}^{\lambda,\nu,n} f(z))' > \delta, z \in U.
$$
Theorem 1. Let \( f \in \mathcal{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta) \) and \( g \in K \), where \( K \) denote the class of convex functions. Then \( f * g \in \mathcal{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta) \).

Proof. Since \( g \) is a convex function, \( \Re \frac{g(z)}{z} > \frac{1}{2} \). Using convolution properties, we have

\[
\Re \left( \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}(f * g)(z) \right)' = \Re \left( \left( \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) \right)' * \frac{g(z)}{z} \right).
\]

We have \( \Re \left( \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) \right)' > \delta \) and applying Lemma 1, for \( p(z) = \frac{g(z)}{z} \), we get

\[
\Re \left( \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}(f * g)(z) \right)' > \delta.
\]

\[ \square \]

Theorem 2. The set \( \mathcal{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta) \) is convex.

Proof. Let the functions

\[
f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}, \\
g(z) = z + \sum_{k=1}^{\infty} b_{k+1} z^{k+1}
\]

be in the class \( \mathcal{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta) \).

Let \( h(z) = \mu_1 f(z) + \mu_2 g(z), \mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1 \). Since

\[
h(z) = z + \sum_{k=1}^{\infty} \left( \mu_1 a_{k+1} + \mu_2 b_{k+1} \right) z^{k+1},
\]

then from (2) we have

\[
\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n} h(z) = z + \sum_{k=1}^{\infty} \left( 1 - \alpha - \beta \right) \frac{(\nu + 1)_k}{(2)_k} (k+1)^{n+1} + \alpha \frac{(\nu + 1)_k}{(2 - \lambda)_k} (k+1) +
\]

\[
+ \beta \frac{(1)_k}{(2 - \lambda)_k} (k+1)^{n+1} \left( \mu_1 a_{k+1} + \mu_2 b_{k+1} \right) z^{k+1}.
\]

Differentiating we obtain

\[
\left( \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n} h(z) \right)' = 1 + \sum_{k=1}^{\infty} \left( 1 - \alpha - \beta \right) \frac{(\nu + 1)_k}{(2)_k} (k+1)^{n+2} + \alpha \frac{(\nu + 1)_k}{(2 - \lambda)_k} (k+1)^2 +
\]

\[
+ \beta \frac{(1)_k}{(2 - \lambda)_k} (k+1)^{n+2} \left( \mu_1 a_{k+1} + \mu_2 b_{k+1} \right) z^{k}.
\]

Then

\[
\Re \left( \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n} h(z) \right)' = \Re \left( 1 + \mu_1 \sum_{k=1}^{\infty} \left( 1 - \alpha - \beta \right) \frac{(\nu + 1)_k}{(2)_k} (k+1)^{n+2} +
\]

\[
+ \beta \frac{(1)_k}{(2 - \lambda)_k} (k+1)^{n+2} \left( \mu_1 a_{k+1} + \mu_2 b_{k+1} \right) z^{k}.
\]
Differential subordination results obtained by using a new operator

\[
\alpha \frac{(\nu + 1)_k}{(2 - \lambda)_k} (k + 1)^2 + \beta \frac{(1)_k}{(2 - \lambda)_k} (k + 1)^{n+2} a_{k+1} z^k + \Re \left( 1 + \mu_2 \sum_{k=1}^{\infty} \left( 1 - \alpha - \beta \right) \frac{(\nu + 1)_k}{(2)_k} (k + 1)^{n+2} a_{k+1} z^k \right) + \Re \left( 1 + \mu_2 \sum_{k=1}^{\infty} \left( 1 - \alpha - \beta \right) \frac{(\nu + 1)_k}{(2 - \lambda)_k} (k + 1)^{n+2} b_{k+1} z^k \right) - 1. 
\]

(4)

Since \( f, g \in \mathcal{R}_{\alpha, \beta}^{\lambda, \nu, n}(\delta) \), using (3) this implies that

\[
\Re \left( 1 + \mu_1 \sum_{k=1}^{\infty} \left( 1 - \alpha - \beta \right) \frac{(\nu + 1)_k}{(2)_k} (k + 1)^{n+2} + \frac{(1)_k}{(2 - \lambda)_k} (k + 1)^{n+2} a_{k+1} z^k \right) > 1 + \mu_1 (\delta - 1),
\]

\[
\Re \left( 1 + \mu_2 \sum_{k=1}^{\infty} \left( 1 - \alpha - \beta \right) \frac{(\nu + 1)_k}{(2)_k} (k + 1)^{n+2} a_{k+1} z^k \right) > 1 + \mu_2 (\delta - 1)
\]

Using (4), (5) and \( \mu_1 + \mu_2 = 1 \), we obtain \( \Re(\mathcal{D}_{\alpha, \beta}^{\lambda, \nu, n} h(z))' > 1 + \mu_1 (\delta - 1) + \mu_2 (\delta - 1) = \delta \). This result implies that \( \mathcal{R}_{\alpha, \beta}^{\lambda, \nu, n}(\delta) \) is a convex set.

**Theorem 3.** Let \( g \) be a convex function, \( g(0) = 1 \) and let \( h \) be a function such that

\[
h(z) = g(z) + zg'(z), \ z \in U.
\]

If \( f \in A \) verifies the differential subordination

\[
(\mathcal{D}_{\alpha, \beta}^{\lambda, \nu, n} f(z))' \prec h(z), \ z \in U
\]

then

\[
\mathcal{D}_{\alpha, \beta}^{\lambda, \nu, n} f(z) \prec g(z), \ z \in U.
\]

The result is sharp.

**Proof.** Let

\[
p(z) = \frac{\mathcal{D}_{\alpha, \beta}^{\lambda, \nu, n} f(z)}{z}, \ z \in U.
\]

Differentiating we obtain

\[
p'(z) = \frac{(\mathcal{D}_{\alpha, \beta}^{\lambda, \nu, n} f(z))'}{z} - \frac{p(z)}{z}.
\]
We get
\[ \left( \mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n} f(z) \right)' = p(z) + zp'(z). \]
The subordination (6) becomes
\[ p(z) + zp'(z) \prec g(z) + zg'(z). \]
Applying Lemma 2, we get
\[ p(z) \prec g(z) \]
or
\[ \frac{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n} f(z)}{z} \prec g(z). \]
This result is sharp.

**Theorem 4.** Let \( g \) be a convex function, \( g(0) = 1 \) and let \( h \) be a function such that
\[ h(z) = g(z) + zg'(z), \quad z \in U. \]
If \( f \in \mathcal{A} \) verifies the differential subordination
\[ (7) \quad \left( \frac{z\mathcal{D}_{a,\beta}^{\lambda,\nu, n+1, n} f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n} f(z)} \right)' \prec h(z), \quad z \in U, \]
then
\[ \frac{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n+1, n} f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n} f(z)} \prec g(z), \quad z \in U. \]
The result is sharp.

**Proof.** Let
\[ p(z) = \frac{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n+1, n} f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n} f(z)}. \]
We obtain
\[ \left( \frac{z\mathcal{D}_{a,\beta}^{\lambda,\nu, n+1, n} f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n} f(z)} \right)' = p(z) + zp'(z). \]
The subordination (7) becomes
\[ p(z) + zp'(z) \prec g(z) + zg'(z). \]
Applying Lemma 2, we get
\[ p(z) \prec g(z) \]
or
\[ \frac{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n+1, n} f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu, n} f(z)} \prec g(z), \quad z \in U. \]
This result is sharp. \( \square \)
Theorem 5. Let $g$ be a convex function, $g(0) = 1$ and let $h$ be a function such that

$$h(z) = g(z) + zg'(z), z \in U.$$ 

If $f \in A$ verifies the differential subordination

$$\left( \frac{zD_{\lambda,\nu,n+1}f(z)}{D_{\lambda,\nu,n}f(z)} \right)' < h(z), z \in U,$$

then

$$\frac{D_{\lambda,\nu,n+1}f(z)}{D_{\lambda,\nu,n}f(z)} < g(z), z \in U.$$ 

The result is sharp.

Proof. Let

$$p(z) = \frac{D_{\lambda,\nu,n+1}f(z)}{D_{\lambda,\nu,n}f(z)}.$$ 

We obtain

$$\left( \frac{zD_{\lambda,\nu,n+1}f(z)}{D_{\lambda,\nu,n}f(z)} \right)' = p(z) + zp'(z).$$

The subordination (8) becomes

$$p(z) + zp'(z) < g(z) + zg'(z).$$

Applying Lemma 2, we get

$$p(z) < g(z)$$

or

$$\frac{D_{\lambda,\nu,n+1}f(z)}{D_{\lambda,\nu,n}f(z)} < g(z), z \in U.$$ 

This result is sharp.

Theorem 6. Let $g$ be a convex function, $g(0) = 0$ and let $h$ be a function such that

$$h(z) = g(z) + zg'(z), z \in U.$$ 

If $f \in A$ verifies the differential subordination

$$D_{\lambda,\nu,n+1}f(z) + D_{\lambda,\nu,n}f(z) + \alpha(D_{1,0}D_{\lambda,\nu,n}f(z) - D_{1,0}^{\lambda,\nu,n}f(z)) < h(z), z \in U,$$

then

$$D_{\lambda,\nu,n}f(z) < g(z), z \in U.$$ 

The result is sharp.
Proof. Let
\[ p(z) = D_\alpha^\lambda \nu \beta^\mu f(z) = (1 - \alpha - \beta) R^{\nu} D^n f(z) + \alpha R^{\nu} \Omega^\lambda_\nu f(z) + \beta D^n \Omega^\lambda_\nu f(z). \]
After a short computation, we obtain
\[ p(z) + zp'(z) = D_\alpha^\lambda \nu \beta^\mu f(z) + D_\alpha^\lambda \nu \beta^\mu f(z) + \alpha \left( z(D_1^\lambda \nu \beta^\mu f(z))' - D_1^\lambda \nu \beta^\mu f(z) \right). \]
The differential subordination (9) becomes
\[ p(z) + zp'(z) \prec h(z) = g(z) + zg'(z). \]
Applying Lemma 2, we have
\[ p(z) \prec g(z), z \in U, \]
or
\[ D_\alpha^\lambda \nu \beta^\mu f(z) \prec g(z), z \in U. \]
This result is sharp. \( \square \)

**Theorem 7.** Let \( h(z) = \frac{1 + (2\delta - 1)z}{1 + z} \) be a convex function in \( U \), where \( 0 \leq \delta < 1 \).

If \( f \in A \) satisfies the differential subordination
\[ D_\alpha^\lambda \nu \beta^\mu f(z) + D_\alpha^\lambda \nu \beta^\mu f(z) + \alpha \left( D_0^\lambda \nu \beta^\mu f(z) - D_1^\lambda \nu \beta^\mu f(z) \right) \prec h(z), z \in U, \]
then
\[ D_\alpha^\lambda \nu \beta^\mu f(z) \prec g(z), z \in U, \]
where \( g \) is given by \( g(z) = 2\delta - 1 + 2(1 - \delta) \frac{\ln(1 + z)}{z} \), \( z \in U \).

The function \( g \) is convex and is the best dominant.

**Proof.** Following the same steps as in the proof of Theorem 6 and considering
\[ p(z) = D_\alpha^\lambda \nu \beta^\mu f(z), \]
the differential subordination (10) becomes
\[ p(z) + zp'(z) \prec h(z) = \frac{1 + (2\delta - 1)z}{1 + z}, z \in U. \]
By using Lemma 3 for \( \gamma = 1 \) and \( n = 1 \), we have
\[ p(z) \prec g(z), z \in U \]
or
\[ D_\alpha^\lambda \nu \beta^\mu f(z) \prec g(z) = \frac{1}{z} \int_0^z h(t) dt = \frac{1}{z} \int_0^z \frac{1 + (2\delta - 1)t}{1 + t} dt = 2\delta - 1 + 2(1 - \delta) \frac{1}{z} \ln(z + 1), z \in U. \]
\( \square \)
**Theorem 8.** Let $g$ be a convex function, $g(0) = 1$ and let $h$ be a function such that

$$h(z) = g(z) + zg'(z), \quad z \in U.$$ 

If $f \in A$ verifies the differential subordination

\[
\frac{1}{z} \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n+2} f(z) + \frac{1}{z} \alpha(D^2 \mathcal{D}_{1,0}^{\lambda,\nu,n} f(z) - \mathcal{D}_{1,0}^{\lambda,\nu,n} f(z)) < h(z), \quad z \in U,
\]

then

\[
(D_{\alpha,\beta}^{\lambda,\nu,n} f(z))' < g(z), \quad z \in U.
\]

The result is sharp.

**Proof.** Let

\[
p(z) = (D_{\alpha,\beta}^{\lambda,\nu,n} f(z))' = \frac{1}{1 - \alpha - \beta}(\mathcal{R}_n^\nu D^n f(z))' + \alpha(\mathcal{R}_n^\nu \Omega_z^\lambda f(z))' + \beta(\mathcal{D}^n \Omega_z^\lambda f(z))'.
\]

After a short computation, we obtain

\[
p(z) + zp'(z) = \frac{1}{z} \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n+2} f(z) + \frac{1}{z} \alpha(D^2 \mathcal{D}_{1,0}^{\lambda,\nu,n} f(z) - \mathcal{D}_{1,0}^{\lambda,\nu,n} f(z)).
\]

The differential subordination (11) becomes

\[
p(z) + zp'(z) < h(z) = g(z) + zg'(z).
\]

By using Lemma 2, we have

\[
p(z) < g(z), \quad z \in U,
\]

or

\[
(D_{\alpha,\beta}^{\lambda,\nu,n} f(z))' < g(z), \quad z \in U.
\]

This result is sharp. \hspace{1cm} \Box

**Theorem 9.** Let $h(z) = \frac{1 + (2\delta - 1)z}{1 + z}$ be a convex function in $U$, where $0 \leq \delta < 1$.

If $f \in A$ satisfies the differential subordination

\[
\frac{1}{z} \mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n+2} f(z) + \frac{1}{z} \alpha(D^2 \mathcal{D}_{1,0}^{\lambda,\nu,n} f(z) - \mathcal{D}_{1,0}^{\lambda,\nu,n} f(z)) < h(z),
\]

then

\[
(D_{\alpha,\beta}^{\lambda,\nu,n} f(z))' < g(z), \quad z \in U,
\]

where $g$ is given by $g(z) = 2\delta - 1 + 2(1 - \delta)\frac{\ln(1 + z)}{z}, \quad z \in U$.

The function $g$ is convex and is the best dominant.
Proof. Following the same steps as in the proof of Theorem 8 and considering
\[ p(z) = \left( D_{\lambda,\nu,\alpha,\beta} f(z) \right)', \]
the differential subordination (12) becomes
\[ p(z) + zp'(z) \prec h(z) = \frac{1 + (2\delta - 1)z}{1 + z}, z \in U. \]
By using Lemma 3 for \( \gamma = 1 \) and \( n = 1 \), we have
\[ p(z) \prec g(z), z \in U, \]
or
\[ D_{\lambda,\nu,\alpha,\beta} f(z) \prec g(z) = \frac{1}{z} \int_0^z h(t)dt = \frac{1}{z} \int_0^z \frac{1 + (2\delta - 1)t}{1 + t} dt = 2\delta - 1 + 2(1 - \delta)\frac{1}{z} \ln(z + 1), z \in U. \]
\[ \square \]

References


Differential subordination results obtained by using a new operator


**Eszter Szatmari**  
Babeş-Bolyai University  
Faculty of Mathematics and Computer Science  
Department of Mathematics  
1, Kogălniceanu Street, 400084 Cluj-Napoca, Romania  
e-mail: szatmari.eszter@math.ubbcluj.ro

**Ágnes Orsolya Páll-Szabó**  
Babeş-Bolyai University  
Faculty of Mathematics and Computer Science  
Department of Mathematics  
1, Kogălniceanu Street, 400084 Cluj-Napoca, Romania  
e-mail: pallszaboagnes@math.ubbcluj.ro
About the sequence of general term $\Omega_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}$

Andrei Vernescu

Dedicated to the memory of the late Professor Alexandru Lupaş Ph. D., ten years after he left us.

