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Optimal control of a stochastic version of the Lotka-Volterra model

Adela Ionescu, Mario Lefebvre, Florian Munteanu

Abstract

We study a controlled dynamical system that reduces to the Lotka-Volterra model of competition between two species if the control variable is taken identically equal to 1. Next, a stochastic version of the feedback linearization of the system is considered. The aim is to maximize the time that the ratio of the number of individuals of each species remains between two acceptable limits, taking the quadratic control costs into account. An explicit solution is found by solving the partial differential equation satisfied by the value function.

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Key words and phrases: Stochastic control, Wiener process, feedback linearization.

1 Introduction

We consider the controlled system

\begin{align}
\dot{x}(t) &= ax(t) - b x(t)y(t) u(t), \\
\dot{y}(t) &= -cy(t) + dx(t)y(t) u(t),
\end{align}

where \(a, b, c\) and \(d\) are positive constants. If \(u(t) \equiv 1\), this system is the Lotka-Volterra model of competition between two species, which is a basic dynamic population model in two dimensions. The term \(ax(t)\) entails that, in the absence of predators, the prey population increases exponentially. Similarly, the term \(-cy(t)\) means that, in the absence of prey, the predator population decreases exponentially. Moreover, the term \(-bx(t)y(t)\) (respectively \(dx(t)y(t)\)) implies that the decrease of the prey population (resp. increase of the predator population) is proportional to
the frequency of the encounters between predators and prey, which is assumed to be a function of the product \( x(t) y(t) \); see [7] and [8].

Suppose that we want to keep the ratio \( x(t)/y(t) \) of the number of individuals of each species between two acceptable limits, \( k_1 \) and \( k_2 \), for as long as possible.

In the next section, we will compute the feedback linearization of the above system. Then, in Section 3, we will consider a stochastic version of this feedback linearization. An optimal control problem will be set up and solved explicitly in one special instance. This type of problem has been termed LQG homing by Whittle [9] and has been considered extensively by the second author in a series of papers; see, for instance, [4] and [5]. In general, the problems that could be solved explicitly so far are for one-dimensional systems. Makasu [6] obtained recently an explicit solution to a two-dimensional problem.

Finally, we will end with a few remarks in Section 4.

## 2 Feedback linearization

We rewrite the system in the form (see [1] and [3])

\[
\dot{x} = f(x) + g(x) \cdot u,
\]

with

\[
f(x) = \begin{pmatrix} ax \\ -cy \end{pmatrix} \quad \text{and} \quad g(x) = \begin{pmatrix} -bxy \\ dxy \end{pmatrix}.
\]

If \( T(x) \) is a diffeomorphism and \( z = T(x) \), then we have:

\[
\dot{z} = \frac{\partial T}{\partial x} \left[ f(x) + g(x) \cdot u \right].
\]

Since \( T \) is a diffeomorphism, \( T^{-1} \) exists and we have:

\[
x = T^{-1}(z).
\]

In order to achieve the feedback linearization of the controlled system, we must choose \( T(x) = (T_1(x), T_2(x))^t \) such that (see [2])

\[
\begin{cases}
\frac{\partial T_1(x)}{\partial x} \cdot g = 0, \\
\frac{\partial T_2(x)}{\partial x} \cdot g \neq 0, \\
\frac{\partial T_1(x)}{\partial x} \cdot f = T_2.
\end{cases}
\]

We find that

\[
T(x) = \begin{pmatrix} dx + by \\ adx - cby \end{pmatrix}.
\]
Using the transformation
\begin{align}
    z_1(t) &= dx(t) + by(t) \quad \text{and} \quad z_2(t) = adx(t) - cby(t),
\end{align}
we obtain that the feedback linearization of the controlled system is
\begin{align}
    \dot{z}_1(t) &= z_2(t), \\
    \dot{z}_2(t) &= a^2 dz_1(t) + bc^2 z_2(t) - (a + c) bdz_1(t) z_2(t) u(t).
\end{align}
Notice that we can write that
\begin{align}
    x(t) &= cz_1(t) + z_2(t) \\
    y(t) &= \frac{az_1(t) - z_2(t)}{b(a + c)}.
\end{align}
In the next section, a stochastic version of the system (10), (11) will be considered.

3 Stochastic control

We consider the controlled stochastic system
\begin{align}
    \dot{z}_1(t) &= z_2(t), \\
    \dot{z}_2(t) &= \frac{az_1(t) + z_2(t)}{d(a + c)} - \frac{az_1(t) - z_2(t)}{b(a + c)} + [v(z_1(t), z_2(t))]^{1/2} \dot{W}(t),
\end{align}
where $v(z_1(t), z_2(t))$ is a non-negative function, and \{W(t), t \geq 0\} is a standard Brownian motion.

Let $z_1(0) = z_1$ and $z_2(0) = z_2$. Based on the acceptable values $k_1$ and $k_2$ (with $0 < k_1 < k_2$) of the ratio $x(t)/y(t)$, we define the first-passage time
\begin{align}
    \tau(z_1, z_2) := \inf \left\{ t \geq 0 : \frac{z_2(t)}{z_1(t)} = \frac{dk_1 - bc}{dk_1 + b} \quad \text{or} \quad \frac{adk_2 - bc}{dk_2 + b} \right\},
\end{align}
and we consider the cost criterion
\begin{align}
    J(z_1, z_2) = \int_0^{\tau(z_1, z_2)} \left\{ \frac{1}{2} q_0 z_1(t) z_2(t) u^2(t) + \gamma \right\} dt,
\end{align}
where $q_0 > 0$ and $\gamma < 0$ are constants. We look for the control $u(t)$ that minimizes the expected value of $J(z_1, z_2)$.

Remark 1 Because $\gamma$ is negative, the aim is to maximize the time that the ratio $z_2(t)/z_1(t)$ remains between the two boundaries, taking the quadratic control costs into account. Moreover, the term $z_1(t) z_2(t)$ in front of $u^2(t)$ is because the larger the number of individuals is, the more expensive it should be to control the two species.
Remark 2 We assume that \(adk_1 - bc \geq 0\), so that \(z_2(t)\) will also be non-negative between the initial time \(t = 0\) and the stopping time \(\tau(z_1, z_2)\).

Let \(F(z_1, z_2)\) be the value function; that is,

\[
F(z_1, z_2) := \inf_{u(t), 0 \leq t \leq \tau(z_1, z_2)} E[J(z_1, z_2)].
\]

Making use of dynamic programming, we find that \(F\) is such that

\[
0 = \frac{1}{2} q_0 z_1 z_2 u^2 + \gamma + z_2 F_{z_1} + (a^2 d z_1 + b c^2 z_2) F_{z_2}^2
- (a + c) b d z_1 z_2 u F_{z_2} + \frac{1}{2} v(z_1, z_2) F_{z_2 z_2},
\]

where \(u = u(0)\)

Differentiating Equation (17) with respect to \(u\), we obtain that the optimal control is given by

\[
u^* = \frac{(a + c) b d}{q_0} F_{z_2} := \kappa F_{z_2}.\]

Substituting this expression into (17), we obtain that \(F\) satisfies the second-order non-linear partial differential equation

\[
0 = -\frac{1}{2} q_0 \kappa^2 z_1 z_2 (F_{z_2})^2 + \gamma + z_2 F_{z_1} + (a^2 d z_1 + b c^2 z_2) F_{z_2}^2 + \frac{1}{2} v(z_1, z_2) F_{z_2 z_2}.
\]

The boundary conditions are

\[
F(z_1, z_2) = 0 \quad \text{if} \quad \frac{z_2}{z_1} = \frac{adk_1 - bc}{dk_1 + b} \quad \text{or} \quad \frac{adk_2 - bc}{dk_2 + b}.
\]

Moreover, because \(\gamma\) is negative, we must have:

\[
F(z_1, z_2) \leq 0.
\]

Next, assume that

\[
v(z_1, z_2) = \sigma^2_0 z_1 z_2 (\geq 0)
\]

and let

\[
\Phi(z_1, z_2) := e^{-\alpha F(z_1, z_2)},
\]

where

\[
\alpha := \frac{[(a + c) b d]^2}{q_0 \sigma^2_0} \quad (> 0).
\]
We find that the function $\Phi(z_1, z_2)$ satisfies the second-order linear partial differential equation

$$-\gamma \alpha \Phi + z_2 \Phi_{z_1} + (a^2 d z_1 + b c^2 z_2) \Phi_{z_2} + \frac{1}{2} \sigma_0^2 z_1 z_2 \Phi_{z_1 z_2} = 0,$$

subject to

$$\Phi(z_1, z_2) = 1 \text{ if } \frac{z_2}{z_1} = \frac{a d k_1 - b c}{d k_1 + b} \text{ or } \frac{a d k_2 - b c}{d k_2 + b}.$$

Finally, let us try a solution of the form

$$\Phi(z_1, z_2) = \Psi(z),$$

where $z := z_2/z_1$. We find that Equation (25) is transformed into the ordinary differential equation

$$-\gamma \alpha \Psi - z^2 \Psi' + (a^2 d + b c^2 z) \Psi' + \frac{1}{2} \sigma_0^2 z \Psi'' = 0.$$

The boundary conditions are

$$\Psi(z) = 1 \text{ if } z = \frac{a d k_1 - b c}{d k_1 + b} \text{ or } \frac{a d k_2 - b c}{d k_2 + b}.$$

Let us consider the following particular case: $a = 1/2$, $c = 1/3$, $b = d = 1$, $q_0 = \sigma_0^2 = 1$, $\gamma = -1$, $k_1 = 2/3$ and $k_2 = 5/3$. We first calculate

$$\alpha = \frac{25}{36}.$$

Equation (28) becomes

$$\frac{25}{36} \Psi - z^2 \Psi' + \left(\frac{1}{4} + \frac{1}{9} z\right) \Psi' + \frac{1}{2} z \Psi'' = 0,$$

and we find that the two boundaries are at 0 and 3/16.

Making use of the mathematical software Maple, we obtain that the general solution of Equation (31) can be written as follows:

$$\Psi(z) = c_1 \text{HeunB} \left( -\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{8}{3}, -z \right) + c_2 \sqrt{z} \text{HeunB} \left( \frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{8}{3}, -z \right),$$

where HeunB is a special function. The constants $c_1$ and $c_2$ are uniquely determined from the boundary conditions $\Psi(0) = \Psi(3/16) = 1$:

$$\Psi(z) = \text{HeunB} \left( -\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{8}{3}, -z \right)$$

$$- \frac{4}{3} \left( -1 + \text{HeunB} \left( -\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{8}{3}, -\frac{3}{16} \right) \sqrt{3z} \text{HeunB} \left( \frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{8}{3}, -z \right) \right).$$
Figures 1 and 2 show respectively the value function $F(z_1, z_2)$ and the optimal control when $z_1 = 1000$. Then, the variable $z_2$ belongs to the interval $[0; 187.5]$. Notice that the value function $F(z_1, z_2)$ is a function of the ratio $z_2/z_1$, but the optimal control is not, because it is expressed in terms of the partial derivative of $F(z_1, z_2)$ with respect to $z_2$.

Figure 1: Value function $F(z_1 = 1000, z_2)$ for $0 \leq z_2 \leq 187.5$.

Figure 2: Optimal control $u^*$ for $z_1 = 1000$ and $0 \leq z_2 \leq 187.5$.

4 Concluding remarks

In this note, we obtained an explicit solution to a stochastic optimal control problem for an important system in a particular case. For a different choice of the function $v(z_1, z_2)$, we could try to solve the appropriate partial differential equation by making use of numerical methods. Finally, we could also try to find a suboptimal control, either by making some approximations, or by choosing the form of the control variable (for instance, a linear control).
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A class of Aczél-Popoviciu type inequality

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Abstract
In this paper we give a new generalized and sharpened version of Aczél-Popoviciu inequality via positive and homogeneous functionals.

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Key words and phrases: Aczél’s inequality, Popoviciu’s inequality, positive functionals, convex functions, Hölder’s inequality.

1 Introduction
The Aczél’s inequality states that if $a_k, b_k, k = 1, n$ are positive numbers such that

$$a_1^2 - \sum_{k=2}^{n} a_k^2 > 0 \quad \text{or} \quad b_1^2 - \sum_{k=2}^{n} b_k^2 > 0$$

then

$$\left( a_1^2 - \sum_{k=2}^{n} a_k^2 \right) \left( b_1^2 - \sum_{k=2}^{n} b_k^2 \right) \leq \left( a_1 b_1 - \sum_{k=2}^{n} a_k b_k \right)^2$$

with equality if and only if $a_k, b_k, k = 1, n$ are proportional.


Theorem 1 (T. Popoviciu) Let $p \geq q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and let $a_i, b_i, i = 1, n$ be positive numbers such that

$$a_1^p - \sum_{i=2}^{n} a_i^p > 0 \quad \text{and} \quad b_1^q - \sum_{i=2}^{n} b_i^q > 0$$
then
\begin{equation}
\left( a_1^p - \sum_{i=2}^{n} a_i^p \right)^{\frac{1}{p}} \left( b_1^q - \sum_{i=2}^{n} b_i^q \right)^{\frac{1}{q}} \leq a_1 b_1 - \sum_{i=2}^{n} a_i b_i \tag{2}
\end{equation}

(\textit{exponential generalization of Aczél’s inequality}).

Many mathematicians have been given different proofs, various generalizations, improvements and applications of inequalities (1) and (2) (see and the references therein).

In this paper we give an inequality of Aczél-Popoviciu type via positive homogeneous functionals.

Let \( I \) be a nonempty subset of \( \mathbb{R} \) and \( L \) be a class of real valued functions on \( I \) such that, if \( f, g \in L \) and \( \lambda \) is a positive real number then:
\begin{enumerate}[a)]
  \item \( f \cdot g \in L \);
  \item \( |f|^p \in L \) for \( p \geq 0 \);
  \item \( \lambda f \in L \).
\end{enumerate}
Let \( A \) be a functional, \( A : L \rightarrow \mathbb{R} \), such that
\begin{enumerate}[i)]
  \item if \( f \in L \) and \( f \geq 0 \), then \( A(f) \geq 0 \);
  \item if \( \lambda \geq 0 \), \( f \in L \) follows that \( A(\lambda f) = \lambda A(f) \).
\end{enumerate}

We note that, if \( L \) is a linear set and \( A \) is a linear positive functional, then the following theorem holds:

**Theorem 2** (Hölder’s inequality) Let \( L \) be a linear class of real valued functions. If \( p, q > 0 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) with \( f, g \geq 0 \) and \( f^p, g^q \in L \), then
\begin{equation}
A(f g) \leq (A(f^p))^{\frac{1}{p}} (A(g^q))^{\frac{1}{q}}. \tag{3}
\end{equation}

**2 Main Result**

Let \( L \) be a class of real valued functions which satisfies the conditions a), b) and c) and \( A \) be a real functional defined on \( L \) which satisfies conditions i) and ii).