Abstract

We give a systematic presentation of the sequence in the title, characterizing it and showing its properties and its role in several areas: the nature of some series, with a closed form for its $n$-th partial sum, the Wallis “domain” (his integrals, formula and inequality), some limits obtained in an elementary way using this sequence, some definite integrals, its deep link with the gamma function, its role in certain Taylor-Maclaurin expansions. We also cite some converse assertions of Professor Alexandru Lupaş which characterize the sequence and remember in a certain manner, but in the discrete domain, the famous theorem of Bohr-Mollerup.

2010 Mathematics Subject Classification: 26A06, 26A09, 26D15.

Key words and phrases: Wallis’s integrals, formula and inequality; convergent/divergent sequences and series, criteria of convergence, closed form of a sum, Stirling’s formula, infinite products, Gamma Function, log convex functions, the middle term, asymptotic expansions, Taylor-Maclaurin expansions, the theorem of Bohr-Mollerup.

1 Introduction

The sequence of real numbers $(\Omega_n)_n$ of general term

$$(1.1) \quad \Omega_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} \quad (n = 1, 2, 3, \ldots)$$

is very useful in several areas of mathematics (combinatorics, numbers theory, special functions, probabilities and others). More elementary, this sequence is especially related to the following three problems:

(a) The expressions of the integrals which conducts to the formula of Wallis;

1) Received 5 July, 2017
Accepted for publication (in revised form) 20 August, 2017
(b) The evaluation of the order of magnitude of the natural number

\[ \max \left\{ \binom{2n}{k} \mid 0 \leq k \leq 2n \right\} = \binom{2n}{n}, \]

also called the middle term of the development of \((1 + 1)^{2n}\);

(c) The calculation of the successive derivatives of the binomial function of exponent \(\alpha = -1/2\), namely \(f : (-1, \infty) \to (0, \infty), f(x) = (1 + x)^{-1/2}\), in connexion with its expansion in Maclaurin series, and also with the expansion of an antiderivative of the function \(x \mapsto f(-x^2)\), namely \(x \mapsto \arcsin x\) (both expansions being considered on the open interval \((-1, 1))\).

Because of this, the elementary study of the sequence \((\Omega_n)_n\) has gained a certain interest and there exists a fairly extensive literature on this sequence (although usually it is mentioned only en passant, as an auxiliary, in texts that pursue other objectives). In this expository survey we intend to present systematically the properties of the sequence \((\Omega_n)_n\) and its involvement in these issues, and in some others, such that the text will have inevitably a somewhat uneven tint.

This paper is written as a pious tribute to the late Professor Dr. Alexandru Lupșă and in remembrance of his suggestions over 20 years ago, when he encouraged me to develop an expository survey about \((\Omega_n)_n\), without disregarding the inclusion of some more elementary parts.

I introduced the notation (1.1) into [40] and I used it in all the papers [41] - [46]. In [38], L. Tóth denoted \(A_n = (2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n)/(1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1))\), i.e. exactly the inverse of the expression which I denote by \(\Omega_n\). Some authors denote the expression \(\Omega_n\) by \(P_n\) and call it ”the product of Wallis”. Using the so-called double factorial, we also can write \(\Omega_n = (2n - 1)!!/(2n)!!\).

## 2 A first double inequality and the limit of the sequence \((\Omega_n)_n\)

We begin by determining the limit of the sequence \((\Omega_n)_n\). The ratio test for limits of sequences fails in this case, because

\[ \frac{\Omega_{n+1}}{\Omega_n} = \frac{2n + 1}{2n + 2} \xrightarrow{n \to \infty} 1. \]

However we have the inequality

\[ 0 < \Omega_n < \frac{1}{\sqrt{2n + 1}}. \]

(N. Dinculeanu and E. Radu [13], problem 6, h), page 106) therefore, because of \(1/\sqrt{2n + 1}\) tends to 0, we obtain, based on the majorization criterion, that

\[ \lim_{n \to \infty} \Omega_n = 0. \]

We note that the majorization of the right part can be proved outside of the induction, directly, using the inequality

\[ \frac{2k - 1}{2k} < \frac{2k}{2k + 1}. \]
by multiplying the particular inequalities obtained for \( k = 1, 2, 3, \ldots, n \) and doing a little calculation. Note that in the inequality obtained by these multiplications appears the square of \( \Omega_n \), and so we find the explanation of the presence of the square root in (2.2). The mathematical induction does not provide this explanation.

The majorization (2.2) can be completed to a double estimate:

\[
\frac{1}{2\sqrt{n}} \leq \Omega_n < \frac{1}{\sqrt{2n + 1}}
\]

(with equality only if \( n = 1 \)). The left part can be also proved by induction, or multiplying the inequalities

\[
\frac{2k - 2}{2k - 1} < \frac{2k - 1}{2k}
\]

written for \( k = 2, 3, 4, \ldots, n \) and performing some elementary calculations.

### 3 An application to the series

The double inequality (2.3) permits to establish the nature of the series

\[
\sum_{n=1}^{\infty} (\Omega_n)^{\alpha} \quad (\alpha > 0)
\]

with a discussion respecting the values of the parameter using the comparison criterion, in relation with the generalized harmonic series \( \sum_{n=1}^{\infty} 1/n^2 \), \( s > 0 \), without the need to apply Raabe-Duhamel’s criterion (as in T. Apostol [5], Ex. 18, p. 403; also see [3]). So, we obtain that:

(a) For \( \alpha > 2 \) we have, by the right part of (2.3),

\[
(\Omega_n)^{\alpha} < \left( \frac{1}{\sqrt{2n + 1}} \right)^{\alpha} < \left( \frac{1}{\sqrt{2n}} \right)^{\alpha} = \frac{1}{2^{\alpha/2}} \cdot \frac{1}{n^{\alpha/2}},
\]

then, by comparison with the convergent series \( \sum_{n=1}^{\infty} 1/n^{\alpha/2} \), with \( \alpha/2 > 1 \), it results, in this case, that the series (3.4) is also convergent.

(b) For \( \alpha \leq 2 \), we have, by the left part of (2.3),

\[
(\Omega_n)^{\alpha} \geq \left( \frac{1}{2\sqrt{n}} \right)^{\alpha} = \frac{1}{2^\alpha} \cdot \frac{1}{n^{\alpha/2}},
\]

then, by comparison with the divergent series \( \sum_{n=1}^{\infty} 1/n^{\alpha/2} \), with \( \alpha/2 \leq 1 \), it results now that the series (3.4) is divergent.

We note, as a curiosity, that, in the particular case \( \alpha = 1 \), namely in the case of the series \( \sum_{n=1}^{\infty} \Omega_n \) we can find a ”closed form” for the partial sum of order \( n \),

\[
\sum_{k=1}^{n} \Omega_k = (2n + 1)\Omega_n - 1,
\]
which is obtained adding from 2 to \( n \) the immediately verifiable identities

\[
\Omega_k = (2k + 1)\Omega_k - (2k - 1)\Omega_{k-1}
\]

(and replacing, after making all reductions, \( 3/2 = 1 + \Omega_1 \)). Using this "closed form" and the inequality (2.2), we obtain

\[
\sum_{k=1}^{n} \Omega_k = (2n + 1)\Omega_n - 1 \geq (2n + 1) \frac{1}{2\sqrt{n}} - 1 \underset{(n \to \infty)}{\longrightarrow} \infty,
\]

so it is again found that \( \sum_{n=1}^{\infty} \Omega_n = \infty \).

The inequality (2.2) also permits us to find immediately that the series

\[
\sum_{n=1}^{\infty} \frac{(n-1)!n!4^n}{(2n)!\sqrt{n}}
\]

of [35] (which can be written with our notation (1.1), as \( \sum_{n=1}^{\infty} 1/(n\sqrt{n}\Omega_n) \)) is divergent. Indeed, based on inequality (2.2), we have

\[
\frac{1}{n\sqrt{n}\Omega_n} > \frac{\sqrt{2n+1}}{n\sqrt{n}} > \frac{\sqrt{2n}}{n\sqrt{n}} = \frac{\sqrt{2}}{n}
\]

and now the comparison with the harmonic series proves the statement.

We also mention, following our paper [46], that, for the convergent series \( \sum_{n=1}^{\infty} (\Omega_n/(n + p)) \), where \( p \) is a positive parameter, we could determine the sum, obtaining that

\[
(3.6) \quad \sum_{n=1}^{\infty} \frac{\Omega_n}{n + p} = 2^{2p-1} \frac{\Gamma^2(p)}{\Gamma(2p)} - \frac{1}{p},
\]

where

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t}dt
\]

is the gamma function, also called the integral of Euler of second species.

Earlier, Gh. Costovici proposed to study, in [11], several series with positive or alternate terms, involving the sequence \((\Omega_n)_n\), requiring to establish the equalities

\[
(3.7) \quad \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{2}} - 1; \quad \sum_{n=1}^{\infty} \frac{(2n - 1)!!}{(2n + 2)!!} = \frac{1}{2}; \quad \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)!!}{(2n + 2)!!} = \frac{1}{\sqrt{2}} - \frac{3}{2}.
\]

The last two are included, as special cases, in our paper [46]; a quick way to obtain the first equality is mentioned in section 13.
Although inequality (2.2) was sufficient to determine the limit of the sequence \((\Omega_n)_n\), and the double inequality (2.3) was sufficient to establish the nature of the series (3.4), respecting the values of the parameter \(\alpha > 0\), inequality (2.3) can be considered in a some sense rough, unsatisfactory to deeper characterizing the order of magnitude of the expression \(\Omega_n\), that would require if it exists, a double inequality of the form

\[
\frac{1}{\sqrt{\alpha n + \beta}} < \Omega_n < \frac{1}{\sqrt{\alpha n + \gamma}},
\]

namely having the same coefficient \(\alpha\) of \(n\) in the two extreme parts. Such inequality exists and is well-known, Wallis’s inequality, and the constant \(\alpha\) is the number \(\pi\). There are also refinements for it. We will review such inequalities a little later. But getting all these results requires the use of Wallis’ formula.

4 The integrals and the formula of Wallis; the relationship with the sequence \((\Omega_n)_n\). The inequality of Wallis.

The integrals of Wallis are

\[
I_n = \int_0^{\pi/2} \sin^n x \, dx \quad n = 0, 1, 2, \ldots
\]

By the change of variable \(x = \pi/2 - t\), we also obtain the dual equality

\[
I_n = \int_0^{\pi/2} \cos^n t \, dt
\]

We have \(I_0 = \pi/2\), \(I_1 = 1\), and after this, by a well known integration by parts, followed by some calculations, we find the recurrence relation

\[
I_k = \frac{k-1}{k} I_{k-2} \quad k \geq 2.
\]

Writing it successively for \(k = 2, 4, 5, \ldots, 2n\) and multiplying the relationships obtained, we find

\[
I_{2n} = \Omega_n \cdot \frac{\pi}{2}.
\]

Writing it for \(k = 3, 5, 7, \ldots, 2n + 1\) and multiplying the relationships obtained, we find

\[
I_{2n+1} = \frac{1}{\Omega_n} \cdot \frac{1}{2n + 1}.
\]
Let us recall in passing that the usual deduction of the formula of Wallis, as it appears today in the majority of treatises (e.g. [21] pp. 466-467, or [16], pag. 138) is based on the strict positivity of the sequence \((I_n)_n\) and that it is (strictly) decreasing. So, \(I_{2n+2} < I_{2n+1} < I_{2n}\) and, because of \(I_p > 0\) for any \(p \in \mathbb{N}\), we can divide the last inequality by \(I_{2n}\) obtaining
\[
\frac{I_{2n+2}}{I_{2n}} < \frac{I_{2n+1}}{I_{2n}} < 1,
\]
and now, applying the recurrence relation (4.10) for \(k = 2n + 2\) and replacing in the precedent double inequality, we find
\[
\frac{2n + 1}{2n + 2} < \frac{I_{2n+1}}{I_{2n}} < 1.
\]
This last relation gives us
\[
\lim_{n \to \infty} \frac{I_{2n+1}}{I_{2n}} = 1.
\]
Replacing the expressions of \(I_{2n+1}\) and \(I_{2n}\) given by (4.12) and (4.11) and making a small calculation, we obtain the result
\[
\lim_{n \to \infty} \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots} \cdot \frac{2n \cdot 2n}{(2n - 1)(2n + 1)} = \frac{\pi}{2},
\]
which is called the formula of Wallis.

It was established by John Wallis in another way in the work "Arithmetica infini-torum" [48], in 1655, before the emergence of the (differential and integral) calculus, initiated by Newton and Leibniz. From a historical point of view, the formula is very important because it constituted the first analytical expression of the number \(\pi\) and allowed the passage of the study of this number from purely geometric methods, based on the approximation of the length of the circle by the perimeters of regular inscribed or circumscribed polygons, inevitably limited methods, to analytical methods. Subsequently, many other analytical expressions of the number \(\pi\) were set, such as series sums, infinite products, continuous fractions, values of defined integrals, etc. The first expression as a serial sum was that of J. Gregory and W. G. Leibniz, who obtained it independently of each other around 1670 ([15], p. 47 and [49], p.140). An interesting account of how Wallis worked was given in the paper [12] of G. A. Dickinson; related to Wallis’ work and the scientific atmosphere of his time, see [2]. An entire monograph [15] is intended for the fundamental number . Also see [7].

These integrals \(I_n\) are called Wallis’ integrals because they are closely related to the formula that bears its name.

Using the notation
\[
W_n = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots} \cdot \frac{2n \cdot 2n}{(2n - 1)(2n + 1)}
\]
for the sequence of the Wallis’s formula, we have the following expression involving \( \Omega_n \)

\[
W_n = \frac{1}{\Omega_n^2} \cdot \frac{1}{2n+1},
\]

so any double inequality related to one of the expressions can be transposed to the other; this reciprocity also takes place for the asymptotic developments regarding these two expressions, developments that we will relate a little further.

We mention that there is also a (laborious) elementary setting of Wallis’ formula, in which the integrals are not used, but only certain trigonometric identities and inequalities (see A. M. Iaglom and I. M. Iaglom, [22]).

We return to presenting the first inequality that satisfies the requirement at the end of Section 3. This is Wallis’ inequality

\[
\frac{1}{\sqrt{\pi (n + 1/2)}} < \Omega_n < \frac{1}{\sqrt{\pi n}}
\]

and is mentioned, under this name (without proof) in the celebrated monograph of D. S. Mitrinović and P. M. Vasić [29], p.192. We quickly recall a demonstration of the inequality (4.15), following the idea of [24], but written with our notation. The inequality

\[
I_{2n+1} < I_{2n} < I_{2n-1}
\]

can be written again (because of the recurrence relation of the integrals of Wallis) so

\[
I_{2n+1} < I_{2n} < I_{2n+1} \cdot \left( \frac{2n}{2n+1} \right)^{-1}
\]

and further, using the expressions (4.11) and (4.12) of the integrals

\[
\frac{1}{\Omega_n} \cdot \frac{1}{2n+1} < \Omega_n \cdot \frac{\pi}{2} < \frac{1}{\Omega_n} \cdot \frac{1}{2n+1} \cdot \left( \frac{2n}{2n+1} \right)^{-1}.
\]

Performing in each part of the double inequality some calculations, we obtain (4.15).