**Theorem 3** Let \( A \) be a positive and homogeneous functional defined on \( L \). Let \( p, q > 0 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and let \( f, g \in L \), \( f, g > 0 \) such that \( a^p \geq A(f^p) \), \( b^q \geq A(g^q) \), where \( a, b \) are two fixed positive numbers. Then the following inequality holds:
\begin{equation}
\left( a^p - A(f^p) \right)^{\frac{1}{p}} \left( b^q - A(g^q) \right)^{\frac{1}{q}} \leq ab - \max \left\{ \left( \frac{1}{p} \cdot \frac{b^p}{a^q} (A(f^p))^{\frac{1}{p}} + \frac{1}{q} \cdot \frac{a^p}{b^q} (A(g^q))^{\frac{1}{q}} \right)^p, \left( \frac{1}{p} \cdot \frac{b^p}{a^q} (A(f^p))^{\frac{1}{p}} + \frac{1}{q} \cdot \frac{a^p}{b^q} (A(g^q))^{\frac{1}{q}} \right)^q \right\}. \tag{4}
\end{equation}
Proof. Let

\[ x = \frac{(A(f^p))^\frac{1}{p}}{a} \in [0, 1] \quad \text{and} \quad y = \frac{(A(g^q))^\frac{1}{q}}{b} \in [0, 1]. \]

Using the Young’s inequality

\[ u^\frac{1}{p} \cdot v^\frac{1}{q} \leq \frac{1}{p} u + \frac{1}{q} v, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{for} \quad u = 1 - x^p \quad \text{and} \quad v = 1 - y^q \]

we have

\[ (1 - x^p)^\frac{1}{p} \leq \frac{1}{p} (1 - x^p) + \frac{1}{q} (1 - y^q) = 1 - \frac{1}{p} x^p - \frac{1}{q} y^q. \]

Because the functions \( t^p \) and \( t^q \) are convex on \([0, 1]\), by Jensen’s inequality we get

\[ \frac{1}{p} x^p + \frac{1}{q} y^q = \frac{1}{p} x^p + \frac{1}{q} y^{\frac{1}{p-1}} = \frac{1}{p} x^p + \frac{1}{q} \left( \frac{1}{x^{\frac{1}{p-1}}} \right)^p \geq \left( \frac{1}{p} x + \frac{1}{q} y^{\frac{1}{p-1}} \right)^p \]

\[ \frac{1}{p} x^p + \frac{1}{q} y^q = \frac{1}{p} x^{\frac{1}{q-1}} + \frac{1}{q} y^q = \frac{1}{p} \left( x^{\frac{1}{q-1}} \right)^q + \frac{1}{q} y^q \geq \left( \frac{1}{p} x^{\frac{1}{q-1}} + \frac{1}{q} y^q \right)^q. \]

So, from (6) and (7) follows

\[ (1 - x^p)^\frac{1}{p} (1 - y^q)^\frac{1}{q} \leq 1 - \max \left\{ \left( \frac{1}{p} x + \frac{1}{q} y^{\frac{1}{p-1}} \right)^p, \left( \frac{1}{p} x^{\frac{1}{q-1}} + \frac{1}{q} y^q \right)^q \right\}. \]

Using (5), the relation is equivalent to that

\[ (a^p - A(f^p))^\frac{1}{p} \cdot (b^q - A(g^q))^\frac{1}{q} \]

\[ \leq ab - \max \left\{ ab \left[ \frac{1}{p} \cdot \frac{(A(f^p))^\frac{1}{p}}{a} + \frac{1}{q} \left( \frac{(A(g^q))^\frac{1}{q}}{b} \right)^\frac{1}{p-1} \right]^p, \right. \]

\[ \left. ab \left[ \frac{1}{p} \left( \frac{(A(f^p))^\frac{1}{p}}{a} \right)^\frac{1}{q-1} + \frac{1}{q} \cdot \frac{(A(g^q))^\frac{1}{q}}{b} \right]^q \right\}. \]

So

\[ ab \left[ \frac{1}{p} \cdot \frac{(A(f^p))^\frac{1}{p}}{a} + \frac{1}{q} \left( \frac{(A(g^q))^\frac{1}{q}}{b} \right)^\frac{1}{p-1} \right]^p \]

\[ = \left[ a^{\frac{1}{p} b^\frac{1}{p}} \cdot \frac{1}{p} \cdot \frac{(A(f^p))^\frac{1}{p}}{a} + a^{\frac{1}{p} b^\frac{1}{p}} \cdot \frac{1}{q} \cdot \left( \frac{(A(g^q))^\frac{1}{q}}{b} \right)^\frac{1}{p-1} \right]^p \]

\[ = \left[ a^{\frac{1}{p} b^\frac{1}{p}} \cdot \frac{1}{p} \cdot \frac{(A(f^p))^\frac{1}{p}}{a} + a^{\frac{1}{p} b^\frac{1}{p}} \cdot \frac{1}{q} \cdot \left( \frac{(A(g^q))^\frac{1}{q}}{b^\frac{1}{q}} \right)^\frac{1}{p} \right]^p \]
\[ \left[ \frac{1}{p} \cdot \frac{b^{\frac{1}{p}}}{a^{\frac{1}{q}}} \left( A(f^p) \right)^{\frac{1}{p}} + \frac{1}{q} \cdot \frac{a^{\frac{1}{q}}}{b^{\frac{1}{p}}} \left( A(g^q) \right)^{\frac{1}{q}} \right]^p \]

and

\[ ab \left[ \frac{1}{p} \left( A(f^p) \right)^{\frac{1}{p}} + \frac{1}{q} \cdot \frac{a^{\frac{1}{q}}}{b^{\frac{1}{p}}} \left( A(g^q) \right)^{\frac{1}{q}} \right]^q \]

\[ = \left[ \frac{a^{\frac{1}{p}}b^{\frac{1}{q}}}{p} + \frac{1}{q} \cdot \frac{a^{\frac{1}{q}}}{b^{\frac{1}{p}}} \left( A(g^q) \right)^{\frac{1}{q}} \right]^q \]

After that, we obtain (5).

**Corollary 1** If the conditions from Theorem 3 are satisfied, then the following Popoviciu-type inequality holds

\[ (a^p - A(f^p))^\frac{1}{q} \leq ab - (A(f^p))^\frac{1}{q} (A(g^q))^\frac{1}{q}. \]

**Proof.** Using the inequality

\[ \frac{1}{p} u + \frac{1}{q} v \geq u^{\frac{1}{p}} \cdot v^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad u, v > 0 \]

we obtain that

\[ \left( \frac{1}{p} \cdot \frac{b^{\frac{1}{p}}}{a^{\frac{1}{q}}} \left( A(f^p) \right)^{\frac{1}{p}} + \frac{1}{q} \cdot \frac{a^{\frac{1}{q}}}{b^{\frac{1}{p}}} \left( A(g^q) \right)^{\frac{1}{q}} \right)^p \]

\[ \geq \left[ \left( \frac{b^{\frac{1}{p}}}{a^{\frac{1}{q}}} \left( A(f^p) \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \cdot \left( \frac{a^{\frac{1}{q}}}{b^{\frac{1}{p}}} \left( A(g^q) \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \right]^p \]

\[ = \frac{b^{\frac{1}{p}}}{a^{\frac{1}{q}}} \left( A(f^p) \right)^{\frac{1}{p}} \cdot \frac{a^{\frac{1}{q}}}{b^{\frac{1}{p}}} \left( A(g^q) \right)^{\frac{1}{q}} = (A(f^p))^\frac{1}{p} \cdot (A(g^q))^\frac{1}{q} \]

and

\[ \left( \frac{1}{p} \cdot \frac{b^{\frac{1}{p}}}{a^{\frac{1}{q}}} \left( A(f^p) \right)^{\frac{1}{p}} + \frac{1}{q} \cdot \frac{a^{\frac{1}{q}}}{b^{\frac{1}{p}}} \left( A(g^q) \right)^{\frac{1}{q}} \right)^q \]
A class of Aczél-Popoviciu type inequality

\[
\left(\frac{b^{\frac{1}{p}}}{a^{\frac{1}{q}}} (A(f^p))^{\frac{1}{q}}\right)^{\frac{1}{p}} \cdot \left(\frac{a^{\frac{1}{p}}}{b^{\frac{1}{q}}} (A(g^q))^{\frac{1}{p}}\right)^{\frac{1}{q}} \geq \left(\frac{b^{\frac{1}{p}}}{a^{\frac{1}{q}}} (A(f^p))^{\frac{1}{q}}\right)^{\frac{1}{p}} \cdot \left(\frac{a^{\frac{1}{p}}}{b^{\frac{1}{q}}} (A(g^q))^{\frac{1}{p}}\right)^{\frac{1}{q}} \frac{b}{a} (A(f^p))^{\frac{1}{p}} \cdot (A(g^q))^{\frac{1}{q}}.
\]

So, from (10), (11) and (5) follows (9).

**Corollary 2** If \( L \) is a linear set of functions and \( A \) is a linear positive functional, then

\[
(a^p - A(f^p))^\frac{1}{p} (b^q - A(g^q))^\frac{1}{q} \leq ab - A(fg).
\]

**Proof.** By (3) and (8) we obtain (11).

### 3 Particular cases

Define

\[ A(f) = \int_0^1 f(x)dx \]

where \( f : [0, 1] \to \mathbb{R} \), \( f \) being an integrable and positive function on \([0, 1]\).

So \( A(f) \) is a positive linear functional.

Taking

\[ a^p \geq \int_0^1 f^p(x)dx \quad \text{and} \quad b^q \geq \int_0^1 g^q(x)dx \]

where \( f, g > 0 \) on \([0, 1]\) and \( a, b \) are two fixed positive numbers, \( p, q > 0 \), \( \frac{1}{p} + \frac{1}{q} = 1 \)

it follows that

\[ \left( a^p - \int_0^1 f^p(x)dx \right)^{\frac{1}{p}} \left( b^q - \int_0^1 g^q(x)dx \right)^{\frac{1}{q}} \leq ab - \int_0^1 f(x)g(x)dx \]

equivalent with

\[ (a^p - A(f^p))^\frac{1}{p} (b^q - A(g^q))^\frac{1}{q} \leq ab - A(fg). \]

For \( p = q = 2 \) we obtain

\[ (a^2 - A(f^2))^\frac{1}{2} (b^2 - A(g^2))^\frac{1}{2} \leq ab - A(fg) \Leftrightarrow (a^2 - A(f^2))(b^2 - A(g^2)) \leq (ab - A(fg))^2 \]

(a Aczél-Popoviciu type inequality).

For this case, the reason of an elementary proof is to define the function

\[ h : (0, +\infty) \to \mathbb{R}, \quad h(t) = t^2(a^2 - A(f^2)) - 2(ab - A(fg))t + (b^2 - A(g^2)). \]
So, is sufficient to remark that the function \( h \) has two real roots if and only if \( \Delta \geq 0 \). But

\[
h(t) = (t^2a^2 - 2abt + b^2) - (t^2A(f^2) - 2A(fg)t + A(g^2)) \\
= (t^2a^2 - 2abt + b^2) - A(t^2f^2 - 2fgt + g^2) \\
= (ta - b)^2 - A((tf - g)^2).
\]

For \( t = \frac{b}{a} \) we have

\[
h \left( \frac{b}{a} \right) = -A \left( \frac{b}{a}f - g \right)^2 < 0, \quad \lim_{t \to \infty} h(t) > 0.
\]

So, the equation \( h(t) = 0 \) has real solutions. It follows that

\[
\Delta \geq 0 \Rightarrow 4(ab - A(fg))^2 - 4(a^2 - A(f^2))(b^2 - A(g^2)) \geq 0
\]

is equivalent with Aczél-Popoviciu's inequality for positive real functions

\[
(a^2 - A(f^2))(b^2 - A(g^2)) \leq (ab - A(fg))^2.
\]

References


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Some properties of Bernstein type Cheney and Sharma Operators

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Abstract

In this paper, we prove that Bernstein type Cheney and Sharma operators preserve modulus of continuity and Lipschitz continuity properties of the attached function \( f \). We also introduce a result for these operators when \( f \) is a convex function.

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Key words and phrases: Modulus of continuity function, Lipschitz continuous function.

1 Introduction

The classical Bernstein operators \( B_n : C[0, 1] \rightarrow C[0, 1] \) are defined by

\[
B_n(f; x) = \sum_{k=0}^{n} f\left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1], n \in \mathbb{N}.
\]

As is well known, some properties of the original function \( f \) are preserved by these operators. Before mention some of them we recall some needful definitions.

A real valued continuous function \( f \) is said to be convex on \([0, 1]\), if the following inequality holds

\[
f\left( \sum_{k=1}^{n} \alpha_k x_k \right) \leq \sum_{k=1}^{n} \alpha_k f(x_k)
\]

for all \( x_1, x_2, \ldots, x_n \in [0, 1] \) and for all non-negative numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that \( \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1 \).

Let \( f \) be a real valued continuous function defined on \([0, 1]\). Then \( f \) is said to be...
Lipschitz continuous of order $\gamma (0 < \gamma \leq 1)$ on $[0, 1]$, if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M |x - y|^{\gamma}$$

for all $x, y \in [0, 1]$. The set of Lipschitz continuous functions of order $\gamma$ with Lipschitz constant $M$ is denoted by $Lip_M(\gamma)$.

A continuous and non-negative function $\omega$ defined on $[0, 1]$ is called a modulus of continuity function, if each of the following conditions is satisfied:

i) $\omega(u + v) \leq \omega(u) + \omega(v)$ for $u, v, u + v \in [0, 1]$, i.e., $\omega$ is semi-additive,

ii) $\omega(u) \geq \omega(v)$ for $u \geq v$, i.e., $\omega$ is non-decreasing,

iii) $\lim_{u \to 0^+} \omega(u) = \omega(0) = 0$.

We now remind which properties of $f$ are preserved by Bernstein operators.

(a) If $f(x)$ is non-decreasing on $[0, 1]$, then $B_n(f; x)$ is non-decreasing on $[0, 1]$ (Theorem 6.3.3 in [6]).

(b) If $f(x)$ is convex on $[0, 1]$, then $B_n(f; x)$ is convex on $[0, 1]$ (Theorem 6.3.3 in [6]) and also $B_n(f; x) \geq B_{n+1}(f; x) \geq f(x)$ for all $n \in \mathbb{N}$ and all $x \in [0, 1]$ (see [12]).

(c) If $f(x) \in Lip_M(\gamma)$, then $B_n(f; x) \in Lip_M(\gamma)$ for all $n \in \mathbb{N}$ and all $x \in [0, 1]$.

This result proved by Lindvall [8] with the help of the probabilistic methods. Later, Brown, Elliott and Paget introduced more elementary proof for the same result in [3].

(d) If $\omega$ is a modulus of continuity function, then for each $n \in \mathbb{N}$ $B_n(\omega; x)$ is a modulus of continuity function also [7].

(e) If $f(x)$ is a non-negative function such that $x^{-1} f(x)$ is non-increasing on $(0, 1]$, then for each $n \in \mathbb{N}$ $x^{-1} B_n(f; x)$ is non-increasing also [7].

By using Abel-Jensen equalities (see [2], p.322 and p.326)

$$(u + v + n\beta)^n = \sum_{k=0}^{n} \binom{n}{k} u (u + k\beta)^{k-1} [v + (n - k)\beta]^{n-k}$$

and

$$(1) \quad (u + v)(u + v + n\beta)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} u (u + k\beta)^{k-1} v [v + (n - k)\beta]^{n-k-1}$$

where $u, v$ and $\beta \in \mathbb{R}$, Cheney and Sharma [4] constructed and investigated two Bernstein type operators for $f \in C[0, 1], x \in [0, 1]$ and $n \in \mathbb{N}$ as follows:

$$Q_n(f; x) = (1 + n\beta)^{-n} \sum_{k=0}^{n} \binom{k}{n} \binom{n}{k} x (x + k\beta)^{k-1} [1 - x + (n - k)\beta]^{n-k}$$
and

\[ G_n(f; x) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \left( \begin{array}{c} n \\ k \end{array} \right) x (x + k\beta)^{k-1} (1-x)^{1-x+(n-k)\beta} n^{-k-1}, \]

where \( \beta \) is a non-negative real parameter. Clear that for \( \beta = 0 \) each of these operators turns out to be the classical Bernstein operators. It is well known that if \( n\beta = n\beta_n \to 0 \) as \( n \to \infty \), then we have \( \lim_{n \to \infty} Q_n(f; x) = f(x) \) for \( f \in C[0,1] \) (see [2], p. 322-326). In this paper, we only concern with the Cheney and Sharma operators defined by \( G_n(f; x) := G_n(f)(x) \). We now give some results related to these operators. Cheney and Sharma [4] proved that these operators reproduce only the constant functions. In [9], Stancu and Cismășiú firstly showed that

\[ G_n(t; x) = x. \]

Later, they established an approximation formula for the function \( f \) by means of such operators and an integral representation for the remainder term of the approximation formula. The authors also introduced an expression for this remainder term in terms of the divided differences. In [5], Crăciun presented a new class of positive linear operators included the operators \( G_n(f; x) \) and examined some approximation properties of them. By using Adolf Hurwitz equality Stancu [10] constructed a generalization of \( G_n(f; x) \) and extended the results given in [9] for these operators. In [1] Agratini and Rus interested a general class of positive linear operators of discrete type that gives the operators \( G_n(f; x) \) as a special case, and studied approximation properties of them and also convergence of the iterates of these operators. By means of Abel-Jensen type combinatorial equalities Stancu and Stoica [11] introduced some algebraic polynomial operators such that one of them involves the Cheney and Sharma operators. They investigated uniform convergence property of these operators and evaluated the remainder term corresponding to approximation formula of \( f \).

2 Main results

In this part, by using the same technique in [3] and [7], we firstly show that if \( \omega \) is a modulus of continuity function, then \( G_n(\omega) \) is also. Later, we prove that \( G_n(f) \) preserves the Lipschitz constant \( M \) and order \( \gamma \) of a Lipschitz continuous function \( f \). Finally, we introduce a property of \( G_n(f) \) under the convexity of \( f \).

**Theorem 1.** If \( \omega \) is a modulus of continuity function, then \( G_n(\omega) \) is also a modulus of continuity function.