Today the formula of Wallis also serves to complete the demonstration of Stirling’s “small” formula

\[
\lim_{n \to \infty} \frac{n!}{n^ne^{-n}\sqrt{2\pi n}} = 1
\]

(whose name is due to the fact that the term ”Stirling’s great formula” usually refers to the asymptotic development of the factorial). Indeed, in the most common proof of it, after establishing that the sequence of general term

\[
a_n = \frac{n!}{n^ne^{-n}\sqrt{n}} = \frac{n!}{n^{n+1/2}e^{-n}}
\]
is convergent to a finite limit $a \neq 0$, the equality

\begin{equation}
(4.19) \quad a = \frac{a^2}{a} = \left( \lim_{n \to \infty} a_n \right)^2 = \lim_{n \to \infty} a_n^2 = \lim \left( \frac{a_n^2}{a_{2n}} \right)
\end{equation}

is obtained and then, using the Wallis’ formula, it is established that $a = \sqrt{2\pi}$.

A simplification of the proof of the convergence of the sequence is presented in [45]. A new and profoundly non trivial demonstration of the Stirling formula with an extension to the gamma function was established by C. P. Niculescu and two of his Ph.d. students, D. E. Dutkay and F. Popovici, in [14].

5 A special limit with an elementary proof suggested by the second step of the demonstration of the Stirling formula

Let us consider the sequence with general term $a_n = (e^n n!) / n^n$, for which we pose the problem of the elementary establishment of nature and limit in $\mathbb{R} \cup \{-\infty, \infty\}$ (the problem being trivial if the Stirling’s formula is used). Because

\[ \frac{a_{n+1}}{a_n} = \frac{e}{(1 + 1/n)^n} (n \to \infty) \]

the ratio criterion for limits of sequences is not applicable. But, because $(1 + 1/n)^n < e$, the last ratio is larger than 1, so the sequence is steadily increasing. Now we will use an inequality that satisfies any sequence of strictly positive and strictly increasing numbers $(x_n)_n$, namely

\begin{equation}
(5.20) \quad x_n > \frac{x_n^2}{x_{2n}}
\end{equation}

(It results immediately from $x_{2n} > x_n$, by multiplying by $x_n$ and dividing by $x_{2n}$.) Applying this inequality to the sequence $(a_n)_n$, and taking into account the end of (2.2), we have

\begin{equation}
(5.21) \quad a_n > \frac{e^n \cdot n!}{n^n} \cdot \frac{(2n)^{2n}}{e^{2n}(2n)!} = \frac{(2^n \cdot n!)^2}{(2n)!} \cdot \frac{1}{[2 \cdot 4 \cdot \ldots \cdot 2n] \cdot [2 \cdot 4 \cdot \ldots \cdot 2n]} = \frac{1}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1)} > \sqrt{2n + 1}.
\end{equation}

\footnote{The problem appears in somewhat less precise form in [10] (problem 32, c), p. 58) because there is a need to show that the sequence is convergent, while, having, as we will see, the infinite limit, it is convergent only in the broad sense, on $\mathbb{R} \cup \{-\infty, +\infty\}$. The wording presented here resumes the form in which it was published in [42].}
Therefore, \( a_n > \sqrt{2n + 1} \), then (because \( \sqrt{2n + 1} \to \infty \) \( \text{as } n \to \infty \)) it results that

\[
\lim_{n \to \infty} a_n = \infty.
\]

The whole solution was inspired by the second step of the usual proof of the Stirling’s formula, where, after showing that the sequence \((a_n)_n\) of (4.18) converges to a limit \( a \neq 0 \), the constant is determined by the process described by formula (4.19). The only difference is that here we used an inequality (5.20), in place of the equality (4.19). A dual problem to the problem presented ([40], issue 84, p. 34) requires the determination of the limit of the sequence of real numbers \((b_n)_n\) given by \( b_n = e^n n!/n^{n+1} \). The solution is similar; the limit is 0.

6 An application for finding the convergence of a sequence of integrals

Consider the sequence of integrals

\[
(6.22) \quad A_n = \int_0^1 \frac{1}{(1 + x^2)^n} \, dx.
\]

We propose to determine in an elementary way the limit of the sequence. By making the change of variable \( x = \tan t \), we get

\[
A_n = \frac{\pi}{4} \int_0^{\pi/4} \frac{1}{(1 + \tan^2 t)^n} \cdot \frac{1}{\cos^2 t} \, dt = \frac{\pi}{4} \int_0^{\pi/4} \left( \frac{1}{\cos^2 t} \right)^n \cdot \frac{1}{\cos^2 t} \, dt =
\]

\[
= \int_0^{\pi/2} \cos^{2n-2} t \, dt < \int_0^{\pi/2} \cos^{2n-2} t \, dt = I_{2n-2} =
\]

\[
= \Omega_{n-1} \cdot \frac{\pi}{2} < \frac{1}{\sqrt{2n - 1}} \cdot \frac{\pi}{2} \to 0 \quad \text{as } n \to \infty.
\]

(We took into account (4.11) and the inequality (2.2) written for \( n - 1 \) instead of \( n \).) So we have

\[
(6.1') \quad \lim_{n \to \infty} A_n = 0.
\]

By the same change of variable, the equality

\[
(6.23) \quad \int_0^\infty \frac{1}{(1 + x^2)^n} \, dx = \int_0^{\pi/2} \cos^{n-2} t \, dt
\]
shows us in this mode that the improper integral is convergent and, moreover

\[ \lim_{n \to \infty} \int_0^\infty \frac{1}{(1+x^2)^n} \, dx = 0. \]  

7 Expressions of Wallis integrals as integrals of irrational functions; consequences to the calculation of two integrals

Is it possible to express Wallis’ integrals as integrals of certain algebraic functions, namely (irrational) functions?

a) Let us do it in the integral \( I_n = \frac{\pi}{2} \int_0^\infty \cos^n t \, dt \), with \( n > 1 \), the change of variable \( x = \sin t \). We express \( I_n \) as an integral of an irrational function

\[ I_n = \int_0^1 (1-x^2)^{(n-1)/2} \, dx \]  

Conversely, the value of the integral \( \int_0^1 \left( \sqrt{1-x^2} \right)^{(n-1)/2} \, dx \) is given by \( I_n \), whose value at its turn, is expressed by the formulas (4.11) and (4.12).

b) Let us do it now in the integral \( I_n = \frac{\pi}{2} \int_0^\infty \cos^n t \, dt \), with the change of variable \( x = \cos t \). We find another expression of \( I_n \) as an integral of a function with radicals

\[ I_n = \int_0^1 \frac{x^n}{\sqrt{1-x^2}} \, dx, \]  

the last integral being improper, because the function to integrate is unbounded when \( x \to 1 \). The formula obtained shows that the improper integral of the right part of (7.2) is convergent and its value is given by \( I_n \), again described by (4.11) and (4.12).

So we obtained the searched expressions for the integrals of Wallis and, moreover, reciprocally, two nontrivial integrals were calculated using them.

8 Some infinite products similar of the one of Wallis, but related to other remarkable constants

The remarkable formula of Wallis can be also written as an infinite product of rational numbers

\[ 2 \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16}{1 \frac{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot \cdots} = \frac{\pi}{2}. \]
About the sequence of general term

Subsequent, similar infinite products have been established, but lead to other remarkable constants, of which we quote only a few. The following three formulas are given by E. Catalan, [9], in 1873:

\[(8.27) \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{16}{15} \cdots = \frac{\pi}{2\sqrt{2}}\]

\[(8.28) \frac{2}{1} \cdot \frac{6}{5} \cdot \frac{10}{9} \cdot \frac{14}{13} \cdots = \sqrt{2}\]

\[(8.29) \frac{2}{1} \left( \frac{4}{3} \right)^{1/2} \left( \frac{6}{5} \right)^{1/4} \left( \frac{8}{7} \right)^{1/8} \left( \frac{10}{9} \right)^{1/16} \cdots = e\]

Also see [37]. The equalities (8.27) and (8.28) give according to [37], a factorization of the Wallis’s formula that could be symbolically written \((8.1) = (8.2) \times (8.3)\). Another remarkable infinite product was established by N. Pippenger in [34], namely

\[\left( \frac{2}{1} \right)^{1/2} \left( \frac{2}{1} \right)^{1/3} \left( \frac{4}{3} \right)^{1/4} \left( \frac{6}{5} \right)^{1/8} \left( \frac{8}{7} \right)^{1/16} \cdots = e^{\frac{1}{2}}\]

A wide range of infinite products as well as comprehensive bibliographic references can be found in [37] and [17].

9 The highest binomial coefficient of the binomial at \(2n\)-th power

The highest binomial coefficient of the development of \((a + b)^{2n}\), or, equivalent, the larger term of the development \((1 + 1)^n\), is \(\binom{2n}{n}\), which is why it is also called the middle term. We have the identity

\[(9.30) \binom{2n}{n} = 4^n \Omega_n.\]

This identity describes the order of magnitude of this highest binomial coefficient and on the other hand, shows that any evaluation for \(\Omega_n\) conducts to an evaluation for \(\binom{2n}{n}\) and reciprocally.

10 Some expressions of \(\Omega_n\) using the gamma function

a) Starting from the definition of \(\Omega_n\), we have successively

\[\Omega_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{1 \cdot 2 \cdot 3 \cdots 2n}{(2 \cdot 4 \cdot 6 \cdots 2n)^2} = \frac{1 \cdot 2 \cdot 3 \cdots 2n}{4^n (1 \cdot 2 \cdot 3 \cdots n)^2},\]
then (taking into account that \( k! = \Gamma(k + 1) \), we obtain a first expression

\[
\Omega_n = \frac{\Gamma(2n+1)}{4^n \Gamma^2(n+1)}.
\]

b) Consider now the equality

\[
\Gamma(n + 1/2) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \Gamma(1/2)
\]

(e.g., \([1]\), the formula 6.1.12). Amplifying again the fraction by the product

\[2 \cdot 4 \cdot 6 \cdots 2n\]

and taking into account that \( \Gamma(1/2) = \sqrt{\pi} \), we obtain

\[
\Gamma(n + 1/2) = \frac{(2n)!}{4^n (n)!} \sqrt{\pi},
\]

then

\[
\Gamma(n + 1/2) = \frac{\sqrt{\pi} \Gamma(2n + 1)}{4^n \Gamma(n + 1)}.
\]

From the formula (10.31) and the last equality it follows that

\[
\Omega_n = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n + 1/2)}{\Gamma(n + 1)},
\]

a formula more useful and more often used than (10.31).

Finally, we note that any of formulas (10.31) and (10.32) lead us to extend the definition of \( \Omega_n \) for \( n = 0 \), by adopting the convention \( \Omega_0 = 1 \).

11 An explanation of Wallis’s inequality based on logarithmic convexity of the gamma function

The expression of \( \Omega_n \) using the gamma function, given by the equality (10.32), together with some of the simplest properties of this function, conduct us to a supplementary explanation of the appearance of the inequality of Wallis. This explanation was communicated to me by the late Professor Lupas \([27]\); we are doing it here with a slight simplification.

The properties we refer to are:

(i) The logarithmic convexity of the gamma function on \((0, \infty)\) [i.e. the fact that, for any \( x > 0 \) (then \( \Gamma(x) > 0 \)), the function \( x \mapsto \ln \Gamma(x) \) is convex].

(ii) The functional equation \( \Gamma(x+1) = x\Gamma(x) \), for all \( x > 0 \)

\[1\]

The two conditions (i) and (ii) checked by the gamma function, together with \( \Gamma(1) = 1 \), determine it uniquely. Indeed, more specifically, there is also a remarkable converse statement, namely the Bohr-Mollerup theorem, which shows that if a function \( g : (0, \infty) \to (0, \infty) \) is logarithmic convex, satisfies the functional equation \( g(x+1) = xg(x) \) for all \( x > 0 \) and \( g(1) = 1 \), then the function \( g \) coincides with Euler’s gamma function, \( g = \Gamma \). This theorem, first published in \([8]\), has been very popular thanks to the short but well-known monograph of E. Artin \([6]\), highly appreciated by Professor Lupas. Professor Lupas also appreciated a less well-known work on the gamma function, \([18]\). For a current fundamental documentation on the Gamma function, see e.g. \([30]\), pp. 50 and \([31]\), p. 263.
Recall that, for a function $g : I \rightarrow (0, \infty)$, the logarithmic convexity being the condition

$$\ln g((1 - t)a + tb) \leq (1 - t) \ln g(a) + t \ln g(b), \ \forall a, b \in I, \ a < b, \ \forall t \in [0, 1],$$

this implies the inequality

$$g((1 - t)a + tb) \leq g^{1-t}(a)g^t(b), \ \forall a, b \in I, \ a < b, \ \forall t \in [0, 1].$$

Particularizing for $t = 1/2$ in the last inequality, it follows that for any logarithmic convex (short, log convex) function we have fulfilled the Jensen condition

$$g\left(\frac{a + b}{2}\right) \leq \sqrt{g(a)g(b)}, \ \forall a, b \in I, \ a < b.$$

If the function is strictly log convex, then all three previous inequalities are strict. The last inequality translates in this way for the strict log convex gamma function,

$$\Gamma\left(\frac{a + b}{2}\right) < \sqrt{\Gamma(a)\Gamma(b)}, \ \forall a, b \in (0, \infty), \ a < b.$$

a) Let us get in (11.33) $a = n$ and $b = n + 1$. We obtain

$$\Gamma(n + 1/2) < \sqrt{\Gamma(n)\Gamma(n + 1)} = \sqrt{\frac{\Gamma^2(n + 1)}{n}} = \frac{\Gamma(n + 1)}{\sqrt{n}},$$

namely

$$\Gamma(n + 1/2) \leq \frac{\Gamma(n + 1)}{\sqrt{n}}.$$

From this, we obtain

$$\frac{1}{\sqrt{\pi}} \frac{\Gamma(n + 1/2)}{\Gamma(n + 1)} \leq \frac{1}{\sqrt{\pi n}},$$

that is, according to (10.32),

(a)$$\Omega_n < \frac{1}{\sqrt{\pi n}}.$$

b) Let us get now in (11.33) $a = n + 1/2$ and $b = n + 3/2$. It results that

$$\Gamma(n + 1) < \sqrt{\Gamma(n + 1/2)\Gamma(n + 3/2)}$$

$$= \sqrt{\Gamma^2(n + 1/2)\cdot(n + 1/2)} = \Gamma(n + 1/2)\sqrt{n + 1/2},$$

namely

$$\Gamma(n + 1) < \Gamma(n + 1/2)\sqrt{n + 1/2}.$$

From this it results that

$$\frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{n + 1/2}} < \frac{1}{\sqrt{\pi}} \frac{\Gamma(n + 1/2)}{\Gamma(n + 1)},$$
and then, using again (10.32),

\[ \frac{1}{\sqrt{\pi(n + 1/2)}} < \Omega_n, \]

(namely the left part of the inequality of Wallis, which concludes recovering (4.15)).

* * *

There is another way of working, closer to the classic deduction, recalled in Section 4. Indeed, from the recurrence relation of the integrals \( I_k \),

\[ I_k = \frac{k - 1}{k} I_{k-2} \quad (k \geq 2) \]

it results

\[ kI_k I_{k-1} = (k - 1)I_{k-1} I_{k-2} \quad (k \geq 2). \]

Then the product \( kI_k I_{k-1} \) is constant; its value is equal with \( 1 \cdot I_1 \cdot I_0 = 1 \cdot 1 \cdot (\pi/2) = \pi/2 \). Due to the fact that the sequence \( (I_k)_k \) is strictly decreasing, we have

\[ I_k < \sqrt{I_k I_{k-1}} < I_{k-1}, \]

or, replacing the product of the middle term by its value \( \pi/2 \), we obtain

\[ I_k < \sqrt{\frac{\pi}{2k}} < I_{k-1}. \]

Taking once \( k = 2n \), replacing \( I_{2n} = \Omega_n (\pi/2) \), \( I_{2n-1} = I_{2n+1} \cdot \cdot ((2n)/(2n + 1))^{-1} \) and making the calculations, either of the two sides of the inequality obtained gives us the right side of the inequality of Wallis.