**Proof.** Let \( x, y \in [0,1] \) such that \( y \geq x \). Then from the definition of \( G_n \) we have

\[ G_n(\omega; y) = (1 + n\beta)^{1-n} \sum_{j=0}^{n} \omega \left( \frac{j}{n} \right) \left( \begin{array}{c} n \\ j \end{array} \right) y (y + j\beta)^{j-1} (1-y)^{1-y+(n-j)\beta} n^{-j-1}. \]
In the equality (1) if we take \( u = x, v = y - x \) and \( n = j \), then one has

\[
y(y + j\beta)^{j-1} = \sum_{k=0}^{j} \binom{j}{k} x (x + k\beta)^{k-1} (y - x)(y - x + (j - k)\beta)^{j-k-1}
\]

and so

\[
G_n (\omega; y) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} \omega \left( \frac{j}{n} \right) \left( \frac{n}{j} \right) \binom{n}{j} x (x + k\beta)^{k-1}
\]

\[
\times (y - x)(y - x + (j - k)\beta)^{j-k-1}(1 - y)[1 - y + (n - j)\beta]^{n-j-1}.
\]

Changing the order of the summations and letting \( j = l \), we obtain

\[
G_n (\omega; y) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} \sum_{j=k}^{n} \omega \left( \frac{j}{n} \right) \frac{n!}{(n-j)!((j-k)!)!} x (x + k\beta)^{k-1}
\]

\[
\times (y - x)(y - x + (j - k)\beta)^{j-k-1}(1 - y)[1 - y + (n - j)\beta]^{n-j-1}.
\]

(2)

(3)

On the other hand,

\[
G_n (\omega; x) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} \omega \left( \frac{k}{n} \right) \binom{n}{k} x (x + k\beta)^{k-1} (1-x)[1-x+(n-k)\beta]^{n-k-1}.
\]

In (1) replacing \( u, v \) and \( n \) by \( y - x, 1 - y \) and \( n - k \), respectively, we get

\[
(1-x)[1-x+(n-k)\beta]^{n-k-1} = \sum_{l=0}^{n-k} \binom{n-k}{l} (y - x)(y - x + l\beta)^{l-1}
\]

\[
\times (1 - y)[1 - y + (n - k - l)\beta]^{n-k-l-1}.
\]

Using this we reach to

\[
G_n (\omega; x) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \omega \left( \frac{k}{n} \right) \frac{n!}{k!(n-k-l)!} x (x + k\beta)^{k-1}
\]

\[
\times (y - x)(y - x + l\beta)^{l-1}(1 - y)[1 - y + (n - k - l)\beta]^{n-k-l-1}.
\]

(3)

Thus from (2) and (3) it follows that

\[
G_n (\omega; y) - G_n (\omega; x) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \left[ \omega \left( \frac{k+l}{n} \right) - \omega \left( \frac{k}{n} \right) \right] \frac{n!}{k!(n-k-l)!}
\]

\[
\times x (x + k\beta)^{k-1}(y - x)(y - x + l\beta)^{l-1}(1 - y)
\]

\[
\times [1 - y + (n - k - l)\beta]^{n-k-l-1}.
\]
Inverting the summations we conclude that

\[
G_n(\omega; y) - G_n(\omega; x) = (1 + \beta)^{1-n} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \omega \left( \frac{k + l}{n} \right) - \omega \left( \frac{k}{n} \right) \frac{n!}{l!(n-l)!} \frac{(n-l)!}{k!(n-k-l)!} \times x (x + k\beta)^{k-1} (y - x) (y - x + l\beta)^{l-1} (1 - y) [1 - y + (n - k - l)\beta]^{n-k-l-1}
\]

\[
= (1 + \beta)^{1-n} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \omega \left( \frac{k + l}{n} \right) - \omega \left( \frac{k}{n} \right) \binom{n}{l} \binom{n-l}{k} x (x + k\beta)^{k-1} \times (y - x) (y - x + l\beta)^{l-1} (1 - y) [1 - y + (n - k - l)\beta]^{n-k-l-1}
\]

Since \( \omega \) is a modulus of continuity function, we have

\[
\omega \left( \frac{k + l}{n} \right) - \omega \left( \frac{k}{n} \right) \leq \omega \left( \frac{l}{n} \right).
\]

Therefore,

\[
G_n(\omega; y) - G_n(\omega; x) \leq (1 + \beta)^{1-n} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \omega \left( \frac{l}{n} \right) \binom{n}{l} \binom{n-l}{k} x (x + k\beta)^{k-1} \times (y - x) (y - x + l\beta)^{l-1} (1 - y) [1 - y + (n - k - l)\beta]^{n-k-l-1}
\]

Now in the equality (1), if we set \( x, 1 - y \) and \( n - l \) in place of \( u, v \) and \( n \), respectively, then we find

\[
(x + 1 - y) [x + 1 - y + (n - l)\beta]^{n-l-1} = \sum_{k=0}^{n-l} \binom{n-l}{k} x (x + k\beta)^{k-1} (1 - y) \times [1 - y + (n - k - l)\beta]^{n-k-l-1}
\]

and so

\[
G_n(\omega; y) - G_n(\omega; x) \leq (1 + \beta)^{1-n} \sum_{l=0}^{n} \omega \left( \frac{l}{n} \right) \binom{n}{l} (y - x) (y - x + l\beta)^{l-1} \times (1 - (y - x)) [1 - (y - x) + (n - l)\beta]^{n-l-1}
\]

\[
= G_n(\omega, y - x)
\]
which shows the semi-additivity of $G_n(\omega)$. From (4) we deduce that $G_n(\omega; y) \geq G_n(\omega; x)$ when $y \geq x$. That is, $G_n(\omega; x)$ is non-decreasing. Finally, by the definition of $G_n$ it is obvious that $\lim_{x \to 0} G_n(\omega; x) = G_n(\omega; 0) = \omega(0) = 0$. Hence, we may conclude that $G_n(\omega)$ is a modulus of continuity function.

Theorem 2. If $f \in \text{Lip}_M(\gamma)$, then $G_n(f) \in \text{Lip}_M(\gamma)$ for all $n \in \mathbb{N}$ and $x \in [0, 1]$.

Proof. Let $x, y \in [0, 1]$ such that $y \geq x$. From (4), one has

$$G_n(f; y) - G_n(f; x) = (1 + n\beta)^{1-n} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \left[ f\left(\frac{k+l}{n}\right) - f\left(\frac{k}{n}\right)\right] \left(\frac{n}{l}\right) \left(\frac{n-l}{k}\right) x (x + k\beta)^{k-1} \times (y-x) (y-x + l\beta)^{l-1} (1-y) \left[1 - y + (n-k-l)\beta\right]^{n-k-l-1}$$

which leads to

$$|G_n(f; y) - G_n(f; x)| \leq M (1 + n\beta)^{1-n} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \left(\frac{l}{n}\right)^\gamma \left(\frac{n}{l}\right) \left(\frac{n-l}{k}\right) x (x + k\beta)^{k-1} \times (y-x) (y-x + l\beta)^{l-1} (1-y) \left[1 - y + (n-k-l)\beta\right]^{n-k-l-1}$$

Since $f \in \text{Lip}_M(\gamma)$, we can get

$$|G_n(f; y) - G_n(f; x)| \leq M (1 + n\beta)^{1-n} \sum_{l=0}^{n} \left(\frac{l}{n}\right)^\gamma \left(\frac{n}{l}\right) x (x + k\beta)^{k-1} \times \sum_{k=0}^{n-l} \left(\frac{n-l}{k}\right) (y-x) (y-x + l\beta)^{l-1} \left[1 - y + (n-k-l)\beta\right]^{n-k-l-1}.$$
Now application of the Hölder inequality with \( p = \frac{1}{\gamma} \) and \( q = \frac{1}{1-\gamma} \) leads to

\[
|G_n(f; y) - G_n(f; x)| \leq M \left\{ (1 + n\beta)^{1-n} \sum_{l=0}^{n} \frac{l}{n} \binom{n}{l} (y - x) (y - x + l\beta)^{l-1} \times (1 - (y - x)) [1 - (y - x) + (n - l)\beta]^{n-l-1} \right\}^\gamma \\
\times \left\{ (1 + n\beta)^{1-n} \sum_{l=0}^{n} \frac{n}{l} \binom{n}{l} (y - x) (y - x + l\beta)^{l-1} \times (1 - (y - x)) [1 - (y - x) + (n - l)\beta]^{n-l-1} \right\}^{1-\gamma} \\
= M \{ G_n(t; y - x) \}^\gamma \{ G_n(1; y - x) \}^{1-\gamma} \\
= M (y - x)^\gamma
\]

which completes the proof.

**Theorem 3.** If \( f \) is convex, then \( G_n(f; x) \geq f(x) \) for all \( n \in \mathbb{N} \) and \( x \in [0, 1] \).

**Proof.** Let

\[
\alpha_k = (1 + n\beta)^{1-n} \binom{n}{k} x (x + k\beta)^{k-1} (1 - x) [1 - x + (n - k)\beta]^{n-k-1}
\]

and

\[
x_k = \frac{k}{n}, \quad k = 0, 1 \ldots, n.
\]

Then, it is clear that

\[
\sum_{k=0}^{n} \alpha_k = G_n(1; x) = 1.
\]

By the hypothesis, we can write

\[
G_n(f; x) = \sum_{k=0}^{n} \alpha_k f(x_k) \geq f \left( \sum_{k=0}^{n} \alpha_k x_k \right) \\
= f(G_n(t; x)) = f(x).
\]

This is the required result.

**References**


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Dual variational principle for a problem of stationary flow of a viscous fluid

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Abstract

In this paper we formulate the dual variational principle for a problem of stationary flow of a viscous fluid in a pipe with transversal section in the L-form represented by a second elliptic equation with Dirichlet boundary conditions.

2010 Mathematics Subject Classification: 35J20.

Key words and phrases: variational principle, stationary flow, Dirichlet boundary conditions.

1 Formulation of the problem

The equation that describes the stationary flow of a viscous fluid in a pipe, with an arbitrary transversal section $\Omega$ with the boundary $\Gamma$, is [3]

$$\mu \Delta u(x, y) = \frac{dp}{dz}, \quad (x, y) \in \Omega$$

$$u(x, y) = 0, \quad (x, y) \in \Gamma$$

where $u$ is the velocity of the fluid, $\mu$ is the coefficient of viscosity and $\Delta p$ is the pressure fall on the length, $l$, of the pipe.

$$\mu = \text{const}; \quad \frac{dp}{dz} = \text{const}; \quad \frac{dp}{dz} = -\frac{\Delta p}{l}$$

The problem is to determine the repartition of the velocity in the section $\Omega$. The problem can be represented in the following mathematical model:

$$-\Delta u = f, \quad \text{in } \Omega$$

$$u = 0, \quad \text{on } \Gamma$$

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We introduced the following elements:

\[ H^1_0(\Omega) = \{ u \in H^1(\Omega) | u = 0 \text{ on } \Gamma \} \]

where \( H^1(\Omega) \) is the Sobolev space on \( \Omega \);

\[ a(u, v) = \int_{\Omega} \nabla u \nabla v \, d\Omega, \forall u, v \in H^1_0(\Omega) \]

(\( a(u, v) \) is a bilinear, symmetrical, bounded and coercive form);

\[ \varphi(v) = \int_{\Omega} fv \, d\Omega, \forall v \in H^1_0(\Omega) \]

(\( \varphi(v) \) is a linear and bounded form).

**Definition 1.** A function \( u \in H^1(\Omega) \) will be called a weak solution of the problem \((1)\) if

\[ a(u, v) = \varphi(v), \forall v \in H^1_0(\Omega) \]

In paper [3] is proved that the equation \((2)\) has a unique solution (by the Lax Milgram theorem) and \( u \) minimizes the functional.

\[ F(u) = \int_{\Omega} \left[ |\nabla u|^2 - 2fu \right] \, d\Omega. \]

Thus the problem \((1)\) is equivalent to the following minimization problem

\[ (P_\nu) \quad \{ \begin{array}{l} \text{Find } u_0 \in H^1_0(\Omega) \text{ such that} \\ F(u_0) \leq F(u), \forall u \in H^1_0(\Omega). \end{array} \]

The solution of the variational problem \((P_\nu)\) with \( \Omega \) in the L-form, is determined using the Ritz method with finite elements through the procedure of local approximation and assembly [2].

## 2 Dual variational principle

We want to find a functional \( F_d(\overline{w}) \) and a class \( W \) of admissible functions such that the equality

\[ \inf_{u \in H^1_0(\Omega)} F(u) = \sup_{w \in W} F_d(\overline{w}) \]
is satisfied.

We introduce the vectorial function $\vec{w}$ and the functional parameter $\vec{a}$ as

$$\vec{w} = \left\{ \begin{array}{c} w_1 \\ w_2 \end{array} \right\} \in W = (H^1(\Omega) \times H^1(\Omega)) \equiv H^{1,2}(\Omega)$$

$$\vec{a} = \left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\} \in C^{1,2}(\overline{\Omega})$$

$$(C^{1,2}(\overline{\Omega}) = C^1(\Omega) \times C^1(\Omega))$$ with $\beta \equiv \nabla \vec{a} - |\vec{a}|^2 \geq \text{const} > 0$ on $\Omega$.

The Green formula can be written in the form

$$\int_{\Omega} \vec{w}^T \cdot \nabla u \, d\Omega + \int_{\partial \Omega} u\vec{w} \cdot \vec{n} \, ds, \quad u \in H^1(\Omega), \quad \vec{w} \in W(\Omega)$$

where $w_n$ is the normal component of $\vec{w}$ on $\Gamma(w_n = \vec{w}^T \cdot \vec{n})$.

Using (5) and the relation

$$\nabla \cdot u \vec{w} = u \nabla \cdot \vec{w} + \vec{w} \cdot \nabla u$$

can be write the following integral identities:

$$\phi(\vec{w}, u) = \int_{\Omega} \nabla \cdot u \vec{w} \, d\Omega - \int_{\Gamma} u\vec{w} \cdot \vec{n} \, ds \equiv 0 \equiv \int_{\Omega} (\vec{w} \cdot \nabla u + u\nabla \cdot \vec{w}) \, d\Omega,$$

$$\phi(u \vec{a}, u) = \int_{\Omega} \nabla \cdot u^2 \vec{a} \, d\Omega - \int_{\Gamma} u^2 \vec{a} \cdot \vec{n} \, ds \equiv 0 \equiv \int_{\Omega} (2u \vec{a} \cdot \nabla u + u^2 \nabla \cdot \vec{a}) \, d\Omega.$$

Then, the functional $F(u)$, (3), $u \in H^1_0(\Omega)$ can be represented

$$F(u) = F(u) - 2\phi(\vec{w}, u) + \phi(u \vec{a}, u)$$

$$= \int_{\Omega} \left[ |\vec{w}|^2 + \beta \left( u - \frac{\gamma}{\beta} \right)^2 - \frac{\gamma^2}{\beta} - |\vec{w}|^2 \right] \, d\Omega$$

where $\vec{v} = \nabla u + u \vec{a} - \vec{w}$ and $\gamma = f + \nabla \cdot \vec{w} - \vec{a} \cdot \vec{w}$. Since $\beta > 0$, it follows that

$$F(u) \geq - \int_{\Omega} \left[ |w|^2 + \frac{1}{\beta} \left( f + \nabla \cdot \vec{w} - \vec{a} \cdot \vec{w} \right)^2 \right] \, d\Omega$$

Let us define the functional $F_d(\vec{w}) : W \rightarrow \mathbb{R}$,

$$F_d(\vec{w}) = - \int_{\Omega} \left[ |\vec{w}|^2 + \frac{\gamma^2}{\beta} \right] \, d\Omega$$

where $W = H^1(\Omega) \times H^1(\Omega)$.

Using (7) we can prove that

$$F(u) \geq F_d(\vec{w})$$ and $$\sup_{\vec{w}} F_d(\vec{w}) \leq \inf_u F(u).$$
Definition 2. If (9) becomes an equality, then, \( F_d(\vec{w}) \), \( w \in W \) is the dual functional for \( F(u) \), \( u \in H^1_0(\Omega) \).

Let now present the conditions for which (9) is an equality.

Theorem 1. The functional \( F_d(\vec{w}) \), \( \vec{w} \in H^{1,2}(\Omega) \) is the dual one for the functional \( F(u) \), \( u \in H^1_0(\Omega) \), if

(10) \[ \vec{w} = \vec{w}_0 = \nabla u_0 + u_0 \vec{a} \]

where \( u_0 \) is the weak solution of the problem (1) and between \( u_0 \) and \( \vec{w}_0 \) there exists the relation

(11) \[ u_0 = \frac{1}{\beta} (f + \nabla \cdot \vec{w} - \vec{a} \cdot \vec{w}) \]

Proof. Let be \( u_0 \) the weak solution of the problem (1) and \( u_0 \vec{a} \in H^{1,2}(\Omega) \), \( \vec{a} \in C^{1,2}(\Omega) \). According to the definition of \( u_0 \), \( \vec{w}_0 \) and \( \vec{a} \), we can write:

\[ \overrightarrow{\nabla}|u = u_0; \overrightarrow{w} = \overrightarrow{w}_0 = \nabla u_0 + u_0 \vec{a} - \overrightarrow{w}_0 = \nabla u_0 + u_0 \vec{a} - \nabla u_0 - u_0 \vec{a} = 0 \]

\[ u_0 - \frac{1}{\beta} [f + \nabla \cdot \overrightarrow{w}_0 - \vec{a} \cdot \overrightarrow{w}_0] = u_0 - \frac{1}{\beta} [f + \nabla \cdot (\nabla u_0 + u_0 \vec{a})] - \vec{a} (\nabla u_0 + u_0 \vec{a}) \]

\[ = ... = u_0 - \frac{u_0}{\beta} \beta = 0 \]

Consequently, in the specified conditions, \( \overrightarrow{\nabla} = 0 \) and \( u = \frac{\gamma}{\beta} \) will be choosen in (7) and then in (11) the equality is fulfilled. This mean that \( F_d(\vec{w}) \) is the dual functional for \( F(u) \).

The dual variational problem is

(P\textsubscript{vd}) \[ \left\{ \begin{array}{l}
\text{Find the variational function } \overrightarrow{w}_0 \in H^{1,2}(\Omega) \text{ such that } \\
F_d(\overrightarrow{w}_0) \geq F_d(\overrightarrow{w}), \\
\forall \overrightarrow{w} \in H^{1,2}(\Omega)
\end{array} \right. \]

Algorithm Ritz for the dual variational problem:

We choose the Ritz approximation in the form \( \overrightarrow{w}_n = (w_{1n}, w_{2n})^T \) where

\[ w_{1n}(x, y) = \sum_{k=1}^{n} b_k \varphi_k(x, y); \ (x, y) \in \overline{\Omega}; \ b_k \in \mathbb{R} \]

\[ w_{2n}(x, y) = \sum_{k=1}^{n} c_k \varphi_k(x, y); \ (x, y) \in \overline{\Omega}; \ c_k \in \mathbb{R} \]

\[ \vec{a} = k(x, y)^T < \frac{2}{R^2} \] where \( R \) is the radius of the circle containing \( \Omega \).

We obtain

\[ F_d(\overrightarrow{w}_n) = \int_\Omega \left[w_{1n}^2 + w_{2n}^2 + \frac{1}{\beta} \left(f + \frac{\partial w_{1n}}{\partial x} + \frac{\partial w_{2n}}{\partial y} - a_1 w_{1n} - a_2 w_{2n}\right)^2\right] d\Omega \]

\[ = (b_1, ..., b_n; c_1, ..., c_n) \]
For $\Phi \rightarrow R^n$ extrem \((-\frac{1}{2} \frac{\partial \Phi}{\partial x_j} = 0; -\frac{1}{2} \frac{\partial \Phi}{\partial y_j} = 0\)$ results the system:

\[
\begin{cases}
\sum_{k=1}^{n} a_{jk} b_k + \sum_{k=1}^{n} \beta_{jk} c_k = r_j^{(1)} \\
\sum_{k=1}^{n} \gamma_{jk} b_k + \sum_{k=1}^{n} \delta_{jk} c_k = r_j^{(2)}
\end{cases}
\]

where

\[
a_{jk} = \int_{\Omega} \left[ \varphi_j \varphi_k + \frac{1}{\beta} (\varphi'_{jx} - a_1 \varphi_j)(\varphi'_{kx} - a_1 \varphi_k) \right] d\Omega
\]

\[
\beta_{jk} = \int_{\Omega} \frac{1}{\beta} \left[ (\varphi'_{jx} - a_1 \varphi_j)(\varphi'_{kx} - a_2 \varphi_k) \right] d\Omega
\]

\[
\gamma_{jk} = \int_{\Omega} \frac{1}{\beta} \left[ (\varphi'_{jy} - a_2 \varphi_j)(\varphi'_{kx} - a_1 \varphi_k) \right] d\Omega
\]

\[
\delta_{jk} = \int_{\Omega} \left[ \varphi_j \varphi_k + \frac{1}{\beta} (\varphi'_{jy} - a_2 \varphi_j)(\varphi'_{kx} - a_2 \varphi_k) \right] d\Omega
\]

\[
\begin{align*}
\varphi_j^{(1)} &= \int_{\Omega} \frac{f}{\beta} (\varphi'_{jx} - a_1 \varphi_j) d\Omega \\
\varphi_j^{(2)} &= \int_{\Omega} \frac{f}{\beta} (\varphi'_{jy} - a_2 \varphi_j) d\Omega
\end{align*}
\]

Numerical example:

\[
\mu = 1,5 \cdot 10^{-4} \text{Ns/m}^2;
\]

\[
\frac{dp}{dz} = -5000 \text{N/m}^3, f = -\frac{1}{\mu} \frac{dp}{dz};
\]

$\Omega$ in the L-form with $a = 0, 1m$

\[
\vec{d} = (80x, 80y)^T; \beta = 160 - 6400(x^2 + y^2)
\]

\[
\varphi_k(x, y) = \omega(x, y) \cdot p_k(x, y)
\]

with

\[
\omega(x, y) = -xy(x - a)(y - a) \left( x + y - a - \sqrt{\left( x - \frac{a}{2} \right)^2 + \left( y - \frac{a}{2} \right)^2} \right)
\]

and

\[
p_k(x, y) = x^jy^j, k = \frac{1}{2}(i + j)(i + j + 1) + j + 1, i, j = 0, 1, 2...
\]
References


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Bi-criteria problems for energy optimization

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Abstract

In this material we consider a new approach for energy optimization based on bi-criteria problems. Similar method was successfully developed for portfolio theory. We managed to extend and improve it. Due to optimization for energy production which has an important impact on greenhouse gases, our models bring some contributions to General Climate Models.

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1 Introduction

Increasing population and climate change is generating a significant increase of power demand. This is even more visible in arid and semiarid areas due to air conditioning. Increase of power demand is generating peaks in consumption, which might be assimilated to extreme events. To prevent brownouts as peak approaches grid capacity, producers have two scenarios, either to increase the grid capacity (requires important investments) or to shave the peak, redistributing this way the consumption to other convenient hour intervals. Even after a brief analysis of energy consumption, fluctuations are visible. To reduce energy fluctuation and extreme consumption, Ruddell, Salamanca and Mahalov ([24]) have created a model which enables a partial shift of power demand from peak load, during extreme events such as heat waves. Reducing daily fluctuation of energy consumption is an objective for the producers, together with the general objective for each company to maximize its economic performance. We have identified a similarity between portfolio selection and energy optimization,
which is visible by assimilating risk to fluctuation and total wealth to economic performance. This class of problems, where risk has to be minimized and total wealth has to be maximized, is playing an important role in portfolio optimization theory. Considered as a milestone for the portfolio optimization, the Mean Variance Model (MVM), introduced by Markowitz ([24]) is using variance to measure risk while total wealth is the amount of money cashed in by the investor. Implementation of MVM is difficult, due to quadratic form of objective function. Mathematicians have tried to extend and improve the MVM (see Smith ([32]), Mossin ([23]), Merton ([22]), Samuelson ([28]), Fama ([10]), Hakkanson ([12]), Elton and Gruber ([8]), ([9]), Li and Ng ([17]) Constantinides ([6]), Perold ([25]), Dumas and Luciano ([7]), Best and Grauer ([1]), ([2]), Chopra, Hensel and Turner ([5]), Sharpe ([29]), ([30]), ([31]), Stone ([33]), Lee, Finnerty and Wort ([15]), Huang and Qiao ([13]), Konno and Yamazaki ([14]), Cai et al ([4])).

Our paper develops a new approach for energy optimization, based on bi-criteria programming.

2 Problem formulation

A power plant focuses the problem of determining the optimum quantity of energy to be produced at certain time intervals, such that fluctuation of energy during a period of time is reduced to minimum, performance indicator is maximized and some constraints imposed by the market and by the power grid are fulfilled. From the way the problem is defined, it’s clear that we are dealing with a bi-criteria problem, where fluctuation has to be minimized and economic performance maximized. During our research together with Professor Mahalov and Professor Duca, we have created two different models. In the first model, fluctuation of energy is the maximum absolute deviation between energy produced at a certain time moment and a predefined level of energy. It is an extension of risk evaluation introduced by Cai et all ([4]) in portfolio theory. The predefined level may be a random value chosen by power plant or the average energy produced during a certain period from the past. Economic performance is evaluated by turnover, calculated as the total, over entire time period, of quantity multiplied with price. For solving reasons it’s required to have both fluctuation and turnover expressed in the same measuring unit. This is why energy fluctuation will be multiplied with price. Introduction of price in the measure of fluctuation is helping the optimization process, because price is an element with a significant impact on demand. We called this minimax model. Denoting by \(1, 2, ..., i, ..., n\) the time horizon for which the energy has to be optimized and

\[
\begin{align*}
\dot{x}_i & = \text{energy produced at hour } i, \ i = 1, n, \\
\dot{p}_i & = \text{price of energy at hour } i, \ i = 1, n, \\
r & = \text{predefined level of energy}, \\
\dot{\varepsilon} & = \text{minimum level of energy which the energy plant has to deliver}, \\
\rho & = \text{maximum production capacity of the energy plant},
\end{align*}
\]

we have the following mathematical expressions for fluctuation of energy
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\[
\max_{i=1,n} |p_ix_i - p_ir|
\]

turnover

\[
\sum_{i=1}^{n} p_ix_i
\]

constraints

\[\varepsilon \leq x_i \leq \rho, \ i = 1, n\]

which are determining the following mathematical model for our problem

(1)

\[
\begin{cases}
\min \left( \max_{i=1,n} |p_ix_i - p_ir|, \ -\sum_{i=1}^{n} p_ix_i \right)^T, \\
\varepsilon \leq x_i \leq \rho, \ i = 1, n.
\end{cases}
\]

Definition 1 For a problem

\[
\begin{cases}
\min f(x) \\
x \in X
\end{cases}
\]

where \( X \subseteq \mathbb{R}^n \) and \( f = (f_1, f_2, \ldots, f_m)^T : X \rightarrow \mathbb{R}^m \), a feasible solution \( x^* \in X \) is said to be efficient solution if \( \exists x \in X \) such that

\[
\begin{align*}
f(x) &\leq f(x^*) \\
f(x) &\neq f(x^*).
\end{align*}
\]

The combination between maximum absolute deviation and turnover in the objective function of the problem will never allow a point situated under the predefined level to be efficient. This is reducing the optimization possibilities and determined us to develop the second model, which we called production index. Fluctuation of energy in the production index model is computed as

\[
\max_{i=1,n} \left\{ \frac{x_i}{\rho} \right\}.
\]

and the mathematical model for our problem is

(2)

\[
\begin{cases}
\min_{i=1,n} \left( \max_{i=1,n} \left\{ \frac{x_i}{\rho} \right\} \right) \; - \sum_{i=1}^{n} p_ix_i \right)^T, \\
\varepsilon \leq x_i \leq \rho, \ i = 1, n.
\end{cases}
\]


3 Computing the optimal solution

This section contains a brief presentation for the minimax model. Readers may refer for an extended proof regarding minimax model to ([19]) and to ([20]) regarding production index model. In order to determine the efficient solution for problem (1) we will introduce the following bi-criteria equivalent problem

\[
\begin{align*}
\min & \left( y, -\sum_{i=1}^{n} p_i x_i \right)^T \\
|p_i x_i - p_i r| & \leq y, \quad i = \overline{1,n} \\
\varepsilon & \leq x_i \leq \rho, \quad i = \overline{1,n}.
\end{align*}
\]

Equivalence between problems (1) and (3) is shown in the following Lemma.

**Lemma 1** If \( x \in \mathbb{R}^n \) is an efficient solution for problem (1), then \((x, y) \in \mathbb{R}^n \times \mathbb{R}\), with \( y = \max_{i=1,n} |p_i x_i - p_i r| \) is an efficient solution for problem (3) and reciprocally.

**Proof.** \( \implies \)

Let \( x \in \mathbb{R}^n \) be an efficient solution for problem (1).

Then \( \exists x_0 \in \mathbb{R}^n \), with \( \varepsilon \leq x_0^i \leq \rho, \quad i = \overline{1,n} \) such that

\[
\max_{i=1,n} \left| p_i x_0^i - p_i r \right| \leq \max_{i=1,n} \left| p_i x_i^i - p_i r \right|
\]

\[
\sum_{i=1}^{n} p_i x_0^i \geq \sum_{i=1}^{n} p_i x_i^i, \quad i = \overline{1,n}
\]

and at least one inequality holds strictly.

Suppose \((x, y) \in \mathbb{R}^n \times \mathbb{R}\), with \( y = \max_{i=1,n} |p_i x_i - p_i r| \) is not an efficient solution for problem (3).

Then there exists \((x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}\) be a feasible solution for problem (3), with

\[
y_0 \leq y \]

\[
\sum_{i=1}^{n} p_i x_0^i \geq \sum_{i=1}^{n} p_i x_i^i, \quad i = \overline{1,n}
\]

and at least one inequality holds strictly.

Because \((x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}\) is a feasible solution for problem (3), it follows that

\[
|p_i x_0^i - p_i r| \leq y_0, \quad i = \overline{1,n}
\]

which means that

\[
\max_{i=1,n} \left| p_i x_0^i - p_i r \right| \leq y_0.
\]
Thus we obtain that
\[
\max_{i=1,n} |p_i x_i^0 - p_i r| \leq y_0 \leq y = \max_{i=1,n} |p_i x_i - p_i r|
\]
\[
\sum_{i=1}^{n} p_i x_i^0 \geq \sum_{i=1}^{n} p_i x_i, \ i = 1, n
\]
and at least one inequality holds strictly.

This way we obtain a contradiction for the efficiency of \( x \in \mathbb{R}^n \) as solution for (1).

In conclusion, \( (x, y) \in \mathbb{R}^n \times \mathbb{R} \), with \( y = \max_{i=1,n} |p_i x_i - p_i r| \) is an efficient solution for problem (3).

\[ \iff \]

Let \( (x, y) \in \mathbb{R}^n \times \mathbb{R} \), with \( y = \max_{i=1,n} |p_i x_i - p_i r| \) be an efficient solution for problem (3).

Suppose \( x \in \mathbb{R}^n \) is not an efficient solution for problem (1) and let \( x_0 \) be a feasible solution, with
\[
\max_{i=1,n} |p_i x_i^0 - p_i r| \leq \max_{i=1,n} |p_i x_i - p_i r|, \ i = 1, n
\]
\[
\sum_{i=1}^{n} p_i x_i^0 \geq \sum_{i=1}^{n} p_i x_i, \ i = 1, n
\]
and at least one inequality holds strictly.

Denoting \( y^0 = \max_{i=1,n} |p_i x_i^0 - p_i r| \), it follows
\[
y^0 \leq y
\]
\[
\sum_{i=1}^{n} p_i x_i^0 \geq \sum_{i=1}^{n} p_i x_i, \ i = 1, n
\]
and at least one inequality holds strictly, which contradicts the efficiency of \( (x, y) \in \mathbb{R}^n \times \mathbb{R} \).

In conclusion \( x \in \mathbb{R}^n \) is an efficient solution for problem (1) and this ends our proof.

Using results of Yu ([34]), Bot ([3]) and Geoffrion ([11]) the bi-criteria problem (3) is equivalent to the following parametric optimization problem

\[
\begin{align*}
(4) \quad \min \left\{ \lambda y - (1 - \lambda) \sum_{i=1}^{n} p_i x_i \right\} \quad & \lambda y - (1 - \lambda) \sum_{i=1}^{n} p_i x_i \\
|p_i x_i - p_i r| & \leq y, \quad i = 1, n \\
\varepsilon & \leq x_i \leq \rho, \quad i = 1, n
\end{align*}
\]
with \( \lambda \in (0, 1) \) and the following Lemma holds.
Lemma 2 \((x, y) \in \mathbb{R}^n \times \mathbb{R}\) is an efficient solution for bi-criteria problem \((3)\) if and only if \(\exists \lambda \in (0, 1)\) such that \((x, y) \in \mathbb{R}^n \times \mathbb{R}\) is an optimal solution for parametric optimization problem \((4)\).