Taking then \( k = 2n + 1 \) and replacing \( I_{2n+1} = 1 / (\Omega_n (2n + 1)) \) and \( I_{2n} \) as before, and again calculating, either of the two parts of inequality gives us the left side of Wallis’ inequality.

12 Refinements of Wallis’ inequality and its asymptotic developments

The inequality of Wallis (4.15) has been improved by setting up various refinements. The first one was that of D. K. Kazarinoff [23] from 1956 (that is, 300 years after Wallis’s discovery!)

\[ \frac{1}{\sqrt{\pi(n + 1/2)}} < \Omega_n < \frac{1}{\sqrt{\pi(n + 1/4)}}, \]

then giving a refinement in the right part. the proof of [23] is based on some consistent elements of the theory of the special functions; an elementary proof is given in [44].
Also in 1956, J. Gurland, [20], established by probabilistic methods, an inequality equivalent to

\[(12.35) \quad \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{4(4n+3)} \right)}} < \Omega_n < \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} \right)}}, \]

then he made a refinement of the left part of the inequality of D. K. Kazarinoff.

The most significant refinement in this matter was obtained in 1985 by L. Panaitopol, in [32]:

\[\frac{4^n}{\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{32n} \right)}} < \left( \frac{2n}{n} \right) < \frac{4^n}{\sqrt{\pi \left( n + \frac{1}{4} \right)}}, \]

then, respecting (9.30),

\[(12.36) \quad \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{32n} \right)}} < \Omega_n < \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} \right)}}. \]

This double inequality shows the adequate additive constant in both square roots of the extreme parts, for a finest evaluation of \(\Omega_n\). It also implies the double inequality

\[(12.3') \quad \frac{1}{4} < \frac{1}{\pi \Omega_n^2} - n < \frac{1}{4} + \frac{1}{32n}, \]

that gives

\[(12.3'') \quad \lim_{n \to \infty} \left( \frac{1}{\pi \Omega_n^2} - n \right) = \frac{1}{4}. \]

Also, (12.36) permitted in [41] the evaluation of the speed of convergence of the increasing sequence \((W_n)_n\) of the formula of Wallis to its limit, namely

\[(12.37) \quad \frac{\pi}{4(2n+1)} \left( 1 - \frac{1}{8n} \right) < \frac{\pi}{2} - W_n < \frac{\pi}{4(2n+1)}. \]

Then the sequence \((W_n)_n\) tends slowly to its limit, with the speed of convergence of order of \(1/n\) the same of the harmonic series. In the monograph [15], at pag. 43 is written: "Le produit infini [de Wallis] converge très lentement. L’admirable formule de Wallis doit susciter les ricanements des chasseurs de décimales..." At the same time, the double estimate (12.36), having under the square root the same additive constant 1/4, gives an idea for the beginning of an asymptotic expansion of \(\Omega_n\) with a square root at the denominator, and which must also contain negative integer powers of \(n\).
In 1993, L. Tóth gave, in [38], the inequality
\[
\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{46n} \right)} < \frac{2 \cdot 4 \cdot \ldots \cdot 2n}{1 \cdot 3 \cdot \ldots \cdot (2n - 1)} < \sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{32n} \right)},
\]
which is equivalent to
\[
(12.5') \quad \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{4} \right)}} < \Omega_n < \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{46n} \right)}},
\]
(with the remark that the constants are the best possible such that the inequality works for any \( n \geq 1 \)).

In 1997, in [43], we established the inequality
\[
(12.39) \quad \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{32(n + 1/4)} \right)}} < \Omega_n < \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{32(n + \theta)} \right)}},
\]
with \( \theta > 1/4 \) and with the right part satisfied for starting with a certain rank \( n_0(\theta) \). This refines the inequality (12.38) and conducts to the double estimate
\[
(12.6') \quad \frac{n}{32(n + \theta)} < n \left( \frac{1}{\pi \Omega_n^2} - n - \frac{1}{4} \right) < \frac{n}{32(n + 1/4)} \quad (n > n_0(\theta))
\]
which gives us the first iterated limit of the sequence with general term \( \frac{1}{\pi \Omega_n^2} - n \), namely
\[
(12.6'') \quad \lim_{n \to \infty} n \left( \frac{1}{\pi \Omega_n^2} - n - \frac{1}{4} \right) = \frac{1}{32}.
\]
This limit, which could not be found using only the inequality (12.38) or (12.5'), could also be achieved by combining the left side of inequality (12.36) with the right side of the inequality (12.39).

Also, in [43], we established the asymptotic expansion of the sequence \((\Omega_n)_n\) with contains a square root, namely
\[
(12.40) \quad \Omega_n = \frac{1}{\sqrt{\pi \left( n + \frac{1}{4} + \frac{1}{32n} - \frac{1}{128n^2} - \frac{5}{2048n^3} + \frac{23}{4096n^4} + \ldots \right)}}.
\]

Previously, in [39], had given an asymptotic development without square root for the sequence of the Wallis formula, which we were alluding to earlier, and for the sequence \((\Omega_n)_n\), namely
\[
(12.41) \quad W_n = \frac{\pi}{2} \left( 1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \ldots \right)
\]
About the sequence of general term

(12.42) \[ \Omega_n = \frac{1}{\sqrt{\pi n}} \left( 1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} + \ldots \right). \]

The development (12.40) was deduced using (12.42).

We mention, as a curiosity, that R. E. Shafer proposed as a problem [36] in the American Mathematical Monthly, 1975 a double inequality, which, with the improvements of J. Grimland jr. and S. Glidewell, conducts to an evaluation of \( \Omega_n \) using some 4th order radicals; for details, see [43]. Also, in [43] we have refined the inequality by the following one

(12.43) \[ \frac{1}{\sqrt{\pi} \sqrt{n^2 + \frac{n}{2} + \frac{1}{8}}} < \Omega_n < \frac{1}{\sqrt{\pi} \sqrt{n^2 + \frac{n}{2} + \frac{1}{8} - \frac{1}{32n}}} \]

which also suggests an asymptotic development

(12.44) \[ \Omega_n = \frac{1}{\sqrt{\pi} \sqrt{n^2 + \frac{n}{2} + \frac{1}{8} - \frac{1}{32n} - \frac{1}{256n^2} + \ldots}}. \]

We close this section by showing how a certain integral inequality allows the deduction of a double inequality similar to (12.41).

In the problem [28], D. Mărghidanu asked as first step to show that if a function \( f : [a, b] \to \mathbb{R}_+ \) is continuous and we denote \( I_n = \int_a^b f^n(x)dx \) \( (n \in \mathbb{N}) \), then

\[ I_{n+k}^2 \leq I_n I_{n+k}, \quad k \in \mathbb{N}, \quad k \leq n. \]

Before we continue our exposure, we notice that, by particularising \( k = 1 \), i.e. considering the inequality

(12.45) \[ I_n^2 \leq I_{n-1} I_{n+1}, \]

we obtain that the sequence of the integrals of the successive powers of a positive and continuous function is log convex.

Particularising further \( a = 0, \ b = \pi/2, \ f(x) = \sin x \), the integrals \( I_n \) become those of Wallis, \( I_n = \int_0^{\pi/2} \sin^n xdx. \)

(a) Substituting now \( n \) in the inequality (12.45), write for the integrals of Wallis, by \( 2n \). We obtain

\[ I_{2n}^2 \leq I_{2n-1} I_{2n+1}; \]

or

\[ I_n^2 \leq I_{2n+1}^2 \cdot \left( \frac{2n}{2n+1} \right)^{-1}, \]
then, expressing in function of $\Omega_n$,

$$
\left( \Omega_n \cdot \frac{\pi}{2} \right)^2 \leq \left( \frac{1}{\Omega_n} \cdot \frac{1}{2n+1} \right)^2 \cdot \frac{2n+1}{2n}.
$$

From here, after a few calculations, we obtain

\begin{equation}
(\alpha) \quad \Omega_n < \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[n]{n^2 + \frac{n}{2}}}. 
\end{equation}

The inequality is strict, for any $n \in \mathbb{N}^*$, because the number $\pi^2$ is irrational.

(\beta) Substituting now $n$ of (12.45), also written for the integrals of Wallis, by $2n + 1$. It results

$$
I_{2n+1}^2 \leq I_{2n} I_{2n+2}
$$

or

$$
I_{2n+1}^2 \leq I_{2n}^2 \cdot \frac{2n + 1}{2n + 2},
$$

then

$$
\left( \frac{1}{\Omega_n} \cdot \frac{1}{2n+1} \right)^2 \leq \left( \Omega_n \cdot \frac{\pi}{2} \right)^2 \cdot \frac{2n + 1}{2n+2}.
$$

Performing the calculations, we obtain

$$
\Omega_n \geq \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[n]{(n + 1/2)^3}},
$$

namely

\begin{equation}
(\beta) \quad \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[n]{n^2 + \frac{n}{2} + \frac{1}{2} - \frac{3}{4(n+1)}}} < \Omega_n,
\end{equation}

where the inequality is also strict.

But the double inequality obtained by assembling (\alpha) with (\beta), namely

\begin{equation}
(12.46) \quad \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[n]{n^2 + \frac{n}{2} + \frac{1}{2} - \frac{3}{4(n+1)}}} < \Omega_n < \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[n]{n^2 + \frac{n}{2}}},
\end{equation}

is weaker than (12.45).
13 Some converse assertions concerning the integrals of Wallis

Bohr-Mollerup’s theorem defines through the three above-mentioned conditions, imposed on an arbitrary function \( g : (0, \infty) \to (0, \infty) \), exactly the gamma function. Thus, the question of whether it could obtain such characterizations also for the integrals \( I_n \) of Wallis. A first result in this direction was established in 1982 by A. Lupuš in [25], by the following

**Problem.** Let \( F \) be the set of continuous functions \( f : [0, 1] \to [0, 1] \) with the properties

\[
\begin{align*}
\int_0^1 f^n(x)dx &= (n-1) \int_0^1 f^{n-2}(x)dx, \quad n = 2, 3, \ldots; \quad \int_0^1 f(x)dx \neq 0.
\end{align*}
\]

Show that \( \int_0^1 f(x)dx = \frac{2}{\pi} \).

Solving is done by highlighting that the sequence of integrals in the statement satisfies exactly the properties of the sequence of integrals of Wallis; then the Wallis formula leads to the result.

Probably the author considered the issue too simple for a paper and has therefore proposed it as a problem.

It is interesting to note, as is shown in the solution published in the periodical in which the problem arose, that the result remains valid for any sequence \((a_n)\) which satisfies the recurrence relationship in the statement, is decreasing, and \( a_0 = 1 \) and \( a_1 > 0 \).

Also, the statement can give the value of \( a_1 \), in place of \( a_0 \), and then we obtain \( a_0 \). An example of a function that satisfies the conditions of the problem is \( f(x) = \sin(\frac{\pi x}{2}) \).

With the change of variable of homothetic type \( t = \pi x/2 \), the sequence of the statement conducts exactly to the sequence of the integrals of Wallis. Then, starting from the problem of A. Lupuš, we obtain the following

**Proposition** ([47]) Let’s \((x_n)_{n \geq 0}\) be a sequence such that:

(a) The sequence satisfies the recurrence relation

\[
x_n = \frac{n-1}{n} x_{n-2}, \quad (n \geq 2);
\]

(b) The sequence is strictly decreasing \((x_n)_{n \geq 0}\);

(c) \( x_1 = 1 \).

Then \( x_n = I_n, \quad \forall n \geq 0 \), namely the sequence \((x_n)_{n \geq 0}\) is exactly that of the integrals of Wallis.

This proposition characterizes the sequence of Wallis’ integrals, being a Bohr-Mollerup type proposition in the field of sequences, that is, of natural variable functions.
Another problem of A. Lupaş [26] leads to the characterization through three similar conditions of a sequence \((x_n)_{n\geq 0}\) which turns out to be \(x_n = 2/\pi I_n\).

Finally, we point out the problem [33] of L. Panaitopol, who also has close ties to Wallis’ integrals.

**14 Developments in Maclaurin series in which \(\Omega_n\) appears**

Some well known Maclaurin series developments can be written in a simple and suggestive way with the expression \(\Omega_n\), using where necessary, the equality \(\Omega_0 = 1\).

We present, without proofs or explanations, some of them; all developments considered are valid over the interval \((-1,1)\).

\[
\sqrt{1 + x} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\Omega_n}{2n-1} x^n
\]  
\[
\frac{1}{\sqrt{1 + x}} = \sum_{n=0}^{\infty} (-1)^n \Omega_n x^n
\]  
\[
\frac{1}{\sqrt{1 + x^2}} = \sum_{n=0}^{\infty} (-1)^n \Omega_n x^{2n}
\]  
\[
\ln \left( x + \sqrt{1 + x^2} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{\Omega_n}{2n + 1} x^{2n+1}
\]  
\[
\sqrt{1 - x} = 1 - \sum_{n=1}^{\infty} \frac{\Omega_n}{2n-1} x^n
\]  
\[
\frac{1}{\sqrt{1 - x}} = \sum_{n=0}^{\infty} \Omega_n x^n
\]  
\[
\frac{1}{\sqrt{1 - x^2}} = \sum_{n=0}^{\infty} \Omega_n x^{2n}
\]  
\[
\arcsin x = \sum_{n=0}^{\infty} \frac{\Omega_n}{2n + 1} x^{2n+1}.
\]
About the sequence of general term

(Remark in passing that formula (14.48) conducts immediately, for \( x = 1 \), to obtain the first equality (3.7)).

Another use of the symbol \( \Omega_n \) when writing a series is that it allows the suggestive writing as the serial sum of the first (complete) elliptical integral

\[
K = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}} \, d\varphi
\]

Using the formula (14.53) in which we consider \( x = k \sin \varphi \), we have

\[
K = \int_0^{\pi/2} \left( \sum_{n=0}^{\infty} \Omega_n (k \sin \varphi)^{2n} \right) \, d\varphi.
\]

Because the Maclaurin series is, as any series of powers, uniformly convergent at any compact included in the interval \((-1, 1)\) the series can be integrated term by term or, as it is said, the integral switches with the summation and we have

\[
K = \int_0^{\pi/2} \left( \sum_{n=0}^{\infty} \Omega_n (k \sin \varphi)^{2n} \right) \, d\varphi = \sum_{n=0}^{\infty} \Omega_n \int_0^{\pi/2} (k \sin \varphi)^{2n} \, d\varphi
\]

\[
= \sum_{n=0}^{\infty} \Omega_n k^{2n} \int_0^{\pi/2} \sin^{2n} \varphi \, d\varphi = \sum_{n=0}^{\infty} \Omega_n k^{2n} I_{2n} =
\]

\[
= \sum_{n=0}^{\infty} \Omega_n k^{2n} \left( \Omega_n \frac{\pi}{2} \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \Omega_n^2 k^{2n},
\]

namely

\[
K = \frac{\pi}{2} \sum_{n=0}^{\infty} \Omega_n^2 k^{2n},
\]

or, not to use

\[
K = \frac{\pi}{2} \left( 1 + \sum_{n=1}^{\infty} \Omega_n^2 k^{2n} \right).
\]

This result appears in some issues such as the study of the arithmetic-geometric mean of two strictly positive numbers (v. [4], pp. 54-57), the problem of the pendulum period (see [19], pp. 10-11) and other. Also see [7].