The meaning of in this context is the sensitivity of energy plant for reducing the fluctuation. The bigger is, the energy plant is more interested in reducing the fluctuation. The smaller is, the energy plant is less interested in reducing the fluctuation and more interested in increasing the turnover.

Considering the equivalence between problems \((1)\) and \((3)\), respectively problems \((3)\) and \((4)\), it follows from transitivity that problems \((1)\) and \((4)\) are equivalent. This means that in order to compute the efficient solution for \((1)\) we have to determine the optimal solution for \((4)\). In the process of computing the optimal solution, we will split the set \(\{1, 2, ..., n\}\) in subsets like \(\{1, 2, ..., l\}\) and \(\{l + 1, l + 2, ..., n\}\), or \(\{1, 2, ..., l\}\), \(\{l + 1, l + 2, ..., m\}\) and \(\{m + 1, m + 2, ..., n\}\). If price is constant on such an interval, it will be denoted by \(\overline{p}\).

Theorem 1 The optimal solutions for parametric optimization problem \((4)\) are:

1. If \(\lambda < \frac{n}{n+1}\), then
   \[
   \begin{align*}
   x^*_i &= \rho, & i &= 1, n \\
   y^* &= \overline{p}(\rho - r)
   \end{align*}
   \]
   or
   
   - if \(\overline{p_i} \leq p_j, \ j = l+1, n\), then
     \[
     \begin{align*}
     x^*_i &= \rho, & i &= 1, l \\
     x^*_j &= r + \frac{p^*}{\overline{p_j}}, & j &= l + 1, n \\
     y^* &= \overline{p}(\rho - r)
     \end{align*}
     \]
     where \(\overline{p_i} = p_i, \ i = 1, l\).

   - else problem has no solution.

2. If \(\lambda = \frac{n}{n+1}\), then
   \[
   \begin{align*}
   x^*_i &= r + \frac{p^*}{\overline{p_i}}, & i &= 1, n \\
   y^* &= \min_{i=1,n} \{p_i(\rho - r)\}.
   \end{align*}
   \]

3. If \(\lambda > \frac{n}{n+1}\), then
   \[
   \begin{align*}
   x^*_i &= r, & i &= 1, n \\
   y^* &= 0.
   \end{align*}
   \]

4. If \(\lambda < \frac{l}{l+1}\), then
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- if \( p_j < \overline{p}, \ j = \overline{l+1,n}, \) then
  \[
  \begin{aligned}
  x_i^* &= \rho, & i &= \overline{1,l} \\
  x_j^* &= \rho, & j &= \overline{l+1,n} \\
  y^* &= \overline{p}(\rho - r)
  \end{aligned}
  \]
  where \( \overline{p}_i = p_i, \ i = \overline{1,l}. \)
- else problem has no solution.

5. If \( \lambda = \frac{1}{l+1}, \) then
  - if \( p_j < \overline{p}_i, \ i = \overline{1,l}, \ j = \overline{l+1,n}, \) then
    \[
    \begin{aligned}
    x_i^* &= \rho + \frac{y^*}{\overline{p}_i}, & i &= \overline{1,l} \\
    x_j^* &= \rho, & j &= \overline{l+1,n} \\
    y^* &= \min_{i=\overline{1,l}} \{p_i(\rho - r)\}, \ \text{if} \ p_j < \overline{p}_i.
    \end{aligned}
    \]
  - else problem has no solution.

6. If \( \lambda < \frac{l+1}{l+(n-m)} \), then
  - if \( p_j \leq \overline{p} \leq \overline{p}_i, \ i = \overline{1,l}, \ j = \overline{l+1,m}, \) then
    \[
    \begin{aligned}
    x_i^* &= \rho + \frac{y^*}{\overline{p}_i}, & i &= \overline{1,l} \\
    x_j^* &= \rho, & j &= \overline{l+1,m} \\
    x_k^* &= \rho, & k &= \overline{m+1,n} \\
    y^* &= \overline{p}(\rho - r)
    \end{aligned}
    \]
    where \( \overline{p}_k = p_k, \ k = \overline{m+1,n}. \)
  - else problem has no solution.

Proof of the Theorem is based on Kuhn-Tucker multipliers and is a 4 steps process. At first step, the possible combinations will be determined, by analyzing, for a fixed \( i \) from \( \{1,2,...,n\} \), the behaviour of all Kuhn-Tucker multipliers related to complementarity slackness and dual feasibility conditions of the Lagrangian.

**Definition 2** Possible combinations are combinations of Kuhn-Tucker multipliers, determined for a fixed \( i \in \{1,2,...,n\} \) for which complementarity slackness and dual feasibility conditions are fulfilled.

At step two, for \( i = \overline{1,n} \), the behavior of possible combinations related to the gradient of Lagrangian is analyzed. The end result are the feasible combinations.

**Definition 3** Feasible combinations are possible combinations for which the gradient of Lagrangian is zero.
At step three, for $i = 1, n$, the combining capacity of feasible combinations is analyzed. The end result are the critical combinations.

**Definition 4** Critical combinations are those feasible combinations for which a solution does not exist if they are combined.

At step four, the optimal solutions are computed based on feasible and critical combinations.

**Proof. of Theorem 1.** Let $\lambda \in (0, 1)$ fixed. Problem (4) is equivalent with the following

\[
\begin{align*}
\min & \left\{ \lambda y - (1 - \lambda) \sum_{i=1}^{n} p_i x_i \right\} \\
- y & \leq p_i x_i - p_i r, \quad i = 1, n \\
p_i x_i - p_i r & \leq y, \quad i = 1, n \\
\varepsilon & \leq x_i, \quad i = 1, n \\
x_i & \leq \rho, \quad i = 1, n
\end{align*}
\]

which is a convex optimization problem with inequality constraints.

The associated Kuhn-Tucker conditions are

\begin{align*}
(\text{KT1}) & \quad \frac{\partial L}{\partial x_i} = -(1 - \lambda) p_i - a_i p_i + b_i p_i - c_i + d_i = 0, \quad i = 1, n \\
(\text{KT2}) & \quad \frac{\partial L}{\partial y} = \lambda - \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = 0 \\
(\text{KT3}) & \quad (y^* - p_i x_i^* + p_i r + c_i) a_i = 0, \quad a_i \geq 0, \quad i = 1, n \\
(\text{KT4}) & \quad (p_i x_i^* - p_i r - y^*) b_i = 0, \quad b_i \geq 0, \quad i = 1, n \\
(\text{KT5}) & \quad (\varepsilon - x_i^*) c_i = 0, \quad c_i \geq 0, \quad i = 1, n \\
(\text{KT6}) & \quad (x_i^* - \rho) d_i = 0, \quad d_i \geq 0, \quad i = 1, n
\end{align*}

where

\[
L : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}
\]

\[
L(x, y, a, b, c, d) = \lambda y - (1 - \lambda) \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} a_i (-y - p_i x_i + p_i r) + \sum_{i=1}^{n} b_i (p_i x_i - p_i r - y) + \sum_{i=1}^{n} c_i (\varepsilon - x_i) + \sum_{i=1}^{n} d_i (x_i - \rho)
\]

is the associated Lagrangian and $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}$ is the optimal solution.
Remark 1 Due to the fact that the optimization problem is a convex one, it follows that Kuhn-Tucker conditions are both necessary and sufficient.

We have to compute \((x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}\) in order to determine the optimal solution for our parametric problem (4).

**Step 1**
Let \(i \in \{1, 2, \ldots, n\}\) fixed. Then, the possible scenarios for Kuhn-Tucker multipliers are

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>(a_i)</th>
<th>(b_i)</th>
<th>(c_i)</th>
<th>(d_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(=0)</td>
</tr>
<tr>
<td>2</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(=0)</td>
</tr>
<tr>
<td>3</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(=0)</td>
</tr>
<tr>
<td>4</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
</tr>
<tr>
<td>5</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
</tr>
<tr>
<td>6</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(=0)</td>
</tr>
<tr>
<td>7</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
</tr>
<tr>
<td>8</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
</tr>
<tr>
<td>9</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
</tr>
<tr>
<td>10</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
</tr>
<tr>
<td>11</td>
<td>(=0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
</tr>
<tr>
<td>12</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
</tr>
<tr>
<td>13</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
</tr>
<tr>
<td>14</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
</tr>
<tr>
<td>15</td>
<td>(&gt; 0)</td>
<td>(=0)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
</tr>
<tr>
<td>16</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
</tr>
</tbody>
</table>

We will analyze the behavior of each scenario related to complementarity slackness and dual feasibility conditions (Kuhn-Tucker conditions (KT3) to (KT6)) and we will determine the solution for each scenario if it exists. We will analyze in detail only scenario 1

\[
\begin{bmatrix}
\ a_i = 0 & \ b_i = 0 & \ c_i = 0 & \ d_i = 0
\end{bmatrix}
\]

The system generated by Kuhn-Tucker conditions (KT3) to (KT6) is

\[
\begin{align*}
-y^* & \leq p_i x_i^* - p_i r \\
 p_i x_i^* - p_i r & \leq y^* \\
x_i^* & \geq \varepsilon \\
x_i^* & \leq \rho.
\end{align*}
\]

(5)

From the first two inequalities of the system we have

\[
\begin{align*}
x_i^* & \geq r - \frac{y^*}{p_i} \\
x_i^* & \leq r + \frac{y^*}{p_i}
\end{align*}
\]
and considering the last two inequalities of the system it follows that

\[ y^* \leq p_i (r - \varepsilon) \]
\[ y^* \leq p_i (\rho - r). \]

Thus, in case of scenario 1, the solution for system (5) is

\[
\begin{cases}
  x_i^* \in \left[ r - \frac{y^*}{p_i}, r + \frac{y^*}{p_i} \right] \\
y^* \leq p_i (r - \varepsilon) \\
y^* \leq p_i (\rho - r).
\end{cases}
\]

From the 16 scenarios, we have proved that only 8 are possible combinations. These are 1, 2, 3, 4, 5, 6, 7 and 10.

**Step 2**

For \( i = \overline{1,n} \) we will analyze the behavior of possible combinations related to the gradient of Lagrangian (Kuhn-Tucker conditions (KT1) and (KT2)). In detail we will present scenario 6.

\[
\begin{bmatrix}
  a_i > 0 \\
b_i > 0 \\
c_i = 0 \\
d_i = 0
\end{bmatrix}
\]

The system generated by Kuhn-Tucker conditions (KT1) and (KT2) is

\[
\begin{align*}
  - (1 - \lambda) p_i - a_i p_i + b_i p_i &= 0, \quad i = \overline{1,n} \\
  \lambda - \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i &= 0
\end{align*}
\]

and thus

\[ a_i = b_i - (1 - \lambda), \quad i = \overline{1,n}. \]

It follows that

\[ \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i - n (1 - \lambda). \]

But \( a_i > 0, \quad i = \overline{1,n} \) and then \( b_i > (1 - \lambda), \quad i = \overline{1,n}. \)

In conclusion we get

\[ \lambda > \frac{n}{n+1}. \]

Synthesizing the behavior of the 8 possible combinations related to Kuhn-Tucker conditions (KT1) and (KT2)) the following situation is obtained:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Solution</th>
<th>(KT1)</th>
<th>(KT2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \exists )</td>
<td>( \times )</td>
<td>( \times )</td>
</tr>
<tr>
<td>2</td>
<td>( \exists )</td>
<td>( \times )</td>
<td>( \times )</td>
</tr>
<tr>
<td>3</td>
<td>( \exists, \text{ if } \lambda = \frac{n}{n+1} )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>4</td>
<td>( \exists )</td>
<td>( \times )</td>
<td>( \times )</td>
</tr>
<tr>
<td>5</td>
<td>( \exists )</td>
<td>( \checkmark )</td>
<td>( \times )</td>
</tr>
<tr>
<td>6</td>
<td>( \exists, \text{ if } \lambda &gt; \frac{n}{n+1} )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>7</td>
<td>( \exists )</td>
<td>( \times )</td>
<td>( \times )</td>
</tr>
<tr>
<td>10</td>
<td>( \exists, \text{ if } \lambda &lt; \frac{n}{n+1} )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
</tbody>
</table>
Remark 2 For scenario 5 we notice that $(KT2)$ is not satisfied, which means that scenario 5 will not generate a solution by it’s own, but may be combined with others to generate solution. 

Thus, the feasible scenarios which will generate the optimal solution for (4) are 3, 5, 6 and 10. 

Step 3

We state that the critical combinations are 6 with 5 and 6 with 10. Readers may find an extended proof in ([19]). Same reference contains a proof that order of scenarios in a combination does not change the solution. 

Step 4

The combinations based on which we will compute the optimal solution for (4) are 3, 6, 10, 6+3, 10+3, 10+5, 3+5 and 3+5+10. We will analyze combination 10+3, the others being similar. They are extensively described in ([19]). For combination 10 with 3, if $i = 1, l$ and $j = l + 1, n$, then Kuhn-Tucker conditions $(KT1)$ and $(KT2)$ will be

$$
\begin{align*}
- (1 - \lambda) p_i + b ip_i + d_i &= 0, \quad i = 1, l \\
- (1 - \lambda) p_j + b jp_j &= 0, \quad j = l + 1, n \\
\lambda - \sum_{i=1}^{l} b_i - \sum_{j=l+1}^{n} b_j &= 0.
\end{align*}
$$

Thus

$$d_i = p_i [(1 - \lambda) - b_i], \quad i = 1, l$$

and because $d_i > 0, i = 1, l$ it follows that

$$\sum_{i=1}^{l} b_i < l (1 - \lambda).$$

Also

$$b_j = (1 - \lambda), \quad j = l + 1, n$$

and then

$$\sum_{j=l+1}^{n} b_j = (n - l) (1 - \lambda)$$

Using now the last equation of the system it follows

$$\lambda < \frac{n}{n + 1}.$$ 

For 10, with $i = 1, l$ the solution is

$$
\begin{align*}
 x_i^* &= p_i, \quad i = 1, l \\
y^* &= p_i (\rho - r), \quad i = 1, l
\end{align*}
$$
and for 3, with \( j = l + 1, n \) the solution is

\[
\begin{align*}
x^*_j &= r + \frac{y^*}{p_j}, & j &= l + 1, n \\
y^* &\leq p_j (\rho - r), & j &= l + 1, n.
\end{align*}
\]

Because \( y^* = p_i (\rho - r), i = 1, l \) we state that all prices are constant on the set \( \{1, 2, \ldots, l\} \) and denote \( \overline{p} = p_i, i = 1, l \). In order to have all conditions for \( y^* \) fulfilled it is necessary that

\[
\overline{p} \leq p_j, \ j = l + 1, n.
\]

Choosing

\[
y^* = \overline{p} (\rho - r)
\]

an optimal solution for (4) is

\[
\begin{align*}
x^*_i &= \rho, & i &= 1, l \\
x^*_j &= r + \frac{y^*}{p_j}, & j &= l + 1, n \\
y^* &= \overline{p} (\rho - r), & \text{if } \overline{p} \leq p_j.
\end{align*}
\]

if \( \lambda < \frac{n}{n+1} \).

Denoting by TR the turnover, using Theorem 1 and the equivalence between problems (1), (3) and (4), the following is true

**Theorem 2** The efficient solution for bi-criteria energy optimization problem (1) is

1. If \( \lambda < \frac{n}{n+1} \), then

\[
\begin{align*}
x^*_i &= \rho, & i &= 1, n \\
y^* &= \overline{p} (\rho - r) \\
TR &= n \overline{p} \rho
\end{align*}
\]

or

- if \( \overline{p} \leq p_j, j = l + 1, n \), then

\[
\begin{align*}
x^*_i &= \rho, & i &= 1, l \\
x^*_j &= r + \frac{y^*}{p_j}, & j &= l + 1, n \\
y^* &= \overline{p} (\rho - r) \\
TR &= \overline{p} \rho + r \sum_{j=l+1}^{n} p_j + (n - l) y^*.
\end{align*}
\]

where \( \overline{p} = p_i, i = 1, l \).

- else problem has no solution.
2. If $\lambda = \frac{n}{n+1}$, then
\[
\begin{cases}
x_i^* = r + \frac{y_i^*}{p_i}, & i = 1, n \\
y^* = \min \{ p_i (\rho - r) \} & i = 1, n \\
TR = r \sum_{i=1}^{n} p_i + ny^*.
\end{cases}
\]

3. If $\lambda > \frac{n}{n+1}$, then
\[
\begin{cases}
x_i^* = r, & i = 1, n \\
y^* = 0 \\
TR = r \sum_{i=1}^{n} p_i.
\end{cases}
\]

4. If $\lambda < \frac{l}{l+1}$, then

- if $p_j < \overline{p}_1$, $j = l+1, n$, then
\[
\begin{cases}
x_i^* = \rho, & i = 1, l \\
x_j^* = \rho, & j = l+1, n \\
y^* = \overline{p} (\rho - r) \\
TR = l \rho + \rho \sum_{j=l+1}^{n} p_j.
\end{cases}
\]

where $\overline{p}_1 = p_i$, $i = 1, l$.
- else problem has no solution.