Acknowledgements. I thank Professor Constantin P. Niculescu for their valuable remarks and suggestions which had hepled me to improve this work.
References


About the sequence of general term


**Andrei Vernescu**
Valahia University (retired)
Bucharest, 17 Arch. Ion Mincu Street
e-mail: avernescu@gmail.com
Sequences which converge to $e$: New insights from an old formula

Matthew F. Causley, Peter Morell

Abstract

One of the most fundamental results in calculus was the discovery of the mathematical constant $e = 2.718\ldots$ by Jacob Bernoulli. Remarkably, new definitions of $e$ are still being discovered, in part due to renewed interest at the advent of modern computing and the quest for more digits.

In this work we review recent discoveries of sequences which tend to $e$, and propose a systematic approach for producing such sequences. In doing so, we establish several classes of sequences, and their generalizations. Our methods use only basic tools of calculus and numerical analysis, such as series expansions and Padé approximants. Numerical results demonstrate that our new sequences rapidly converge to $e$.

2010 Mathematics Subject Classification: 97N40, 65B05.

Key words and phrases: Mathematical constant, Closed form, Sequence, Bernoulli numbers, Quadrature, Padé approximants.

1 Introduction

Amateur and professional mathematicians alike have put great effort into computing digits of special mathematical constants such as $\pi = 3.14\ldots$, $e = 2.71\ldots$, and the Euler-Mascheroni constant $\gamma = 0.57\ldots$. Of these, $\pi$ has received the most attention, as is evident by the depth and breadth of algorithms now available for its computation (a good survey of these can be found in [1]). Nonetheless, some work has been directed at $e$, and in fact literature is growing steadily.

As of this year, trillions of digits of $e$ have been computed (i.e., within a tolerance of $10^{-10^{12}}$), and most records rely on the Maclaurin series (2), combined with modern computational techniques such as binary splitting [14].

1Received 10 July, 2017
Accepted for publication (in revised form) 5 September, 2017
However, very recently many modifications of the sequence (1) have been identified, which are of mathematical and technical interest [5]. Below we will give a brief survey of methods for computing \(e\), from old to new [3, 7, 8, 9, 10, 11]. Then, we will introduce several generalizations, which we could not find elsewhere in the literature, despite the fact that their construction relies only on basic techniques of calculus.

1.1 \(e\) from its Origins.

Although the modern definition of \(e\) is the base of the natural logarithm, this was not the case when John Napier invented the logarithm in the early 1600’s. Consequently, Jacob Bernoulli first defined \(e\) in 1683, instead by studying compound interest. Bernoulli considered the sequence

\[ a_n = \left(1 + \frac{1}{n}\right)^n, \quad n = 1, 2, \ldots \]

and deduced the limiting value was between 2 and 3. Our modern definition of the constant reads

\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.
\]

As pointed out in [2, 12], for fixed \(n\) the sequence (1) converges to its limit rather slowly. For example when \(n = 10\), the sequence yields \(a_{10} \approx 2.6\ldots\), which is not even accurate to the first decimal.

However it was Euler who first employed the binomial expansion to the sequence (1), leading to the Maclaurin series definition

\[ s_n = \sum_{k=0}^{n-1} \frac{1}{k!} = 1 + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{(n-1)!}, \quad \lim_{n \to \infty} s_n = e, \]

which converges much more rapidly. Again setting \(n = 10\), we now have \(s_{10} = 2.7182815\ldots\), which is accurate to 6 decimals! Perhaps it is fitting that we still use Euler’s original notation to represent this mathematical constant. More generally for any large but finite \(n\), we see from equation (2) that \(|s_n - e| = \mathcal{O}\left(\frac{1}{n!}\right)\), which can be proven using Taylor’s theorem.

1.2 \(e\) from Averaging.

It turns out that for the sequence (1), \(|a_n - e| = \mathcal{O}\left(\frac{1}{n}\right)\), which explains the aforementioned slow convergence. More generally, it can be shown that

\[ a_n = e^{b_n} = e^{1 - \frac{1}{2n} + \frac{1}{3n^2} + \ldots} = e\left(1 - \frac{1}{2n} + \frac{11}{24n^2} + \ldots\right). \]

Calculating the coefficients on the right hand side of this expression is itself an interesting exercise [4, 13]. In practice, this result tells us that to obtain the first \(m\) decimal digits of \(e\), we require \(n = 10^m\).
Brothers and Knox [3, 11] have proposed some very creative alternatives to improve the rate of convergence (3). For example, they define the reflected sequence (1)

\[ a_{-n} = \left( 1 - \frac{1}{n} \right)^{-n} = e \left( 1 + \frac{1}{2n} + \frac{11}{24n^2} + \ldots \right), \]

and average the two, to produce the complimentary addition method (CAM)

\[ \frac{1}{2} (a_n + a_{-n}) = e \left( 1 + \frac{11}{24n^2} + \ldots \right). \]

The CAM method therefore converges like \( O \left( \frac{1}{n^2} \right) \). This is also the power ratio method (PRM)

\[ (1 + n) a_n + (1 - n) a_{-n} = e \left( 1 + \frac{1}{24n^2} + \ldots \right). \]

Further use of averaging will remove the next leading order error term, thereby producing

\[ \left( \frac{1}{2} + \frac{11}{24n} \right) a_n + \left( \frac{1}{2} - \frac{11}{24n} \right) a_{-n} = e \left( 1 - \frac{2593}{5760n^4} + \ldots \right). \]

Brothers and Knox also introduce the mirror image method (MIM)

\[ \tilde{a}_n = \left( \frac{2n+1}{2n-1} \right)^n = \left( 1 + \frac{1}{n - \frac{1}{2}} \right)^n = e \left( 1 + \frac{1}{12n^2} + \ldots \right), \]

and form an average of 3 sequences which further improve the accuracy. This latter improvement is dubbed the acronym-intensive CMPAM, and we express it as

\[ \frac{3829(a_n + a_{-n})}{2418} + \frac{1941n(a_n - a_{-n})}{1209} - \frac{679\tilde{a}_n}{1209} \approx e - e \left( \frac{383443}{83566080n^6} \right). \]

Equation (8) produces 8 decimal digits of \( e \) when evaluated at \( n = 10 \), which is a dramatic improvement over Bernoulli’s sequence. It is possible to follow this approach to obtain even more accurate sequences, e.g.

\[ \frac{1237 (a_n - a_{-n})}{2400n} + \left( 1 + \frac{137}{2400n^2} \right) a_n + a_{-n} \approx e - e \left( \frac{375451}{72576000n^6} \right). \]

Clearly the resulting sequences, or even the approach to finding them, quickly becomes cumbersome.
1.3  $e$ from Integration

Khattri [7, 9] has pursued a different line of thought, by considering

$$I_n = \int_n^{n+1} \frac{dx}{x} = \ln \left(1 + \frac{1}{n}\right).$$

If we divide by $I_n$ and exponentiate both sides, then we have an identity reminiscent of (1)

$$e = \exp \left( \frac{1}{I_n} \ln \left(1 + \frac{1}{n}\right) \right) = \left(1 + \frac{1}{n}\right)^{c(n)}, \quad c(n) = \frac{1}{I_n}.$$

Khattri observes that by approximating $I_n$ with $\tilde{I}_n$ using numerical integration, one obtains a rational function $\tilde{c}(n) \approx c(n)$. For instance, the trapezoidal rule yields

$$\int_n^{n+1} \frac{dx}{x} \approx \tilde{I}_n = \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n+1}\right) = \frac{2n+1}{2n(n+1)},$$

which produces the sequence

$$\left(1 + \frac{1}{n}\right)^{\frac{2n(n+1)}{2n+1}} = e \left(1 - \frac{1}{6n^2} + \ldots\right).$$

Here we see that the relative error is $O \left(\frac{1}{n^2}\right)$. In the special case where the approximation is based on the left endpoint rule

$$\int_n^{n+1} \frac{dx}{x} \approx \tilde{I}_n = \frac{1}{n},$$

Khattri’s approach recovers Bernoulli’s sequence (1). Khattri obtains explicit constructions in [7, 9], using as many as 5 Gaussian quadrature points, to produce the dramatically improved “GLEE5” sequence

$$\left(1 + \frac{1}{n}\right)^n \approx e \left(1 - \frac{1}{698544n^{10}}\right),$$

which achieves 16 decimal digits of accuracy when $n = 10$. In principle the number of points can be increased arbitrarily; however the exponents can no longer be written as rational functions with integer coefficients beyond 5 points, which will eventually limit the accuracy.

2  New Approximations for $e$

We now propose several generalizations of (1), which will rapidly converge to $e$. The ideas presented so far can be sorted into three categories:

1. Use a weighted average of several less accurate sequences.
2. Modify the exponent in the traditional sequence (1).
3. Modify the base in the traditional sequence (1).

The sequences (4), (5) and (8) due to Brothers and Knox, as well as (9) illustrate the first approach, which we will not consider any further below. Khattri’s approach falls into the second category, and the mirror image method (7) is, at the moment, the only representative of the third category. Below, we will focus on the second and third strategies, by independently developing sequences of the form

\[ a_n = \left(1 + \frac{1}{n}\right)^{c(n)}, \quad (11) \]

\[ a_n = \left(1 + \frac{1}{b(n)}\right)^n. \quad (12) \]

We will also briefly comment on a more general, combined approach, of the form

\[ a_n = \left(1 + \frac{1}{b(n)}\right)^{c(n)}. \quad (13) \]

### 2.1 Sequences with Modified Exponents

Consider the sequence (11). We want to find an exponent \(c(n)\) for which the sequence \(a_n\) approaches \(e\). But what if we simply impose that the sequence is exact for all integers \(n\)? Then

\[ a_n = \left(1 + \frac{1}{n}\right)^{c(n)} = e, \quad (14) \]

and so taking the natural log of both sides yields

\[ c(n) \ln \left(1 + \frac{1}{n}\right) = 1. \quad (15) \]

Our goal is to form an asymptotic series expansion for \(c(n)\) in powers of \(x = \frac{1}{n}\). We will therefore make use of the Taylor series of the natural logarithm (40). For large \(n\), the quantity \(x = \frac{1}{n}\) is small, and so

\[ \ln \left(1 + \frac{1}{n}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)n^{k+1}} = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots \]

Likewise we expand \(c(n)\) in a series, which for reasons that will become clear, we include a leading term proportional to \(n\)

\[ c(n) = \sum_{m=0}^{\infty} \frac{c_m}{n^{m-1}} = c_0n + c_1 + \frac{c_2}{n} + \ldots \quad (16) \]
In view of equation (15), we then have a product of two infinite series

\[
1 = \left( \sum_{m=0}^{\infty} \frac{c_m}{n^{m+1}} \right) \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)n^{k+1}} \right) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{c_m d_k}{n^{m+k}}, \quad d_k = \frac{(-1)^k}{k+1}.
\]

In fact this is referred to as a Cauchy product.

**Definition 1** (Cauchy Product). The Cauchy product of two series is

\[
\alpha(n) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{c_m d_k}{n^{m+k}} = \sum_{p=0}^{\infty} \frac{\alpha_p}{n^p},
\]

where for each \( p = 0, 1, \ldots \)

\[
\alpha_p = \sum_{k=0}^{p} c_k d_{p-k} = c_0 d_p + c_1 d_{p-1} + \cdots + c_{p-1} d_1 + c_p d_0.
\]

Now we identify equation (17) as a Cauchy product, with \( \alpha(n) = 1 \), so that

\[
1 = \sum_{p=0}^{\infty} \frac{\alpha_p}{n^p} = \alpha_0 + \frac{\alpha_1}{n} + \ldots,
\]

where \( d_{p-k} = (-1)/(p-k+1) \), hence

\[
\alpha_0 = c_0, \quad \alpha_1 = \frac{c_0}{2} + c_1, \quad \alpha_2 = \frac{c_0}{3} - \frac{c_1}{2} + c_2, \quad \alpha_3 = -\frac{c_0}{4} + \frac{c_1}{3} - \frac{c_2}{2} + c_3,
\]

and more generally

\[
\alpha_p = \sum_{k=0}^{p} c_k d_{p-k} = (-1)^p \frac{c_0}{p+1} + (-1)^{p-1} \frac{c_1}{p} + \ldots + c_p.
\]

We obtain the coefficients \( c_0, c_1, \ldots \) in turn by fixing the values of \( \alpha_p \), which we see from equation (18) must be \( \alpha_0 = 1 \), and \( \alpha_1 = \alpha_2 = \ldots = 0 \). Thus

\[
1 = c_0, \quad 0 = -\frac{c_0}{2} + c_1, \quad 0 = \frac{c_0}{3} - \frac{c_1}{2} + c_2, \quad 0 = -\frac{c_0}{4} + \frac{c_1}{3} - \frac{c_2}{2} + c_3,
\]

and so on. Note that if we had not chosen our series expansion according to equation (16), then the algebra here wouldn’t have worked out.

In each new equation, a single coefficient is added, and so if we solve for the highest coefficients in order we can find them all up to say \( c_m \) for some \( m > 1 \)

\[
c_0 = 1, \quad c_1 = \frac{c_0}{2}, \quad \ldots, \quad c_m = \frac{c_{m-1}}{2} - \frac{c_{m-2}}{3} + \ldots + \frac{(-1)^{m+1}c_0}{m+1}.
\]

If we truncate this procedure at \( m = M \), then the approximate exponent is

\[
c^M(n) = \sum_{m=0}^{M} \frac{c_m}{n^{m-1}},
\]
Sequences which converge to $e$ and we have an infinite family of sequences which approximate $e$. When $M = 1$, we recover the Bernoulli sequence; and so the first few corresponding to $M = 2, 3, 4, 5$ and 6 are

\begin{align*}
  (20) \quad & \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} = e \left(1 + \frac{1}{12n^2} + \ldots\right) \\
  (21) \quad & \left(1 + \frac{1}{n}\right)^{n+\frac{1}{12n^2}} = e \left(1 - \frac{1}{24n^3} + \ldots\right) \\
  (22) \quad & \left(1 + \frac{1}{n}\right)^{n+\frac{1}{12n^2} + \frac{1}{24n^2}} = e \left(1 + \frac{19}{720n^4} + \ldots\right) \\
  (23) \quad & \left(1 + \frac{1}{n}\right)^{n+\frac{1}{12n^2} + \frac{1}{24n^2} - \frac{19}{720n^4}} = e \left(1 - \frac{3}{160n^5} + \ldots\right) \\
  (24) \quad & \left(1 + \frac{1}{n}\right)^{n+\frac{1}{12n^2} + \frac{1}{24n^2} - \frac{19}{720n^4} + \frac{3}{160n^5}} = e \left(1 + \frac{863}{60480n^6} + \ldots\right).
\end{align*}

We observe that the error coefficient of each previous sequence coincides with the next coefficient to appear in the exponent.

**Theorem 1.** For each $M > 1$, the sequence resulting from the exponent $c^M(n)$ will have the following asymptotic behavior

\[
\left(1 + \frac{1}{n}\right)^{c^M(n)} = e \left(1 - \frac{c_{M+1}}{n^{M+1}} + \ldots\right),
\]

where $c_{M+1}$ is the next coefficient appearing in $c^{M+1}(n)$.