5. If $\lambda = \frac{l}{l+1}$, then

- if $p_j < p_i$, $i = 1, l$, $j = l+1, n$, then
\[
\begin{cases}
x_i^* = r + \frac{y_i^*}{p_i}, & i = 1, l \\
x_j^* = \rho, & j = l+1, n \\
y^* = \min \{ p_i (\rho - r) \} \\
TR = ly^* + r \sum_{i=1}^{l} p_i + \rho \sum_{j=l+1}^{n} p_j.
\end{cases}
\]

- else problem has no solution.

6. If $\lambda < \frac{l+(n-m)}{l+(n-m)+1}$, then
• if \( p_j \leq \bar{p}_3 \leq p_i, i = 1, l, j = l + 1, m, \) then

\[
\begin{align*}
    x^*_i &= r + \frac{y^*}{p_i}, & i &= 1, l \\
    x^*_j &= r, & j &= l + 1, m \\
    x^*_k &= r, & k &= m + 1, n \\
    y^* &= \bar{p}(\rho - r) \\
    TR &= ly^* + r \sum_{i=1}^{l} p_i + \rho \sum_{j=l+1}^{n} p_j + (n - m) \bar{p} \rho.
\end{align*}
\]

where \( \bar{p}_3 = p_k, k = m + 1, n. \)

• else problem has no solution.

4 Conclusion

Both models are sensitive to input data. A small change of parameters in the constraint system can change completely the solution. Due to predefined level and the form for measure of fluctuation, Minimax model has a limited optimization range. Our models may provide the framework for optimization of power grids.

References


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On a Markov method

Ioan Țîncu

Abstract
This paper contains a new approach a transformed Markov.

2010 Mathematics Subject Classification: 40A05, 40B05.
Key words and phrases: Method Markov, convergent series.

1 Introduction
Let \( \sum_{k=1}^{\infty} A^{(k)} \) a series of real numbers convergence with the sum \( \sum_{k=1}^{\infty} A^{(k)} = A \).

The method of A.A.Markov consists in expansion of every term \( A^{(k)} \) in, convergent series,
\[
A^{(k)} = \sum_{k=1}^{\infty} a^{(k)}_i, \quad A = \sum_{k=1}^{\infty} A^{(k)} = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} a^{(k)}_i = \sum_{i=1}^{\infty} A_i, \quad A_i = \sum_{k \geq 1}^{\infty} a^{(k)}_i,
\]
when all the series from the columns \( A_i \) are convergent.

2 Main result
Continue presents a new approach of method Markov.

Theorem 1 Let \( \sum_{n=0}^{\infty} a_n \) a real series convergent and \( f, g : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \) two functions which verify:
\[ i) \ f(0, n) = a_n, \ (\forall)n \in \mathbb{N}, \]
\[ ii) \ f(i + 1, j) - f(i, j) = g(i, j + 1) - g(i, j), \ (\forall)i \in \{0, 1, ..., j\}, \ j \in \mathbb{N}. \]
If \( \lim_{n \to \infty} \sum_{j=0}^{n} g(j, n + 1) = 0 \) then \( \sum_{j=0}^{\infty} a_j = \sum_{j=0}^{\infty} [f(j + 1, j) + g(j, j)]. \)

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**Proof.** We have

\[ f(j + 1, j) - f(0, j) = \sum_{i=0}^{j} [f(i + 1, j) - f(i, j)] = \sum_{i=0}^{j} [g(i, j + 1) - g(i, j)]. \]

Note

\[ a_j = f(j + 1, j) + \sum_{i=0}^{j} [g(i, j) - g(i, j + 1)]. \]

Therefore,

\[ \sum_{j=0}^{n} a_j = \sum_{j=0}^{n} f(j + 1, j) + \sum_{j=0}^{n} \sum_{i=0}^{j} [g(i, j) - g(i, j + 1)]. \]

In formula

\[ \sum_{j=0}^{n} \sum_{i=0}^{j} A_{i,j} = \sum_{j=0}^{n} \sum_{i=j}^{n} A_{j,i}, \]

we will consider

\[ A_{i,j} = g(i, j) - g(i, j + 1), \]

and we will obtain

\[ \sum_{j=0}^{n} a_j = \sum_{j=0}^{n} f(j + 1, j) + \sum_{j=0}^{n} \sum_{i=j}^{n} [g(j, i) - g(j, i + 1)] \]

\[ = \sum_{j=0}^{n} f(j + 1, j) + \sum_{j=0}^{n} [g(j, j) - g(j, n + 1)] \]

\[ = \sum_{j=0}^{n} [f(j + 1, j) + g(j, j)] - \sum_{j=0}^{n} g(j, n + 1). \]

Since \( \lim_{n \to \infty} \sum_{j=0}^{n} g(j, n + 1) = 0 \), it follows \( \sum_{j=0}^{n} a_j = \sum_{j=0}^{n} [f(j + 1, j) + g(j, j)]. \)

**Propertie 1** The following relation

\[ (n + j - 2) \binom{n}{j} \geq j, \quad (\forall) j \in \mathbb{N} \]

holds.

**Proof.** In order to prove this relation we will use mathematical induction.

For \( j = 1 \) we have \( \binom{n + 3}{1} \geq 1, \ 1 \geq 1. \)
We suppose that for \( j = k \) we have \( \binom{n+k+2}{k} \geq k \) and we prove that for \( j = k+1 \) we obtain \( \binom{n+k+3}{k+1} \geq k+1 \).

Since
\[
\binom{n+k+3}{k+1} = \frac{n+k+3}{k+1} \binom{n+k+2}{k} \geq \frac{n+k+3}{k+1} \cdot k \geq k+1,
\]
it follows \( \binom{n+k+3}{k+1} \geq k+1 \).

In conclusion the inequality (1) is verified.

**Example 1** Let \( \sum_{n \geq 0} a_n \) with \( a_n = \frac{1}{(n+1)^2}, n \in \mathbb{N} \).

We consider
\[
f(i, j) = \frac{i!j!}{(j+1)(i+j+1)!}, \quad g(i, j) = \frac{i!j!}{(i+1)(i+j+1)!}, \quad (i, j) \in \mathbb{N} \times \mathbb{N}.
\]
The functions verify the conditions i) and ii) from the Theorem 1.

We will prove \( \sum_{j=0}^{n} g(j, n+1) \to 0 \) for \( n \to \infty \).

We have
\[
\sum_{j=0}^{n} g(j, n+1) = \sum_{j=0}^{n} \frac{1}{(n+2)(j+1)} \cdot \frac{1}{(n+1)^2}
\]
\[
\leq \frac{1}{n+2} \left[ \frac{1}{n+2} + \sum_{j=1}^{n} \frac{1}{j(j+1)} \right] = \frac{1}{n+2} \left( 2 - \frac{1}{n+1} \right) \to 0, \quad \text{for} \quad n \to \infty.
\]

From Theorem 1 it follows
\[
\sum_{j=0}^{\infty} \frac{1}{(j+1)^2} = \sum_{j=0}^{\infty} [f(j+1, j) + g(j, j)]
\]
\[
= \sum_{j=0}^{\infty} \left[ \frac{(j+1)!j!}{(j+1)(2j+2)!} + \frac{(j!)^2}{(2j+1)!(j+1)} \right] = 3 \sum_{j=0}^{\infty} \frac{(j!)^2}{(2j+2)!}.
\]

In conclusion
\[
\frac{\pi^2}{6} = \sum_{j=0}^{\infty} \frac{1}{(j+1)^2} = 3 \sum_{j=0}^{\infty} \frac{(j!)^2}{(2j+2)!}
\]
\[
= 3 \sum_{j=0}^{\infty} \frac{1}{(2j+1)(2j+2)} \frac{(2j)!}{j!} = 3 \sum_{j=0}^{\infty} \frac{1}{(j+1)^2 \binom{2j+2}{j}}.
\]

If we apply the Markov method we will consider:
i) \[ \sum_{n \geq 1} \frac{1}{n(n + 1)(n + k)} = \frac{1}{k \cdot k!} \]

ii) \[ a_i^{(k)} = \frac{(i - 1)!}{k(k + 1)...(k + i)} \]

iii) \[ A_i = \sum_{k \geq 1} a_i^k = (i - 1)! \sum_{k \geq 1} \frac{1}{k(k + 1)...(k + i)} = (i - 1)! \sum_{k \geq 1} \left( \frac{1}{k(k + 1)...(k + i)} - \frac{1}{(k + 1)...(k + i)} \right) \cdot \frac{1}{i} = \frac{(i - 1)!}{i \cdot i!} = \frac{1}{i^2}, \]

\[ \sum_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} = \sum_{i \geq 1} \sum_{k \geq 1} a_i^{(k)}. \]

References


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About a category of abelian groups

Teodor Dumitru Vălcănc

Abstract

We say that an \( R \)-module (abelian group) \( M \) has the direct summand intersection property (in short D.S.I.P.) if the intersection of any two direct summands of \( M \) is again a direct summand in \( M \). In this work we will present three classes of abelian groups (torsion, divisible, respectively torsion-free) which have the property that any proper subgroup has D.S.I.P. and we are going to show that there are not such mixed groups.

2010 Mathematics Subject Classification: 13D30, 43A70.
Key words and phrases: \( R \)-module, abelian group, Frattini subgroup.

1 Short history and introduction

Let \( R \) be an associative ring, with unity. We say that an \( R \)-module (abelian group) \( M \) has the direct summand intersection property, in short D.S.I.P., if the intersection of any two direct summands of \( M \) is again a direct summand in \( M \). The \( R \)-modules (the abelian groups) with D.S.I.P. have been studied in [1]-[3], [5], [6] and [8]-[17]. In this work we will present three classes of abelian groups (torsion, divisible, respectively torsion-free) which have the property that any proper subgroup has D.S.I.P. and we are going to show that there are not mixed groups with this property. We shall call these groups ”groups with the property (P)”.

In this context, all through this paper by group we mean abelian group in additive notation and we will use the classic notations:
- \( P \) - for the set of all prime numbers,
- \( r(A) \) - for the rank of a group \( A \),
- \( t(A) \) - for the type of a torsion-free group \( A \),
- \( F(A) \) - for the Frattini subgroup of a group \( A \) (see [10]),
- \( |I| \) - for the cardinal of a set \( I \).

For the beginning we make the remark that, according to the proofs in [10], it follows:

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Remark 1 Let $G$ be an abelian group.

1) If $G$ has D.S.I.P., then no any their (proper) subgroup has D.S.I.P..

2) If any proper subgroup of $G$ has D.S.I.P. and $G$ has a (direct) decomposition in fully invariant direct summands, then $G$ has D.S.I.P..

3) If $G$ has D.S.I.P., then no any quotient group of $G$ has D.S.I.P..

4) If any nontrivial quotient group of group $G$ has D.S.I.P. and $G$ has a (direct) decomposition in fully invariant direct summands, then $G$ has D.S.I.P..

5) Let $B$ be a (proper) subgroup of $G$. If $B$ and $G=B$ have D.S.I.P., then it does not follow that $G$ has D.S.I.P..

Proof. 1) Counterexample: let $p$ be a prime number and let be $G=\mathbb{Z}(p)\oplus \mathbb{Z}(p^\infty)$ and $B=\mathbb{Z}(p)\oplus C$, where $C=\langle c_2 \rangle$, and $c_2$ is a generator of $\mathbb{Z}(p^\infty)$ with the property that $p^2c_2=0$; so $C \cong \mathbb{Z}(p^2)$. Then, according to [6,Theorem 2], $G$ has D.S.I.P., but $B$ doesn’t have this property anymore.

2) Let $G$ be a group with the property that any proper subgroup has D.S.I.P. and let $G=\bigoplus_{i\in I} G_i$ be a direct decomposition of $G$ in fully invariant direct summands. Then the hypothesis and [6,Lemma 1] show that $G$ has D.S.I.P., too.

3) Counterexample: let $G=Q \oplus Q$ and $B=Z$. Then, according to [8,Lemma 4.1 and Corollary 2.2], $G$ and $B$ have D.S.I.P., but [6,Theorem 2] shows that $G/B=(Q/Z)\oplus (Q/Z) = \bigoplus_{p \in P} (\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty))$ does not have this property.

4) Let $G$ be a group with the property that any quotient group (of $G$) has D.S.I.P. and, as to the point 2), let $G=\bigoplus_{i\in I} G_i$ be a direct decomposition of $G$ in fully invariant direct summands. Then the hypothesis shows that, for every $i \in I$, $G_i$ has D.S.I.P.. Again [6,Lemma 1] shows that $G$ has D.S.I.P., too.

5) Counterexample: let be $G=\mathbb{Z}(p^2)\oplus \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(q^\infty)$, where $p$ and $q$ are two distinct prime numbers. Then $F(G) = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(q^\infty)$ and $G/F(G) \cong \mathbb{Z}(p^2)$. According to [8,Remark 3.6] and to [6,Theorem 2], $F(G)$ and $G/F(G)$ have D.S.I.P., but $G$ does not have this property.

2 Torsion groups

We begin the determination of the torsion groups with the property $(P)$ with the determination of the $p$-groups with this property.

Theorem 1 The following statements are equivalent for a $p$-group $G$:

a) $G$ has the property $(P)$.

b) either
   
i) $G$ is indecomposable non-elementary, \hspace{1cm} (1)
   or
   
   ii) $G$ is elementary, \hspace{1cm} (2)

iii) \[ G = Z(p^2) \oplus Z(p). \] (3)

**Proof.** a) implies b) Let \( G \) be a \( p \)-group with the property that any proper subgroup has D.S.I.P.. If \( G \) is either indecomposable or elementary, then this gives the required result. So that we suppose that \( G \) is not a such group. If there is \( g \in G[p] \) such that \( h_p(g) = k \), where \( 1 \leq k < \infty \), then \( g \in p^kG \) and \( g \notin p^{k+1}G \). It follows that there is \( a \in G \) such that \( g = p^ka \). Since \( pg = 0 \), it follows that \( o(a) = k + 1 \) and according to [4,Corollary 27.2], \( \langle a \rangle \) is a direct summand in \( G \), that is \( G = \langle a \rangle \oplus B \). Now we distinguish two cases.

**Case 1:** For any \( b \in B[p] \), \( \langle b \rangle \) is a direct summand in \( B \). Then \( B[p] \) is a direct summand in \( B \). We suppose that \( B = B[p] \oplus C \). But \( C[p] = C \cap B[p] = 0 \); so \( C = 0 \) and \( B = B[p] \). It follows that there is a cardinal \( m_p \) such that, up to isomorphism, \( G = Z(p^{k+1}) \oplus Z(p) \). If \( k = m_p = 1 \), then \( G = Z(p^2) \oplus Z(p) \) is of the form (3). If \( k \geq 2 \), the subgroup \( Z(p^{k+1}) \oplus Z(p) \) does not have D.S.I.P., which is a contradiction to the hypothesis.

**Case 2:** There is a \( b_1 \in B[p] \) such that \( \langle b_1 \rangle \) is not a direct summand in \( B \). We choose such a \( b_1 \in B[p] \). Then the subgroup \( \langle a \rangle \oplus \langle b_1 \rangle \) does not have D.S.I.P. and \( G \) does not have the property \( (P) \) - again we obtain a contradiction to the hypothesis.

It follows that, for any \( g \in G[p] \), either \( h_p(g) = 0 \) or \( h_p(g) = \infty \). So the direct summands of \( G \) are either \( p \)-bounded or isomorphic to \( Z(p^\infty) \). So, \( G = E \oplus F \), where \( pE = 0 \) and \( F \) is divisible. Then, according to the hypothesis and to [6,Theorem 2], it follows that \( F = Z(p^\infty) \) and \( G \) is a \( p \)-group with D.S.I.P.. So there is a cardinal \( m_p \) such that \( G = (\oplus m_p Z(p)) \oplus Z(p^\infty) \). We suppose that \( Z(p^\infty) = \langle c_1, c_2, \ldots, c_n, \ldots \rangle \), with relationships \( pc_1 = 0, pc_2 = c_1, \ldots, pc_n = c_{n-1}, \ldots \) and also we consider the subgroup \( K = Z(p) \oplus \langle c_2 \rangle \). Then the subgroup \( K \) does not have D.S.I.P. - again we obtain a contradiction to the hypothesis. In conclusion \( F = 0 \) and \( G \) is elementary.

b) implies a) If \( G \) is a group as in the statement, then either

i) \( G = Z(p^n) \), \hspace{1cm} (4)

or

ii) \( G = Z(p^\infty) \), \hspace{1cm} (5)

or

iii) \( G = \oplus m_p Z(p) \), \hspace{1cm} (6)

or \( G \) is of the form (3), where \( n \in \mathbb{N}^* \), \( 2 \leq n < \infty \) and \( m_p \) is any cardinal. If \( G \) is of the form (3) it has the property (P), because any proper subgroup of \( G \) is either indecomposable or elementary. If \( G \) is either of the form (4) or of the form (5), then any subgroup of \( G \) is indecomposable, so any subgroup of \( G \) has, in a trivial way, D.S.I.P.. If \( G \) is of the form (6), then any subgroup of \( G \) is of the same form, so any subgroup of \( G \) has D.S.I.P..

From the above theorem and from [6,Theorem 2] it follows the following remarks:

**Remark 2** a) If \( G \) is a \( p \)-group with D.S.I.P., then no any subgroup of \( G \) has the same property.
b) Generally, the p-groups with the property (P) not coincide with these with D.S.I.P. with this property.

Now we can present the structure of torsion groups with the property (P).

Corollary 1 If $G$ is a torsion group, then the following statements are equivalent:

a) $G$ has the property (P).

b) $G$ is of the form

$$G = \left( \bigoplus_{p \in P_1} A_p \right) \oplus \left( \bigoplus_{p \in P_2} B_p \right) \oplus \left( \bigoplus_{p \in P_3} C_p \right) \oplus \left( \bigoplus_{p \in P_4} D_p \right),$$

(7)

where:

- $P_1, P_2, P_3$ and $P_4$ are subsets of the set $P$ of all prime numbers, with the property that $P_1 \cap P_2 = P_1 \cap P_3 = P_1 \cap P_4 = P_2 \cap P_3 = P_2 \cap P_4 = P_3 \cap P_4 = \emptyset$,

- for every $p \in P_1$, $A_p$ is a reduced (non-elementary) indecomposable $p$-group,

- for every $p \in P_2$, $B_p$ is an elementary $p$-group,

- for every $p \in P_3$, $C_p$ is a $p$-group of the form (3),

- for every $p \in P_4$, $D_p$ is an indecomposable divisible $p$-group.

Proof. a) implies b) Let $G = \bigoplus_{p \in P} G_p$ be a torsion group, decomposed according to [4,Theorem 8.4]. According to the hypothesis, for every $p \in P$, any proper subgroup of $G_p$ has D.S.I.P.. So, for every $p \in P$, $G_p$ is either of the form (3) or (4) or (5) or (6) and $G$ is of the form (7).

b) implies a) Conversely, let $G$ be a group of the form (7) and let $H$ be any subgroup of $G$. Then [4,Theorem 18.1] shows that $H = \bigoplus_{p \in P} H_p$, where, for every $p \in P$, $H_p = H \cap G_p$, and $G_p$ is the $p$-component of $G$. Since, for every $p \in P$, $H_p$ satisfies the conditions from (2.1), it follows that, for every $p \in P$, $H_p$ has D.S.I.P.. Then [6,Lemma 1] shows that $H$ has D.S.I.P., too and $G$ has the property (P).

From the above proved facts and from [8,Corollary 3.3] we obtain:

Remark 3 a) If $G$ is a torsion group with D.S.I.P., then no any proper subgroup of $G$ has this property.

b) Generally, the torsion groups which have the property (P) not coincide with these with D.S.I.P. with this property.

3 Divisible groups

Now we shall pass to determination of the divisible groups with the property (P). In this context we have the following result:

Proposition 1 If $G$ is a divisible group, then the following statements are equivalent:
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a) $G$ has the property $(P)$.
b) $G$ has one from the following forms:
   i) $G = Q$, $\quad (8)$
or
   ii) $G = Q \oplus Q$, $\quad (9)$
or
   iii) $G = \bigoplus_{p \in P_0} \mathbb{Z}(p^\infty)$, $\quad (10)$
   or

   where $P_0$ is a subset of the set $P$ of all prime numbers.

Proof. a) implies b) Let $G = (\oplus_{m_0} Q) \oplus \bigoplus_{p \in P_0} D_p$ be a divisible group, where: $Q$

is the additive group of rational numbers, $m_0$ is the torsion-free rank of $G$, $P_0$ is a

subset of the set $P$ of all prime numbers and, for every $p \in P_0$, $D_p$ is a divisible

$p$-group. According to the hypothesis and to [17,Proposition 6], either $G = \oplus_{m_0} Q$
or $G = \bigoplus_{p \in P_0} D_p$.

We suppose that $G = \oplus_{m_0} Q$ and $m_0 \geq 3$. In this case we consider the subgroup

$H = Q \oplus E \oplus F$, where $E$ and $F$ are (proper) subgroups of $Q$ such that $t(E)$ and

$t(F)$ are incomparable. Then [5,Theorem 3.3] shows that $H$ does not have D.S.I.P.. It

follows that if $G$ is a divisible torsion-free group with the property $(P)$, then it is

either of the form (8) or of the form (9).

Otherwise: According to (4.1), $Q \oplus Q$ has an indecomposable subgroup $H$, of

rank 2. Then the subgroup $B = H \oplus Q$ does not have D.S.I.P., because there are homomorphisms $0 \neq f : H \to Q$ which are not monomorphisms.

We suppose that $G$ is torsion. According to the hypothesis and to [8,Theorem

4.4], $G$ is of the form (10).

b) implies a) If $G = Q$, then any subgroup $H$ of $G$ is indecomposable, so $H$ has

D.S.I.P.. If $G$ is of the form (9), then any subgroup of $G$ is either indecomposable

(see (4.1)) or completely decomposable of rank $r \leq 2$. In this case [6,Theorem 6]
completes the proof. If $G$ is torsion and $H$ is any subgroup of $G$, then, following
the same reasoning as in (2.3), we obtain that $H = \bigoplus_{p \in P} H_p$, where, for every $p \in P$,

$H_p$ is a subgroup of $\mathbb{Z}(p^\infty)$. It follows that, for every $p \in P$, $H_p$ has D.S.I.P.. Again

[8,Theorem 4.4] shows that $H$ has D.S.I.P..

From (3.1) and [8,Theorem 4.4] we obtain:

Remark 4 a) If $G$ is a divisible group with D.S.I.P., then not any subgroup of $G$

has the same property.

b) The divisible groups which have the property that any proper subgroup has

D.S.I.P. coincide with these with D.S.I.P. with this property.
4 Torsion-free groups

Before passing to determine the structure of the torsion-free abelian groups with the property (P), we will present several results, obtained in [16], concerning the indecomposable subgroups of some torsion-free groups, results which we need here.

(4.1): If \( I \) is any index set, with the property that \( |I| \leq |P| \), then the group \( Q^* = \oplus_I Q \) has indecomposable subgroups of every rank \( m \leq |I| \).

(4.2): Let \( \{H_i | i \in I\} \) be a family of torsion-free groups such that, for every \( i \in I \), there is \( G_i \leq H_i \), where \( \{G_i | i \in I\} \) is a family of reduced of rank one groups, with the property that, for every \( i_1, i_2 \in I \), \( i_1 \neq i_2 \), \( t(G_{i_1}) \) and \( t(G_{i_2}) \) are incomparable. Then the group \( H = \bigoplus_{i \in I} H_i \) has indecomposable subgroups of every rank \( m \leq |I| \).

(4.3): Let \( G = B \oplus C \) be any group and \( A \) a subgroup of \( G \). If \( C \) is free and \( A \) is indecomposable, then either \( r(A) = 1 \) or \( A \subseteq B \).

Now we shall pass to the determination of torsion-free groups with the property (P). We begin with the following elementary result:

Remark 5 If \( G \) is a free group, then any subgroup of \( G \) has D.S.I.P..

Proof. According to [4,Theorem 14.5], any subgroup of a free group is a free group, too. Now [8,Corollary 2.2] completes the proof.

Proposition 2 Let be \( G = D \oplus F \), where \( D \neq 0 \) is divisible and \( F \) is free. Then the following statements are equivalent:

a) \( G \) has the property (P).

b) \( r(D) = 1 \) and \( F \) is of finite rank.

Proof. a) implies b) Let be \( G = D \oplus F \), where \( D \neq 0 \) is divisible and \( F \) is free. According to the hypothesis and to [5,Theorem 3.3], it follows that \( F \) is of finite rank.

If \( r(D) \geq 2 \), then let \( B \) and \( C \) be two subgroups of \( D \), such that \( t(Z) < t(B) < t(C) \) and let \( K = Z \oplus B \oplus C \). Then \( K \) is a subgroup of \( G \), which, according to [6,Theorem 6], does not have D.S.I.P. It follows that \( r(D) = 1 \).

Otherwise: If \( r(D) \geq 2 \), then, according to (4.1), \( D \) has an indecomposable subgroup \( H \), of rank two. Since there are homomorphisms \( 0 \neq f : H \rightarrow Z \) which are not monomorphisms, in this case [5,Proposition 1] shows that the subgroup \( H \oplus Z \) does not have D.S.I.P..

b) implies a) Let \( G = Q \oplus F \) be a group as in the statement and let \( B \) be any subgroup of \( G \). According to (4.3), if \( B \) is indecomposable, then \( r(B) = 1 \) and \( B \) has D.S.I.P.. We suppose that \( r(B) \geq 2 \). Then we distinguish three cases:

Case 1: Suppose that \( Q \leq B \). Then \( B = Q \oplus E \), where \( E = F \cap B \) is a free group and \( r(E) \geq 1 \). According to [5,Theorem 3.3], \( B \) has D.S.I.P..

Case 2: Suppose that \( F \leq B \). Then \( B = C \oplus F \), where \( C = Q \cap B \) is a reduced of rank one group. According to [5,Theorem 5.5], \( B \) has D.S.I.P..

Case 3: Suppose that \( Q \cap B = C \) is a reduced of rank one group and \( F \cap B = E \) is a free group with \( r(E) < r(F) \). In this case, according to [4,p.44], there is \( H \) a
In this case (4.2) shows that \( K \) is an indecomposable subgroup of rank two and (thus) \( G \) has an indecomposable subgroup of rank two and (thus) \( G \) does not have the property (P).

Therefore, also in this case, \( r(G) \leq 2 \).

\[ \text{Theorem 2} \] Let \( G \) be a completely decomposable torsion-free group, which has no free direct summand. Then the following statements are equivalent:

a) The group \( G \) has the property (P).

b) \( r(G) \leq 2 \).

\[ \text{Proof.} \] a) implies b) We distinguish two cases:

Case 1: \( G \) is not reduced. In this case \( G = D \oplus B \), where \( D \) is divisible and \( B \) is reduced. According to the hypothesis and to (3.1), it follows that \( r(D) \leq 2 \).

From [5, Theorem 3.3] it follows that \( B \) is homogeneous, completely decomposable of finite rank.

We suppose that \( G = Q \oplus B \) and \( r(B) \geq 2 \), and we consider the subgroup \( K = Q \oplus C \oplus E \), where \( C \) is a direct summand of rank one in \( B \), and \( E \) is a subgroup of \( B \) which is isomorphic with \( Z \). According to [5, Theorem 3.3], \( K \) does not have D.S.I.P. - contradiction to the hypothesis. It follows that if \( G = Q \oplus B \), then \( r(B) \leq 1 \).

Now we suppose that \( G = Q \oplus Q \oplus B \). From above proved facts it follows that \( r(B) \leq 1 \). If \( r(B) = 1 \), then let \( U \) and \( V \) be two subgroups of \( Q \) such that \( t(B) < t(U) < t(V) \) and \( K = V \oplus U \oplus B \). Then, again, [5, Theorem 3.3] shows that this subgroup \( K \) does not have D.S.I.P.. It follows that \( B = 0 \) and \( r(G) = 2 \).

Otherwise: If \( G = Q \oplus Q \oplus B \), then, according to (4.1), \( Q \oplus Q \) has an indecomposable subgroup of rank two \( L \), in which case \( L \oplus B \) does not have D.S.I.P..

Case 2: \( G \) is reduced. Then \( G = \bigoplus_{i \in I} G_i \), where, for every \( i \in I \), \( r(G_i) = 1 \). We suppose that \( r(G) \geq 3 \).

If there are \( i_1, i_2 \in I \), \( i_1 \neq i_2 \), such that \( t(G_{i_1}) \) and \( t(G_{i_2}) \) are comparable, then the subgroup \( K = G_{i_1} \oplus G_{i_2} \oplus H \) does not have D.S.I.P.; here \( H \) is a subgroup of \( G \) which is isomorphic to \( Z \).

We suppose that, for every \( i_1, i_2 \in I \), \( i_1 \neq i_2 \), \( t(G_{i_1}) \) and \( t(G_{i_2}) \) are incomparable. In this case (4.2) shows that \( G \) has an indecomposable subgroup of rank two and (thus) \( G \) does not have the property (P).

Therefore, also in this case, \( r(G) \leq 2 \).
b) implies a) Let $G$ be a group as in the statement, with $r(G) \leq 2$ and let $B$ be any subgroup of $G$. Then either $B$ is indecomposable or $B = H \oplus K$, where $r(H) = r(K) = 1$. In both cases $B$ has D.S.I.P..

**Corollary 2** If $G$ is a torsion-free group, then the following statements are equivalent:

a) The group $G$ has the property $(P)$.
b) i) If $G$ is divisible, then $r(G) \leq 2$;
   ii) If $G$ is reduced, then:
   either
   a) $G$ is indecomposable and $r(G) \leq 2$,
   or
   b) $G$ is completely decomposable and it has no free direct summand and $r(G) \leq 2$,
   or
   $\gamma)$ $G = B \oplus F$
   where: $F$ is free of finite rank, $r(B) = 1$ and $t(B) \geq t(F)$.
   iii) $G = Q \oplus H$
   where $H$ is reduced and either
   a) if $H$ is free, then $r(H)$ is finite,
   or
   $\beta)$ if $H$ is not free, then $r(H) = 1$.

**Proof** a) implies b) Let $G = D \oplus H$ be a torsion-free group, where $D$ is divisible and $H$ is reduced. According to proved facts in this section, we distinguish three cases:

**Case 1:** Suppose that $H = 0$. In this case (3.1) shows that $G$ has the property $(P)$ if and only if $r(G) \leq 2$.

**Case 2:** Suppose that $D = 0$. If $G$ is indecomposable and $r(G) \geq 3$, then, following the same reasoning as in the proof of (4.6) - Case 2, we obtain that $G$ has a subgroup which does not have D.S.I.P.. Therefore, in this case $r(G) \leq 2$.

We suppose that $G$ is not indecomposable. In this case we consider, according to Zorn’s Lemma, a maximal independent set $\{G_i\}_{i \in I}$ of direct summands of rank one of $G$. If $G^* = \bigoplus_{i \in I} G_i$, then either $G = G^*$ or $G = G^* \oplus K$, where $K$ is a direct summand of $G$ and $r(K) \geq 2$. According to the hypothesis, $K$ and any his subgroup has D.S.I.P.. But $K$ is not completely decomposable, because then we contradict the maximality of $\{G_i\}_{i \in I}$. If $K$ has an indecomposable subgroup $L$, then $r(L) \geq 2$ and, for every $i \in I$, $G_i \oplus L$ does not have the property $(P)$. Therefore $K = 0$ and $G$ is completely decomposable.

If $G$ has no direct summand isomorphic to $Z$, then (4.6) shows that $G$ has the property $(P)$ if and only if $r(G) \leq 2$.

We suppose that $G$ has a direct summand isomorphic to $Z$. Then, according to [6,Theorem 6], $G = (\oplus n Z) \oplus (\bigoplus_{i \in J} H_i)$, where: $n \in N^*$, $J$ is a subset of $I$, for every
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\[ i_1, i_2 \in J, i_1 \neq i_2, t(H_{i_1}) \text{ and } t(H_{i_2}) \text{ are incomparable types. Now, following the same reasoning as in the proof of (4.6) - Case 2, we obtain that } |J| \leq 2. \text{ If } |J| = 2, \text{ then (4.2) and [5,Theorem 5.5] show that } G \text{ has a subgroup which does not have D.S.I.P.. Now (4.5) shows that, in this case, } G \text{ has the property (P) if and only if it is of the form (14).}

Case 3: Suppose that \( D \neq 0 \) and \( H \neq 0 \). According to the hypothesis, \( D, H \) and any their subgroup have D.S.I.P.. According to (3.1) and to [5,Theorem 3.3], \( r(D) \leq 2 \) and \( H \) is completely decomposable of finite rank. If \( H \) is free, then (4.5) completes the proof for this case. If \( H \) is not free, then, according to [5,Theorem 3.3], it has no direct summands isomorphic to \( \mathbb{Z} \). In this case (4.6) completes the proof.