**Proof.** We replace $c(n)$ with the truncated series $c^M(n)$. Recall that the coefficients are chosen so that $\alpha_0 = 1$, and $\alpha_1 = \alpha_2 = \ldots = \alpha_M = 0$ in equation (18). Thus, we have

\[
C^M(n) \ln \left(1 + \frac{1}{n}\right) = \left[ \sum_{m=0}^{M} \frac{c_m}{n^{m-1}} \right] \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)n^{k+1}} \right] = 1 + \tilde{\alpha}_{M+1} + \ldots
\]

where the first nonzero coefficient due to truncation is

\[
\tilde{\alpha}_{M+1} = -\frac{1}{2}c_M + \frac{1}{3}c_{M-1} + \ldots + \frac{(-1)^{M+2}}{M+2} c_0.
\]

The exact coefficient is given by

\[
\alpha_{M+1} = c_{M+1} - \frac{1}{2}c_M + \frac{1}{3}c_{M-1} + \ldots + \frac{(-1)^{M+2}}{M+2} c_0,
\]

and $c_{M+1}$ is likewise chosen to produce $\alpha_{M+1} = 0$. Consequently $\tilde{\alpha}_{M+1} = -c_{M+1}$, and we exponentiate both sides to produce

\[
\left(1 + \frac{1}{n}\right)^{c^M(n)} = \exp \left[1 - \frac{c_{M+1}}{n^{M+1}} + \ldots\right] = e \left(1 - \frac{c_M}{n^{M+1}} + \ldots\right),
\]
Table 1: Relative absolute errors for approximating $e$ using (20)-(24).

<table>
<thead>
<tr>
<th></th>
<th>$n = 1$</th>
<th>$n = 10$</th>
<th>$n = 100$</th>
<th>$n = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. (20)</td>
<td>0.04052</td>
<td>0.00075717</td>
<td>8.2508 $\times 10^{-6}$</td>
<td>8.3250 $\times 10^{-8}$</td>
</tr>
<tr>
<td>Eq. (21)</td>
<td>0.01788</td>
<td>3.7363 $\times 10^{-5}$</td>
<td>4.1199 $\times 10^{-8}$</td>
<td>4.1619 $\times 10^{-11}$</td>
</tr>
<tr>
<td>Eq. (22)</td>
<td>0.010899</td>
<td>2.349 $\times 10^{-6}$</td>
<td>2.6073 $\times 10^{-10}$</td>
<td>2.6357 $\times 10^{-14}$</td>
</tr>
<tr>
<td>Eq. (23)</td>
<td>0.0074241</td>
<td>1.6611 $\times 10^{-7}$</td>
<td>1.8516 $\times 10^{-12}$</td>
<td>1.8726 $\times 10^{-17}$</td>
</tr>
<tr>
<td>Eq. (24)</td>
<td>0.0055602</td>
<td>1.2599 $\times 10^{-8}$</td>
<td>1.4086 $\times 10^{-14}$</td>
<td>1.4251 $\times 10^{-20}$</td>
</tr>
</tbody>
</table>

as desired.

The relative errors are presented in Table 1, for $n = 1, 10, 100, 1000$. The results validate Theorem 1, as the errors in each row to drop by a factor of $10^{-M}$, for $M = 2, 3, 4, 5$ and 6. Across the rightmost column, the error coefficients on the right hand side of equations (20)-(24) emerge: $1/12 \approx 0.0833$, $1/24 \approx 0.0416$, etc.

2.2 Sequences with Modified Bases

We next fix the exponent, and consider varying the base of the Bernoulli sequence (12), so that

$$e = \left(1 + \frac{1}{b(n)}\right)^n \implies b(n) = \frac{1}{e^{1/n} - 1}.$$ 

As before, the desired series is the reciprocal of a well-known Taylor series (41), and so we could use Cauchy products to find its reciprocal. But in this case, setting $x = 1/n$ produces equation (42) to within a multiple of $x$, which produces

$$\sum_{m=0}^{\infty} b_m n^{1-m} = \frac{1}{e^{1/n} - 1} = n - \frac{1}{2} + \frac{1}{12n} - \frac{1}{720n^3} + \ldots = \sum_{m=0}^{\infty} \frac{B_m}{m!} n^{1-m}.$$ 

Coincidentally, $B_m$ denotes the $m$th Bernoulli number, named after the same Jacob Bernoulli who defined $e$ using (1)! The Cauchy product would lead us to the recursion relation for Bernoulli numbers (43), which we simplify by observing that odd Bernoulli numbers vanish, except for $B_1 = -\frac{1}{2}$. Thus, we solve for each coefficient recursively up to $b_{2m} = \frac{B_{2m}}{(2m)!}$.

$$b_0 = 1, \quad b_1 = -\frac{1}{2}, \quad b_{2m} = \frac{2m - 1}{2(2m+1)!} - \sum_{k=1}^{m-1} \frac{b_{2k}}{(2m - 2k + 1)!}.$$ 

Now define the truncated series

$$b^M(n) = n - \frac{1}{2} + \sum_{m=1}^{M-1} b_{2m} n^{1-2m}, \quad b_{2m} = \frac{B_{2m}}{(2m)!}.$$
Theorem 2. For large finite integers $n$, the sequence with base $b^M(n)$ satisfies

\[(27) \quad \left(1 + \frac{1}{b^M(n)}\right)^n \approx e \left(1 + \frac{b_{2M}}{n^{2M}} + O\left(\frac{1}{n^{2M+2}}\right)\right) .\]

Proof. It is easily shown that for the full and partial sums

\[b(n) = n - \frac{1}{2} + \sum_{m=1}^{\infty} b_{2m} n^{1-2m}, \quad b^M(n) = n - \frac{1}{2} + \sum_{m=1}^{M-1} b_{2m} n^{1-2m},\]

their difference is

\[b(n) - b^M(n) \approx b_{2M} n^{1-2M} + O\left(n^{-1-2M}\right) .\]

We will also require the approximation

\[(b(n) + 1)b^M(n) \approx \left(n + \frac{1}{2} + \frac{1}{12n}\right) \left(n - \frac{1}{2} + \frac{1}{12n}\right) = n^2 - \frac{1}{12} + O\left(\frac{1}{n^2}\right) .\]

as well as the binomial approximation

\[(1 + \beta)^n \approx 1 + n\beta, \quad -1 < \beta < 1.\]

Since the error bound involves $e$, we obtain the identity

\[\left(1 + \frac{1}{b(n)}\right)^n = e \implies 1 = \frac{e}{\left(1 + \frac{1}{b(n)}\right)^n},\]

and introduce it into the sequence

\[\left(1 + \frac{1}{b^M}\right)^n = e \left[\frac{1 + \frac{1}{b^M}}{1 + \frac{1}{b}}\right]^n = e \left[1 + \frac{1}{b^M} - \frac{1}{b}\right]^n = e \left(1 + \frac{b - b^M}{(b + 1)b^M}\right)^n .\]

The binomial approximation produces

\[e \left(1 + \frac{b - b^M}{(b + 1)b^M}\right)^n \approx e \left(1 + n \left(\frac{b - b^M}{(b + 1)b^M}\right)\right) .\]
We insert the approximations for the numerator and denominator and simplify
\[ n \left( \frac{b - b^M}{(b + 1)b^M} \right) \approx \frac{b_{2M}n^{2-2M} + \mathcal{O}(n^{-2M})}{n^2 - \frac{1}{12} + \mathcal{O}(n^{-2})} \approx b_{2M}n^{-2M} + \mathcal{O}(n^{-2-2M}), \]
and this term is precisely that found in equation (27).

**Remark 1.** After some algebra to expose symmetry, the approximations can also be written as
\[
e \approx \left( \frac{b^M + 1}{b^M} \right)^n = \left( \frac{S^M(n)}{S^M(-n)} \right)^n,
\]
\[S^M(\pm n) = 1 \pm \frac{1}{2n} + \frac{M-1}{2n} b_{2m},\]

The resulting sequences for \( M = 1, 2, 3, 4 \) and 5 are
\[
(29) \quad \left( \frac{1 + \frac{x}{2}}{1 - \frac{x}{2}} \right)^n \approx e \left( 1 + \frac{x^2}{12} \right)
\]
\[
(30) \quad \left( \frac{1 + \frac{x}{2} + \frac{x^2}{12}}{1 - \frac{x}{2} + \frac{x^2}{12}} \right)^n \approx e \left( 1 - \frac{x^4}{720} \right)
\]
\[
(31) \quad \left( \frac{1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720}}{1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720}} \right)^n \approx e \left( 1 + \frac{x^6}{30240} \right)
\]
\[
(32) \quad \left( \frac{1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240}}{1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240}} \right)^n \approx e \left( 1 - \frac{1}{1209600x^8} \right)
\]
\[
(33) \quad \left( \frac{1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600}}{1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600}} \right)^n \approx e \left( 1 + \frac{x^{10}}{47900160} \right),
\]
where \( x = \frac{1}{n} \). When compared to the modified exponent sequences (20)-(22), we observe the doubly-rapid error improvement! As shown in Table (2), the error decreases by a factor of \( 10^{-2M} \) in each row. Scanning the bottom row, we again see the error coefficients \( \frac{1}{12} \approx 0.0833, \frac{1}{720} \approx 0.00139 \), etc.

### Table 2: Relative absolute errors for approximating \( e \) using (29)-(33).

<table>
<thead>
<tr>
<th>( n = 1 )</th>
<th>( n = 10 )</th>
<th>( n = 100 )</th>
<th>( n = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. (29)</td>
<td>( 1.0364 \times 10^{-1} )</td>
<td>( 8.3493 \times 10^{-4} )</td>
<td>( 8.3335 \times 10^{-6} )</td>
</tr>
<tr>
<td>Eq. (30)</td>
<td>( 1.4701 \times 10^{-3} )</td>
<td>( 1.3897 \times 10^{-7} )</td>
<td>( 1.3889 \times 10^{-11} )</td>
</tr>
<tr>
<td>Eq. (31)</td>
<td>( 3.5044 \times 10^{-9} )</td>
<td>( 3.3088 \times 10^{-11} )</td>
<td>( 3.3069 \times 10^{-17} )</td>
</tr>
<tr>
<td>Eq. (32)</td>
<td>( 8.7583 \times 10^{-1} )</td>
<td>( 8.2720 \times 10^{-15} )</td>
<td>( 8.2672 \times 10^{-25} )</td>
</tr>
<tr>
<td>Eq. (33)</td>
<td>( 2.2116 \times 10^{-8} )</td>
<td>( 2.0889 \times 10^{-18} )</td>
<td>( 2.0877 \times 10^{-28} )</td>
</tr>
</tbody>
</table>
3 Combined Strategy

Finally, we briefly consider a full generalization of the form (13). While the computations are more involved, the methods are a natural extension of those shown above.

3.1 Generalized Bernoulli Approximations

We compare two different modifications of the base. First, we will use the new approach following from Bernoulli numbers, obtaining

\[ e \approx \left( \frac{S^M(n)}{S^M(-n)} \right)^{c(n)}. \]

The first few generalizations using Bernoulli-bases are obtained by replacing \( n \) with \( c(n) \) in equations (29)-(31). In principle, the exponents can be chosen to give any desired order of accuracy. We will compare our results with those of Khattri’s Glee5 (10), and so we obtain 10th order accurate sequences

\[
\begin{align*}
(34) & \quad \left(\frac{1 + \frac{x}{2} + \frac{x^2}{22}}{1 - \frac{x}{2} + \frac{x^2}{22}}\right)^{\frac{1}{2} + \frac{x^3}{720} + \frac{x^5}{11440} + \frac{x^7}{95040}} \approx e \left(1 + \frac{2549x^{10}}{119750400}\right) \\
(35) & \quad \left(\frac{1 + \frac{x}{2} + \frac{x^2}{12}}{1 - \frac{x}{2} + \frac{x^2}{12}}\right)^{\frac{1}{2} + \frac{x^3}{720} + \frac{x^5}{11440} + \frac{x^7}{95040}} \approx e \left(1 + \frac{13x^{10}}{95800320}\right) \\
(36) & \quad \left(\frac{1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720}}{1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720}}\right)^{\frac{1}{2} + \frac{x^5}{720} + \frac{x^7}{11440} + \frac{x^9}{95040}} \approx e \left(1 + \frac{x^{10}}{22809600}\right) \\
(37) & \quad \left(\frac{1 + \frac{x}{2} + \frac{3x^2}{28} + \frac{x^3}{84} + \frac{x^4}{1188} + \frac{3x^6}{39276}}{1 - \frac{x}{2} + \frac{3x^2}{28} - \frac{x^3}{84} + \frac{x^4}{1188} + \frac{3x^6}{39276}}\right)^{\frac{1}{2} + \frac{x^7}{1209600}} \approx e \left(1 - \frac{23x^{10}}{479001600}\right),
\end{align*}
\]

where \( x = \frac{1}{n} \).

3.2 Generalized Padé Approximants

A further comparison is possible if we make another choice for the base. Here, we utilize the \([m,m]\) Padé rational approximants of the exponential function (44) evaluated at \( x = 1/n \). Coincidentally, \( m = 1 \) and \( m = 2 \) correspond to equations (34) and (35) respectively. For \( m = 3 \) and \( 4 \), we have

\[
\begin{align*}
(38) & \quad \left(\frac{1 + \frac{x}{2} + \frac{3x^2}{28} + \frac{x^3}{84} + \frac{x^4}{1188} + \frac{3x^6}{39276}}{1 - \frac{x}{2} + \frac{3x^2}{28} - \frac{x^3}{84} + \frac{x^4}{1188} + \frac{3x^6}{39276}}\right)^{\frac{1}{2} + \frac{x^7}{2592000}} \approx e \left(1 + \frac{x^{10}}{190080000}\right) \\
(39) & \quad \left(\frac{1 + \frac{x}{2} + \frac{3x^2}{28} + \frac{x^3}{84} + \frac{x^4}{1188} + \frac{3x^6}{39276}}{1 - \frac{x}{2} + \frac{3x^2}{28} - \frac{x^3}{84} + \frac{x^4}{1188} + \frac{3x^6}{39276}}\right)^{\frac{1}{2} + \frac{x^7}{2592000}} \approx e \left(1 + \frac{x^{10}}{869299200}\right).
\end{align*}
\]
Remark 2. Following the same line of thought as Knox and Brothers, we could also consider taking a linear combination of the sequences (34)-(39), to produce more accurate approximations of $e$.

4 Numerical Results

In Table (3) we compare sequences (34)-(39) to Khattri’s GLEE5 formula (10). While we see 10th order convergence in all cases, the error constants vary by up to four orders of magnitude. The most accurate among these are the $[3, 3]$ and $[4, 4]$ Padé approximant sequences (38), (39). The most accurate result is given by the [4, 4] Padé sequence (39) with $n = 1000$, and yields

\[
\left( \frac{168084018002000001}{16791601799800001} \right)^{1000+\frac{10^{-23}}{254016}} = 2.718281828459045235360287471352662497754 \ldots
\]

This gives $e$ to 39 decimals of accuracy, and all but the final digit shown are correct.

5 Conclusion

We have shown how straightforward use of Taylor series and Cauchy products can be used to produce sequences which converge rapidly to the constant $e$. In our sequences, the error is of the form $O(n^{-M})$, where $M > 0$ is any integer. While our methods are similar to those of Knox and Brothers or of Khattri, our results are distinguished by the fact that there is no ambiguity about how to proceed for increasing $M$.

Such sequences often find practical use in the proof of monotonicity bounds in number theory [5, 6], and may be useful for verification of large-digit calculations of $e$, such as those based on the Maclaurin series (2) [14]. Finally, it is very likely that this work can be further extended in new directions to produce more sequences
Sequences which converge to \( e \). For example, another way to proceed is to replace the exponent \( c(n) \) with a Padé approximant for the function \( 1/[\ln(1+1/n)] \), producing yet another variant of equations (20)-(24)

\[
\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{2}{1 + \frac{2n}{m}}\right) = e \left(1 - \frac{1}{36n^3} + \ldots\right),
\]

\[
\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{6}{1 + \frac{12n}{m\pi}}\right) = e \left(1 - \frac{1}{600n^5} + \ldots\right),
\]

\[
\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{20}{1 + \frac{20n}{m}} - \frac{2}{3m} - \frac{4}{15m^3}\right) = e \left(1 - \frac{1}{9800n^7} + \ldots\right).
\]

We leave this subject to future work.

Acknowledgment

This paper is dedicated to the memory of Brian J. McCartin, a great mentor, professor and scholar, who also shared a deep appreciation for \( e \).

A Useful formulas

The Taylor series of the natural log is

\[(40) \ln (1 + x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k + 1} = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots,\]

and that of the exponential function is

\[(41) e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\]

The Bernoulli numbers \( B_k \) are defined by

\[(42) \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.\]

The first few Bernoulli numbers are \( 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42} \). They satisfy the following recurrence relation for each integer \( m \geq 1 \)

\[(43) \sum_{k=0}^{m} \frac{B_k}{k!(m - k + 1)!} = 0.\]
The \([m,n]\) Padé approximant of \(f(x)\) is a rational function of degree \(m\) in the numerator, and \(n\) in the denominator which agrees with the Taylor series of \(f\) up to \(m+n+1\) terms. The \([m,m]\) Padé approximant of the exponential function yields

\[
e^x \approx \frac{P_m(x)}{P_m(-x)} + \mathcal{O}(x^{2m+1}), \quad P_m(x) = \sum_{k=0}^{m} \frac{m!(2m-k)!}{(2m)!(m-k)!} x^k.
\]

References


Sequences which converge to $e$


Matthew F Causley
Kettering University
Assistant Professor
Mathematics Department
Flint, MI
e-mail: mcausley@kettering.edu

Peter Morell
Kettering University
Mathematics Department
Flint, MI
e-mail: more1885@kettering.edu
Some computational aspects of polycyclic aromatic hydrocarbons

V. Lokesha, Sushmitha Jain, T. Deepika and K.M. Devendraiah

Abstract
Topological indices acts a significant role in the study of physicochemical properties of chemical compounds. Polycyclic aromatic hydrocarbons $PAH_k$ have been the focus of great recognition since long time owing to their implication on public health and environment. $M$-polynomials is rich in producing closed forms of many degree-based topological indices which correlate chemical properties of the material under investigation. In this article, we established closed forms of various degree-based topological indices and M-polynomial for the semi total (line and point) and total graph of $PAH_k$.

2010 Mathematics Subject Classification: 05C05, 05C35, 05C12.
Key words and phrases: Polycyclic aromatic hydrocarbons, topological indices, $M$-polynomials, graph operators.

1 Introduction

Even though Wiener was working on boiling point of paraffin in 1947 but in first time the idea of topological index popularly known from his work. This index he called as Wiener index and then theory of topological index emerged. Topological indices are numerical tendencies that often depict quantitative structural activity/property/toxicity relationships and correlate certain physico-chemical properties, such as boiling point, stability, and strain energy of respective nanomaterial. Cheminformatics is one more emerging field in which quantitative structure activity relationships and quantitative structure property relationships [4] predict the biological activities and properties of the nano-material. In these studies, some physicochemical properties and topological indices such as zagreb index, generalized randić index, general sum-connectivity index, atom-bond connectivity index,

---

Received 20 July 2017
Accepted for publication (in revised form) 6 September 2017

175
geometric-arithmetic index and harmonic index are used to predict bioactivity of the chemical compounds. Algebraic polynomials have also more applications in chemistry, such as Hosoya polynomial, which plays a key role in measuring distance based topological indices. Among other algebraic polynomials, the $M$-polynomial \[3\], introduced in 2015, plays the same role in determining the closed form of certain degree-based topological indices. The main advantage of $M$-polynomial is the gain of information it contains about degree-based graph invariants. Polycyclic aromatic hydrocarbons (PAH$_k$) are consists of large group of carcinogenic compounds. These compounds consist of two or more fused aromatic rings made up of carbon and hydrogen atoms in their chemical structures. PAH$_k$ are aromatic hydrocarbons with two or more fused benzene rings with natural as well as anthropogenic sources. They are generally distributed environmental contaminants that have detrimental biological effects, toxicity, mutagenecity and carcinogenicity. It is known that the general representation of the polycyclic aromatic hydrocarbon PAH$_k$ has $6k^2$ carbon(C) atoms and $6k$ hydrogen(H) atoms. This polycyclic aromatic hydrocarbons (or PAH$_k$ family) have very similar properties to the benzenoid system (circumcoronene homologous series of benzenoid). The $M$-polynomial \[3\], introduced in 2015, plays the same role in determining the closed form of certain degree-based topological indices. The main advantage of $M$-polynomial is the gain of information it contains about degree-based graph invariants.

The first and second zagreb indices were introduced more than thirty years ago by Gutman and Trinajstic \[11\] which are defined as

\[
M_1(G) = \sum_{uv \in E(G)} \left[ d(u) + d(v) \right] \\
M_2(G) = \sum_{uv \in E(G)} \left[ d(u)d(v) \right]
\]

On the other hand, the third zagreb index \[6\], was introduced by Fath-tabar.

\[
M_3(G) = \sum_{uv \in E(G)} \left| d(u) - d(v) \right|
\]

In the study of structure-dependency of the total $\pi$-electron energy, the zagreb index was most studied topological index. Recently, Furtula and Gutman \[8\] renamed this topological index as a forgotten index or the $F$-index in 2015 and defined it as

\[
F(G) = \sum_{uv \in E(G)} \left( d(u)^2 + d(v)^2 \right)
\]
A new version of zagreb indices named hyper-zagreb index was introduced by Shirdel et. al. in 2013 \cite{17} defined as,

\begin{equation}
HM(G) = \sum_{uv \in E(G)} [d(u) + d(v)]^2
\end{equation}

The symmetric division degree index is one of the discrete adriatic indices that is good predictor of total surface area for polychlorobiphenyls. The symmetric division degree index of a connected graph $G$, is defined as

\begin{equation}
SDD(G) = \sum_{uv \in E(G)} \frac{d_u^2 + d_v^2}{d_ud_v}
\end{equation}

For more details, we refer to \cite{9, 10, 14, 15}.

The general sum-connectivity index is defined as

\begin{equation}
\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}
\end{equation}

One of the well-known degree based topological indices is atom-bond connectivity index, proposed by Estrada et. al., In \cite{5}, which was defined for modeling the enthalpy for formation of alkanes.

\begin{equation}
ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_ud_v}}
\end{equation}

The Randić index, \cite{16} denoted by $R_\frac{1}{2}(G)$ and introduced by Milan Randić in 1975, is also one of the oldest topological index. The Randić index is defined as

\begin{equation}
R_\frac{1}{2}(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_ud_v}}
\end{equation}

In 1998, working independently, Bollobas and Erdos \cite{11} and Amic et. al., proposed the generalized randić index, defined as

\begin{equation}
R_\alpha(G) = \sum_{uv \in E(G)} \frac{1}{(d_ud_v)^\alpha}
\end{equation}

For more details, we refer to \cite{13, 20}.

Another variant of the Randić index is the harmonic index \cite{19} defined as

\begin{equation}
H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}
\end{equation}
The $M$-polynomial \[3\] of graph $G$ is defined as

\[
M(G, x, y) = \sum_{i \leq j} m_{ij}(G) x^i y^j
\]

where $m_{ij}(G)$, $(i, j \geq 1)$ be the number of edges $e = uv$ of $G$ such that $(d_u, d_v) = (i, j)$.

The following Table 1 relates some well-known degree-based topological indices with $M$-polynomials.

<table>
<thead>
<tr>
<th>Topological index</th>
<th>Derivation from $M(G; x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>First zagreb</td>
<td>$(D_x + D_y)(M(G; x, y)) \mid_{x=y=1}$</td>
</tr>
<tr>
<td>Second zagreb</td>
<td>$(D_x D_y)(M(G; x, y)) \mid_{x=y=1}$</td>
</tr>
<tr>
<td>Second modified zagreb</td>
<td>$(S_x S_y)(M(G; x, y)) \mid_{x=y=1}$</td>
</tr>
<tr>
<td>Randić</td>
<td>$(D^\alpha_x D^\alpha_y)(M(G; x, y)) \mid_{x=y=1}$</td>
</tr>
<tr>
<td>Generalized randić</td>
<td>$(S^\alpha_x S^\alpha_y)(M(G; x, y)) \mid_{x=y=1}$</td>
</tr>
<tr>
<td>Symmetric Division Degree</td>
<td>$(D_x S_y + S_x D_y)(M(G; x, y)) \mid_{x=y=1}$</td>
</tr>
</tbody>
</table>

where $D_x = \frac{\partial (f(x,y))}{\partial x}$, $D_y = \frac{\partial (f(x,y))}{\partial y}$, $S_x = \int_0^x \frac{f(t,y)}{t} dt$ and $S_y = \int_0^y \frac{f(x,t)}{t} dt$.

Recently, R. Farahani, Wang et. al. \[7\] computed omega polynomial of polycyclic aromatic hydrocarbons, motivated from these we computed topological indices and $M$-polynomials for polycyclic aromatic hydrocarbons.

This paper is organized as follows. Section 1 consists of a brief introduction and literature review which is essential for the development of main results. forthcoming three sections, we shall give the topological indices and $M$-polynomials of the graph semi total (line and point) and total graphs for polycyclic aromatic hydrocarbons.

### 2 The case on semi total line graph

In this section, we computed the some degree based topological indices and its $M$-polynomials of the semi total line graphs of $PAH_k$.

<table>
<thead>
<tr>
<th>Number of edges</th>
<th>$(d_u, d_v)$ where $uv \in E(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6k$</td>
<td>(1,4)</td>
</tr>
<tr>
<td>$6k$</td>
<td>(3,4)</td>
</tr>
<tr>
<td>$6k(3k-1)$</td>
<td>(3,6)</td>
</tr>
<tr>
<td>$12k$</td>
<td>(6,4)</td>
</tr>
<tr>
<td>$3(4k^2 + k - 3)$</td>
<td>(6,6)</td>
</tr>
</tbody>
</table>
Theorem 1. Let $G_1$ be the semi total line graph of the $PAH_k$ then

1. $M_1(G_1) = 306k^2 + 174k - 108$.
2. $M_2(G_1) = 594k^2 + 384k - 324$.
3. $M_3(G_1) = 54k^2 + 30k$.
4. $F(G_1) = 1674k^2 + 1794k - 1944$.
5. $H(G_1) = \frac{1}{10}1260k^2 + 1298k - 315$.
6. $HM(G_1) = 3186k^2 + 1590k - 1296$.
7. $R(G_1) = 2k^2 + 10.731k - 0.08$.
8. $\chi(G_1)) = 7.732k^2 + 7.611k - 2.598$.
9. $ABC(G_1) = 17.549k^2 + 11k - 4.743$.
10. $SDD(G_1) = 117k^2 + 136k - 261$.

Proof. The semi total line graph of $PAH_k$ for $k=2$ is shown in Figure 1. It can easily be calculated that the total number of vertices/atoms in $PAH_k$ molecular graph is $6k(k+1)$ and among them $6k^2$ are carbon atoms and $6k$ are hydrogen atoms. Thus, the number of edges in this hydrocarbon molecule $PAH_k$ is $3k(3k+1)$. Now, if we partition the edges of $E_{(1,4)}, E_{(3,4)}, E_{(3,6)}, E_{(6,4)}$ and $E_{(6,6)}$. 

![Figure 1: PAH$_2$ of semi-total line graph operator](image-url)
Let, \( M_1(G) = \sum_{u,v \in E(G)} [d(u) + d(v)] \)

\[ M_1(G_1) = |E(1,4)| \sum_{u,v \in E(1,4)(G_1)} [d(u) + d(v)] + |E(3,4)| \sum_{u,v \in E(3,4)(G_1)} [d(u) + d(v)] \]

\[ + |E(3,6)| \sum_{u,v \in E(3,6)(G_1)} [d(u) + d(v)] + |E(6,4)| \sum_{u,v \in E(6,4)(G_1)} [d(u) + d(v)] \]

\[ + |E(6,6)| \sum_{u,v \in E(6,6)(G_1)} [d(u) + d(v)] \]

\[ = 306k^2 + 174k - 108. \]

Let, \( M_2(G) = \sum_{u,v \in E(G)} [d(u)d(v)] \)

\[ M_2(G_1) = |E(1,4)| \sum_{u,v \in E(1,4)(G_1)} [d(u)d(v)] + |E(3,4)| \sum_{u,v \in E(3,4)(G_1)} [d(u)d(v)] \]

\[ + |E(3,6)| \sum_{u,v \in E(3,6)(G_1)} [d(u)d(v)] + |E(6,4)| \sum_{u,v \in E(6,4)(G_1)} [d(u)d(v)] \]

\[ + |E(6,6)| \sum_{u,v \in E(6,6)(G_1)} [d(u)d(v)] \]

\[ = 594k^2 + 384k - 324. \]

Let, \( M_3(G) = \sum_{u,v \in E(G)} |d(u) - d(v)| \)

\[ M_3(G_1) = |E(1,4)| \sum_{u,v \in E(1,4)(G_1)} |d(u) - d(v)| + |E(3,4)| \sum_{u,v \in E(3,4)(G_1)} |d(u) - d(v)| \]

\[ + |E(3,6)| \sum_{u,v \in E(3,6)(G_1)} |d(u) - d(v)| + |E(6,4)| \sum_{u,v \in E(6,4)(G_1)} |d(u) - d(v)| \]

\[ + |E(6,6)| \sum_{u,v \in E(6,6)(G_1)} |d(u) - d(v)| \]

\[ = 54k^2 + 30k. \]
Let, $F(G) = \sum_{u,v \in E(G)} [d(u)^2 + d(v)^2]$

$$F(G_1) = |E_{(1,4)}| \sum_{u,v \in E_{(1,4)}(G_1)} [d(u)^2 + d(v)^2] + |E_{(3,4)}| \sum_{u,v \in E_{(3,4)}(G_1)} [d(u)^2 + d(v)^2] + |E_{(3,6)}| \sum_{u,v \in E_{(3,6)}(G_1)} [d(u)^2 + d(v)^2] + |E_{(6,4)}| \sum_{u,v \in E_{(6,4)}(G_1)} [d(u)^2 + d(v)^2] + |E_{(6,6)}| \sum_{u,v \in E_{(6,6)}(G_1)} [d(u)^2 + d(v)^2]$$

$$= 1674k^2 + 1794k - 1944.$$

Let, $H(G) = \sum_{u,v \in E(G)} \frac{2}{(d_u + d_v)}$

$$H(G_1) = |E_{(1,4)}| \sum_{u,v \in E_{(1,4)}(G_1)} \frac{2}{(d_u + d_v)} + |E_{(3,4)}| \sum_{u,v \in E_{(3,4)}(G_1)} \frac{2}{(d_u + d_v)} + |E_{(3,6)}| \sum_{u,v \in E_{(3,6)}(G_1)} \frac{2}{(d_u + d_v)} + |E_{(6,4)}| \sum_{u,v \in E_{(6,4)}(G_1)} \frac{2}{(d_u + d_v)} + |E_{(6,6)}| \sum_{u,v \in E_{(6,6)}(G_1)} \frac{2}{(d_u + d_v)}$$

$$= \frac{1}{210} (1260k^2 + 1298k - 315).$$

Let, $HM(G) = \sum_{u,v \in E(G)} [d(u) + d(v)]^2$

$$HM(G_1) = |E_{(1,4)}| \sum_{u,v \in E_{(1,4)}(G_1)} [d(u) + d(v)]^2 + |E_{(3,4)}| \sum_{u,v \in E_{(3,4)}(G_1)} [d(u) + d(v)]^2 + |E_{(3,6)}| \sum_{u,v \in E_{(3,6)}(G_1)} [d(u) + d(v)]^2 + |E_{(6,4)}| \sum_{u,v \in E_{(6,4)}(G_1)} [d(u) + d(v)]^2 + |E_{(6,6)}| \sum_{u,v \in E_{(6,6)}(G_1)} [d(u) + d(v)]^2$$

$$= 3186k^2 + 1590k - 1296.$$
Let, \( R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}} \)

\[
R(G_1) = |E_{(1,4)}| \frac{1}{\sqrt{d_u d_v}} + |E_{(3,4)}| \frac{1}{\sqrt{d_u d_v}} + |E_{(3,6)}| \frac{1}{\sqrt{d_u d_v}} + |E_{(6,4)}| \frac{1}{\sqrt{d_u d_v}} + |E_{(6,6)}| \frac{1}{\sqrt{d_u d_v}} = 2k^2 + 10.731k - 0.08.
\]

Let, \( \chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}} \)

\[
\chi(G_1) = |E_{(1,4)}| \frac{1}{\sqrt{d_u + d_v}} + |E_{(3,4)}| \frac{1}{\sqrt{d_u + d_v}} + |E_{(3,6)}| \frac{1}{\sqrt{d_u + d_v}} + |E_{(6,4)}| \frac{1}{\sqrt{d_u + d_v}} + |E_{(6,6)}| \frac{1}{\sqrt{d_u + d_v}} = 7.732k^2 + 7.611k - 2.598.
\]

Let, \( ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \)

\[
ABC(G_1) = |E_{(1,4)}| \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} + |E_{(3,4)}| \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} + |E_{(3,6)}| \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} + |E_{(6,4)}| \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} + |E_{(6,6)}| \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} = 17.549k^2 + 11k - 4.743.
\]
Theorem 2. Let $G$ be the semi total line graph of PAH$_K$ then $M$-polynomial is
1. $M(G_1, x, y) = 6kxy^4 + 6kx^3y^4 + 6k(3k - 1)x^3y^6 + 12kx^3y^6 + 3(4k^2 + k - 3)x^6y^6. 
2. $M_1(G_1) = 306k^2 + 174k - 108.
3. $M_2(G_1) = (126k^2 + 72k - 54)(180k^2 + 102k - 54).
4. $mM_2(G_1) = \frac{1}{4}(160k^3 + 380k^3 + 138k^2 - 99k + 9).
5. $R_a(G_1) = [(126k^2 + 72k - 54)(180k^2 + 102k - 54)]^a.
6. $R'_a(G_1) = [\frac{1}{4}(160k^4 + 380k^3 + 138k^2 - 99k + 9)]^a.
7. $SDD(G_1) = 3(690k^4 + 1301k^3 + 129k^2 + 231k + 54).

Proof. Let, $M(G_1, x, y) = \sum_{i,j} m_{ij}(G)x^iy^j$
$$= \sum_{1 \leq i \leq 4} m_{14}(G)x^iy^4 + \sum_{3 \leq i \leq 6} m_{34}(G)x^3y^4 + \sum_{3 \leq i \leq 6} m_{36}(G)x^3y^6$$
$$+ \sum_{4 \leq i \leq 6} m_{46}(G)x^4y^6 + \sum_{6 \leq i \leq 6} m_{66}(G)x^6y^6.$$
\[ D_x = 6kxy^4 + 18kx^2y^4 + 18k(3k - 1)x^3y^6 + 48kx^4y^6 + 18(4k^2 + k - 3)x^6y^6 \]
\[ D_x|_{x=y=1} = 126k^2 + 72k - 54. \]
\[ D_y = 24kxy^4 + 24kx^3y^4 + 36k(3k - 1)x^3y^6 + 72kx^4y^6 + 18(4k^2 + k - 3)x^6y^6 \]
\[ D_y|_{x=y=1} = 180k^2 + 102k - 54. \]
\[ S_x = 6ky^4 + \frac{6kx^3y^4}{3} + \frac{6k(3k - 1)x^3y^6}{3} + \frac{12kx^4y^6}{4} + \frac{3(4k^2 + k - 3)x^6y^6}{6} \]
\[ S_x|_{x=y=1} = \frac{16k^2 + 19k - 3}{2}. \]
\[ S_y = \frac{6ky^4}{4} + \frac{6kx^3y^4}{4} + \frac{6k(3k - 1)x^3y^6}{6} + \frac{12kx^4y^6}{6} + \frac{3(4k^2 + k - 3)x^6y^6}{6} \]
\[ S_y|_{x=y=1} = \frac{10k^2 + 9k - 3}{2}. \]

3 The case on semi total point graph

In this section, we computed the topological indices and \( M \)-polynomials of the semi total point graphs of \( \text{PAH}_k \).

Figure 2: \( \text{PAH}_2 \) of semi-total point graph operator
**Theorem 3.** Let $G_2$ be the semi total point graph of the PAH<sub>k</sub> then

1. $M_1(G_2) = 252k^2 + 36k.$
2. $M_2(G_2) = 540k^2 - 12k.$
3. $M_3(G_2) = 72k^2 + 24k.$
4. $F(G_2) = 1368k^2 + 72k.$
5. $H(G_2) = 3k^2 - 2k.$
6. $HM(G_2) = 2448k^2 + 48k.$
7. $R(G_2) = 6.69k^2 + 4.233k.$
8. $\chi(G_2)) = 8.95k^2 + 4.26k.$
9. $ABC(G_2) = 17.479k^2 + 5.666k.$
10. $SDD(G_2) = 84k^2 + 14k.$

**Proof.** The semi total point graph of PAH<sub>k</sub> for $k = 2$ is shown in Figure 2. We partition the size of $G_2$ into edges of the type $E_{(d(u), d(v))}$ where $uv$ is an edge. In $G_2$ edges of the type $E_{(2,2)}, E_{(2,6)}$ and $E_{(6,6)}$. The number of the edges of these types are given in the table 3. The proof technique is similar as theorem 1 and utilizing the contents of the table 3.

**Theorem 4.** Let $G_2$ be the semi total point graph of PAH<sub>K</sub> then $M$-polynomial is

1. $M(G_2, x, y) = 6kx^2y^2 + 6k(3k - 1)x^2y^6 + 3(3k^2 - k)x^6y^6.$
2. $M_1(G_2)) = 258k^2 + 36k.$
3. $M_2(G_2) = 15552k^4 + 3852k^3 + 180k^2.$
4. $mM_2(G_2) = \frac{1}{3}(261k^4 - 99k^3 - 77k^2 + 247k).$
5. $R_\alpha(G_2) = [\frac{1}{3}(261k^4 - 99k^3 - 77k^2 + 247k)]^\alpha.$
6. $R'_\alpha(G_2) = [\frac{1}{3}(261k^4 - 99k^3 - 77k^2 + 247k)]^\alpha.$
7. $SDD(G_2) = 3(711k^4 - 71k^3 - 48k^2).$

**Proof.** Let, $M(G_2, x, y) = \sum_{i+j} m_{ij}(G)x^iy^j$

<table>
<thead>
<tr>
<th>Number of edges</th>
<th>$(d_u, d_v)$ where $uv \in E(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6k$</td>
<td>$(2,2)$</td>
</tr>
<tr>
<td>$6k(3k + 1)$</td>
<td>$(2,6)$</td>
</tr>
<tr>
<td>$3(3k^2 - k)$</td>
<td>$(6,6)$</td>
</tr>
</tbody>
</table>


\[
M(G_2, x, y) = \sum_{2 \leq i \leq 6} m_{i,j}(G)x^i y^j
= \sum_{2=2} m_{22}(G)x^2 y^2 + \sum_{2 \leq 6} m_{26}(G)x^2 y^6 + \sum_{6=6} m_{66}(G)x^6 y^6
= \sum_{uv \in E_{(2,2)}} m_{22}(G)x^2 y^2 + \sum_{uv \in E_{(2,6)}} m_{26}(G)x^2 y^6
+ \sum_{uv \in E_{(6,6)}} m_{66}(G)x^6 y^6
= |E_{(2,2)}| x^2 y^2 + |E_{(2,6)}| x^2 y^6 + \frac{1}{3}(261k^4 - 99k^3 - 77k^2 + 247k)^\alpha.\]
Hence, we get the required $M$-polynomial equation. Thus,

$$
D_x = 12kx^2y^2 + 12k(3k + 1)x^2y^6 + 18k(3k - 1)x^6y^6
$$

$$
D_x|_{x=y=1} = 6k(16k + 1).
$$

$$
D_y = 12kx^2y^2 + 36k(3k + 1)x^2y^6 + 18k(3k - 1)x^6y^6
$$

$$
D_y|_{x=y=1} = 180k^2 + 102k - 54.
$$

With these cardinalities substituting in topological indices definitions we get required results.

4 The case on total graph

In this section, we computed the topological indices and $M$-polynomials of the total graphs of $PAH_k$.

**Theorem 5.** Let $G_3$ be the total graph of the $PAH_k$ then

1. $M_1(G_3) = 6(158k^2 - 232k + 156).$
2. $M_2(G_3) = 4(645k^2 - 907k + 510).$
3. $M_3(G_3) = 12(6k^2 + k + 3).$
4. $F(G_3) = 12(454k^2 - 690k + 420).$
5. $H(G_3) = 15k^2 - 19k + 18.$
6. $HM(G_3) = 16(663k^2 - 1041k + 630).$
7. $R(G_3) = 17.19k^2 - 18.20k + 18.$
8. $\chi(G_3) = 18.363k^2 - 15.85k + 62.$
9. $ABC(G_3) = 29.75k^2 - 21.111k + 50.055.$
10. $SDD(G_3) = 12(6k^2 + k + 6).$

**Proof.** The total graph of $PAH_k$ for $k = 2$ is shown in Figure 3. We partition the size of $G$ into edges of the type $E_{(d(u),d(v))}$ where $uv$ is an edge. In $G$ edges of the type $E_{(2,2)}$, $E_{(2,4)}$, $E_{(2,6)}$, $E_{(4,6)}$, $E_{(4,4)}$ and $E_{(6,6)}$. The number of the edges of these types are given in the table 4. The proof is analogous to the above theorem 1.
Some computational aspects of polycyclic aromatic hydrocarbons

Figure 3: PAH\(_2\) of total graph operator

Table 4: Edge partition of total graph PAH\(_k\)

<table>
<thead>
<tr>
<th>Number of edges</th>
<th>((d_u, d_v)) where (uv \in E(G))</th>
</tr>
</thead>
<tbody>
<tr>
<td>6k</td>
<td>(2,2)</td>
</tr>
<tr>
<td>6k</td>
<td>(2,4)</td>
</tr>
<tr>
<td>18k(^2)</td>
<td>(2,6)</td>
</tr>
<tr>
<td>18k</td>
<td>(4,6)</td>
</tr>
<tr>
<td>6(k(^2) - 5k + 6)</td>
<td>(4,4)</td>
</tr>
<tr>
<td>63k(^2) - 117k + 54</td>
<td>(6,6)</td>
</tr>
</tbody>
</table>

Theorem 6. Let \(G_3\) be the total graph of PAH\(_k\) then M-polynomial is

1. \(M(G_3, x, y) = 6kx^2y^2 + 6kx^2y^4 + 18k^2x^2y^6 + 6(k^2 - 5k + 6)x^4y^4 + (63k^2 - 117k + 54)x^6y^6.\)
2. \(M_1(G_3) = 12(79k^2 - 132k + 78).\)
3. \(M_2(G_3) = 36(6205k^4 - 20868k^3 + 29747k^2 - 20592k + 26).\)
4. \(mM_2(G_3) = \frac{3}{2}(210k^4 - 525k^3 + 642k^2 - 417k + 126).\)
5. \(R_\alpha(G_3) = (36(6205k^4 - 20868k^3 + 29747k^2 - 20592k + 26))^\alpha.\)
6. \(R'_\alpha(G_3) = (\frac{3}{2}(210k^4 - 525k^3 + 642k^2 - 417k + 126))^\alpha.\)
7. \(SDD(G_3) = 9(1920k^4 + 5449k^3 + 6140k^2 - 4765k + 1482).\)
Proof. Let, \( M(G, x, y) = \sum_{i \leq j} m_{ij}(G)x^iy^j \)
\[
= \sum_{2=2} m_{22}(G)x^2y^2 + \sum_{2 \leq 4} m_{24}(G)x^2y^4 + \sum_{2 \leq 6} m_{26}(G)x^2y^6 \\
+ \sum_{4 \leq 4} m_{44}(G)x^4y^4 + \sum_{6 \leq 6} m_{66}(G)x^6y^6. \\
= \sum_{uv \in E(2,2)} m_{22}(G)x^2y^2 + \sum_{uv \in E(2,4)} m_{24}(G)x^2y^4 \\
+ \sum_{uv \in E(2,6)} m_{26}(G)x^2y^6 + \sum_{uv \in E(4,4)} m_{44}(G)x^4y^4 \\
+ \sum_{uv \in E(6,6)} m_{66}(G)x^6y^6. \\
= |E(2,2)| x^2y^2 + |E(2,4)| x^2y^4 + |E(2,6)| x^2y^6 \\
+ |E(4,4)| x^4y^4 + |E(6,6)| x^6y^6.
\]

Hence, we get the required \( M \)-polynomial equation. We have,
\[
D_x = 12kx^2y^2 + 12kx^2y^4 + 36k^2x^2y^6 + 24(k^2 - 5k + 6)x^4y^4 + 378k^2 - 702k + 324x^6y^6 \\
D_x|_{x=y=1} = 3(146k^2 - 266k + 156) \\
D_y = 12kx^2y^2 + 24kx^2y^4 + 108k^2x^2y^6 + 24(k^2 - 5k + 6)x^4y^4 + 378k^2 - 702k + 324x^6y^6 \\
D_y|_{x=y=1} = 3(170k^2 - 262k + 156). \\
S_x = \frac{6kx^2y^2}{2} + \frac{6kx^2y^4}{2} + \frac{18k^2x^2y^6}{2} + \frac{6(k^2 - 5k + 6)x^4y^4}{4} + \frac{(63k^2 - 117k + 54)x^6y^6}{6} \\
S_x|_{x=y=1} = 21k^2 - 21k + 18 \\
S_y = \frac{6kx^2y^2}{2} + \frac{6kx^2y^4}{2} + \frac{18k^2x^2y^6}{4} + \frac{6(k^2 - 5k + 6)x^4y^4}{4} + \frac{(63k^2 - 117k + 54)x^6y^6}{6} \\
S_y|_{x=y=1} = \frac{30k^2 - 45k + 21}{2}.
\]

Hence applying these results in equations in topological indices definitions, we obtained required results.

Conclusion. In the field of chemical graph theory under the framework of semi-total(line and point) and total graph operator is a new phenomena in the area of structural chemistry. In this paper, we dealt with polycyclic aromatic hydrocarbons \( PAH_k \), studied certain topological indices and \( M \)-polynomials on these molecular structures which will be helpful in computational chemistry. Moreover, we have also computed the edge partition of each \( PAH_k \) based on end vertices of each edge, which can be used to compute many other topological indices.

Acknowledgement. The second author is thankful to Maulana Azad National Fellowship (MANF) for minority students F1-17.1 2017-18 /MANF-2017-18-KAR-76148.
References


**V. Lokesha, Sushmitha Jain, T. Deepika and K.M. Devendraiah**

Vijayanagara Sri Krishnadevaraya University  
Faculty of Science  
Mathematics  
Vijayanagara Sri Krishnadevaraya University, Ballari, India  
v.lokesha@gmail.com, sushmithajain9@gmail.com  
sastry.deepi@gmail.com and devendraiahkm@gmail.com