Two remarks should be made here, see [8]:

Remark 6 a) If \( G \) is a torsion-free group with D.S.I.P., then no any subgroup of \( G \) has the same property.

b) The torsion-free groups which have the property that any proper subgroup has D.S.I.P. coincide with these with D.S.I.P. with this property.

5 Splitting mixed groups

In this section we are going to show that the problem of mixed groups with the property (P) has no solution, that is:

Proposition 3 There are not mixed groups with the property (P).

Proof. Indeed, if \( G \) is a mixed group, then there are \( g, t \in G \), where \( g \) is an element of infinite order and \( t \) is an element of order \( p^n \), with \( p \)-prime number and \( n \in \mathbb{N}^* \). According to [5,Proposition 1.4], it follows that the subgroup \( H = \langle g \rangle \oplus \langle t \rangle \cong \mathbb{Z} \oplus \mathbb{Z}(p^n) \) does not have D.S.I.P. and then \( G \) does not have the property (P) anymore.

Open problem: Find the structure of abelian groups which have the property that any proper quotient group has the direct summand intersection property.

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Note on a conditional inequality

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Abstract

In this paper we present some considerations on a conditional inequality.

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1 Introduction

In [3] E. Păltănea has presented some interesting proofs for the following conditional inequality:

**Theorem 1** Let $a_1, a_2, \ldots, a_n$ be arbitrary positive numbers, $n \geq 2$, such that

\[ \frac{1}{1 + a_1} + \frac{1}{1 + a_2} + \cdots + \frac{1}{1 + a_n} = s, \quad s \in (0, 1]. \]

Then

\[ a_1 a_2 \cdots a_n \geq \left( \frac{n}{s} - 1 \right)^n. \]

The main purpose of this note is to give a new proof of this theorem and we will establish the following related inequalities.

**Theorem 2** Let $a_1, a_2, \ldots, a_n$ be arbitrary positive numbers such that

\[ \frac{1}{1 + a_1} + \frac{1}{1 + a_2} + \cdots + \frac{1}{1 + a_n} = s, \quad s \in (n - 1, n). \]

Then

\[ a_1 a_2 \cdots a_n \leq \left( \frac{n}{s} - 1 \right)^n, \quad n \geq 2. \]
Theorem 3 For \((a_1, a_2, \ldots, a_n) \in [1, \infty)^n\) and \(n \geq 1\), the inequalities

\[
\prod_{i=1}^{n} a_i \geq e^p \left(2n - e \sum_{i=1}^{n} \frac{1}{a_i}\right)^p,
\]

\[
\prod_{i=1}^{n} a_i > e^p \left(2n - 4 \sum_{i=1}^{n} \frac{1}{1 + a_i}\right)^p
\]

are valid for all \(p \in (0, \infty)\).

2 Proofs of theorems

Now we are in a position to prove our theorems.

**Proof of Theorem 1.** If \(s = 1\), we denote \(\frac{1}{1 + a_i} = x_i\) and we have \(a_i = \frac{1 - x_i}{x_i}\), \(x_i \in (0, 1)\). But \(\sum_{i=1}^{n} x_i = 1\), and hence

\[
\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} \frac{1 - x_i}{x_i} = \prod_{i=1}^{n} \frac{x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n}{x_i}
\]

\[
\geq \prod_{i=1}^{n} \left(\frac{n-1}{x_i} \right)^{n-1} = (n-1)^n.
\]

For \(s \in (0, 1)\) we denote

\[
\frac{1}{s} = x_i, \ a_i = \frac{1 - x_i}{x_i}
\]

and

\[
\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} \frac{1 - x_i}{x_i}.
\]

Let \(f(x_1, x_2, \ldots, x_n) = \varphi(x_1) \varphi(x_2) \cdots \varphi(x_n)\), where \(\varphi(x) = \frac{1}{s} - \frac{x\phi'(x)}{\phi(x)}\), \(\sum_{i=1}^{n} x_i = 1\).

We have \(\varphi'(x_i) = -\frac{1}{sx_i^2}, \ \varphi''(x_i) = \frac{2}{sx_i^3}\).

Now we consider

\[
L(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) + \lambda(x_1 + x_2 + \cdots + x_n - 1).
\]

The system of equations

\[
\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \lambda = 0, \quad i = 1, 2, \ldots, n.
\]
Note on a conditional inequality

which is equivalent to

\[- \frac{1}{s x_1^2} \varphi(x_2) \ldots \varphi(x_n) = - \frac{1}{s x_2^2} \varphi(x_1) \varphi(x_3) \ldots \varphi(x_n) = \ldots = \frac{1}{s x_n^2} \varphi(x_1) \varphi(x_2) \ldots \varphi(x_{n-1}),\]

or

\[x_1^2 \varphi(x_1) = x_2^2 \varphi(x_2) = \ldots = x_{n-1}^2 \varphi(x_{n-1}) = x_n^2 \varphi(x_n),\]

and

\[
\begin{cases}
(x_1 - x_2) (x_1 + x_2 - \frac{1}{s}) = 0 \\
(x_2 - x_3) (x_2 + x_3 - \frac{1}{s}) = 0 \\
\ldots \\
(x_{n-1} - x_n) (x_{n-1} + x_n - \frac{1}{s}) = 0,
\end{cases}
\]

has a unique nonzero solution \(x_i = \frac{1}{n}, 1 \leq i \leq n\). Thus, the point \(\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)\) is a unique critical point which is located in the interior of \(\left\{(x_1, x_2, \ldots, x_n) \in (0, 1)^n \mid \sum_{i=1}^n x_i = 1\right\}\).

We observe that \(x_1 + x_2 = \frac{1}{s} \in (1, \infty)\) is impossible, and so on.

Straightforward computation gives us

\[
d^2 L \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) = \frac{n^4}{s^n} (n-s)^{n-2} \left[\frac{2(n-s)}{n} (dx_1^2 + dx_2^2 + \cdots + dx_n^2) + 2(dx_1 dx_2 + dx_1 dx_3 + \cdots + dx_{n-1} dx_n)\right] > \frac{n^4}{s^n} (n-s)^{n-2} (dx_1 + dx_2 + \cdots + dx_n)^2 = 0,
\]

because \(\frac{2(n-s)}{n} > 1\) for \(n \geq 2\).

Hence \(\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)\) is a extremal point (minimum) of the function \(f(x_1, x_2, \ldots, x_n)\).

Therefore, it follows that

\[f(x_1, x_2, \ldots, x_n) \geq f \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) = \left(\frac{n-s}{s}\right)^n.\]

**Proof of Theorem 2.** We denote \(\frac{1}{a_i} = b_i \in (0, \infty)\) and we have

\[
\sum_{i=1}^n \frac{1}{1 + a_i} = \sum_{i=1}^n \frac{b_i}{1 + b_i} = n - \sum_{i=1}^n \frac{1}{1 + b_i},
\]

But \(\sum_{i=1}^n \frac{1}{1 + a_i} = s \in (n-1, n)\), hence \(\sum_{i=1}^n \frac{1}{1 + b_i} = n - s \in (0, 1)\) with \(b_i \in (0, \infty)\).
Using the Theorem 1 we obtain
\[ \prod_{i=1}^{n} b_i \geq \left( \frac{n}{n-s} - 1 \right)^n = \left( \frac{s}{n-s} \right)^n. \]

Hence \( \prod_{i=1}^{n} \frac{1}{a_i} \geq \left( \frac{s}{n-s} \right)^n \), and \( \prod_{i=1}^{n} a_i \leq \left( 1 - \frac{n}{s} \right)^n \).

**Proof of Theorem 3.** Using probability methods, the following sharp inequality is established in [2]
\[ \exp \left( \sum_{i=1}^{n} x_i \right)^p \leq \exp \left( \sum_{i=1}^{n} x_i \right), \]
where \( p \in (0, \infty) \), \( n \in \mathbb{N}^* \), \( x_i \geq 0 \) for \( 1 \leq i \leq n \).

For \( e^{x_i} = a_i \in [1, \infty) \) we obtain:
\[ \prod_{i=1}^{n} a_i \geq \frac{e^p}{p^p} \left( \sum_{i=1}^{n} \ln a_i \right)^p. \]

Letting \( f(x) = x \ln x - 2x \), \( x \in [1, \infty) \), it is easy to obtain \( \ln x \geq 2 - \frac{e}{x} \), for \( x \in [1, \infty) \).

Hence \( \sum_{i=1}^{n} \ln a_i \geq 2n - e \sum_{i=1}^{n} \frac{1}{a_i} \) and \( \prod_{i=1}^{n} a_i \geq \frac{e^p}{p^p} \left( 2n - e \sum_{i=1}^{n} \frac{1}{a_i} \right)^p \).

Now, for \( g(x) = (x+1) \ln x - 2(x-1) \), \( x \in [1, \infty) \) it is easy to find \( \ln x \geq 2 - \frac{4}{x+1} \) for \( x \in [1, \infty) \).

Hence,
\[ \sum_{i=1}^{n} \ln a_i \geq 2n - 4 \sum_{i=1}^{n} \frac{1}{1+a_i}, \]
and
\[ \prod_{i=1}^{n} a_i \geq \frac{e^p}{p^p} \left( 2n - 4 \sum_{i=1}^{n} \frac{1}{1+a_i} \right)^p. \]

**References**


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Shape preserving properties of generalized Szász operators of max-product kind

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Abstract

In this study, we present the nonlinear generalized Szász operators of max-product kind and we give a better error estimate for the large subclasses of functions. Also we study some shape preserving properties of concerned operators.

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Key words and phrases: nonlinear max-product operators, max-product generalized Szász operators, shape preserving properties.

1 Introduction

In the Korovkin-type approximation theory, one of the important problems is approximate to a continuous function by linear positive operators. In order to define these approximating operators mainly the addition and multiplication of the reals are used. Therefore all of approximating operators are linear operators.

In [1], [2] and the open problem given in [11](pp. 324-326, Open Problem 5.5.4), the following questions were raised:

- Is the linear structure the only one which allows us to construct approximation operators?
- Do all the approximation operators have to be linear?

The answer to these questions is negative. As a solution to this problem “max-product kind operators” were presented using maximum instead of sum in usual linear operators and gave error estimate in terms of modulus of continuity in [3], [4], [5], [6], [7], [9], [10].
The nonlinear Favard-Szász–Mirakjan operators of max-product type were introduced and the error estimate was obtained in [3],[8], [9].

In [12], we defined the nonlinear generalized Szász operators of max-product type as the following

\[
S_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} c_{n,k}(x) f\left(\frac{b_nk}{a_n}\right)}{\bigvee_{k=0}^{\infty} c_{n,k}(x)}
\]

with

\[
c_{n,k}(x) = \frac{(a_n x)^k}{b_n^k k!}
\]

where \(x \in [0, \infty)\), \((a_n)\) and \((b_n)\) are increasing and unbounded sequences of positive real numbers such that \(\lim_{n \to \infty} \sqrt{\frac{b_n}{a_n}} = 0\), \(f : [0, \infty) \to \mathbb{R}_+\) is a continuous function. Also we gave an error estimate as \(\omega_1\left(f, \sqrt{\frac{b_n x}{a_n}}\right)\) for the operators \(S_n^{(M)}(f) : CB_+( [0, \infty)) \to CB_+( [0, \infty))\) given by (1) in terms of the modulus of continuity. Here \(CB_+( [0, \infty)) \equiv \{ f \mid f : [0, \infty) \to \mathbb{R}_+ \text{ continuous and bounded}\}\).

Let \(S_n^{(M)}(f)(x)\) is the generalizad Szász operators of max-product kind defined in (1). Then,

- For a continuous function \(f : [0, \infty) \to \mathbb{R}_+\), the operators \(S_n^{(M)}(f)(x)\) are positive and continuous on \([0, \infty)\).

- The operators \(S_n^{(M)}(f)(x)\) satisfy the pseudo-linearity property. That is, for every \(f, g \in CB_+( [0, \infty))\) and for any \(\alpha, \beta \in \mathbb{R}_+\),

\[
S_n^{(M)}(\alpha f \vee \beta g) = \alpha S_n^{(M)}(f)(x) \vee \beta S_n^{(M)}(g)(x).
\]

Moreover, these operators are positive homogenous, i.e., \(S_n^{(M)}(\lambda f) = \lambda S_n^{(M)}(f)\) for all \(\lambda \geq 0\).

- Since \(S_n^{(M)}(f)(0) - f(0) = 0\) for all \(n\), we may suppose, throughout the paper, that \(x \in (0, \infty)\).

For each \(k, j \in \{0, 1, 2, \ldots\}\) and \(x \in \left[\frac{b_nj}{a_n}, \frac{b_n(j+1)}{a_n}\right]\), we will denote by

\[
m_{k,n,j}(x) = \frac{c_{n,k}(x)}{c_{n,j}(x)}.
\]
2 Better Error Estimates for Some Subclasses of Functions

In this section we will show that for some subclasses of functions, for example bounded, nondecreasing and concave functions, the order of approximation can be improved.

Consider the functions $f_{k,n,j} : \left[ \frac{b_n}{a_n}, \frac{b_n(j+1)}{a_n} \right] \rightarrow \mathbb{R}$,

$$f_{k,n,j} (x) = m_{k,n,j}(x) f \left( \frac{b_n k}{a_n} \right)$$

$$= c_{n,k}(x) f \left( \frac{b_n k}{a_n} \right)$$

$$= \frac{j!}{k!} \left( \frac{b_n}{a_n} \right)^{k-j} f \left( \frac{b_n k}{a_n} \right)$$

for all $k, j \in \{0, 1, 2, \ldots \}$. Then we can write $S_n^{(M)} (f) (x) = \bigvee_{k=0}^{\infty} f_{k,n,j} (x)$, for any $j \in \{0, 1, 2, \ldots \}$ and $x \in \left[ \frac{b_n}{a_n}, \frac{b_n(j+1)}{a_n} \right]$.

Lemma 1 Let $f : [0, \infty) \rightarrow [0, \infty)$ be a bounded function such that

$$S_n^{(M)} (f) (x) = \max \{ f_{j,n,j} (x), f_{j+1,n,j} (x) \} \text{ for all } x \in \left[ \frac{b_n}{a_n}, \frac{b_n(j+1)}{a_n} \right],$$

then

$$|S_n^{(M)} (f) (x) - f (x)| \leq \omega_1 \left( f, \frac{b_n}{a_n} \right) \text{ for all } x \in \left[ \frac{b_n}{a_n}, \frac{b_n(j+1)}{a_n} \right],$$

where $\omega_1 (f, \delta) = \sup \{|f (x) - f (y)| : |x - y| \leq \delta; x, y \in [0, \infty) \}$.

Proof. We have the following two cases:

Case 1) Let $x \in \left[ \frac{b_n}{a_n}, \frac{b_n(j+1)}{a_n} \right]$ such that $S_n^{(M)} (f) (x) = f_{j,n,j} (x)$.

Since $x \in \left[ \frac{b_n}{a_n}, \frac{b_n(j+1)}{a_n} \right]$ we have $0 \leq x - \frac{b_n}{a_n} \leq \frac{b_n}{a_n}$ and $f_{j,n,j} (x) = m_{j,n,j} (x) f \left( \frac{b_n j}{a_n} \right) = f \left( \frac{b_n j}{a_n} \right)$.

Hence, we get

$$|S_n^{(M)} (f) (x) - f (x)| = |f_{j,n,j} (x) - f (x)| = \left| f \left( \frac{b_n j}{a_n} \right) - f (x) \right| \leq \omega_1 \left( f, \frac{b_n}{a_n} \right).$$
Case 2) Let \( x \in \left[ \frac{b_nj}{a_n}, \frac{b_n(j+1)}{a_n} \right] \) such that \( S_n^{(M)}(f)(x) = f_{j+1,n,j}(x) \). Since \( 0 \leq \frac{b_n(j+1)}{a_n} - x \leq \frac{b_n}{a_n} \) and \( m_{k,n,j}(x) \leq 1 \) for all \( k, j \in \{0, 1, 2, \ldots\} \) we have

\[
\left| S_n^{(M)}(f)(x) - f(x) \right| = \left| f_{j+1,n,j}(x) - f(x) \right|
\]

\[
= \left| m_{j+1,n,j}(x)f \left( \frac{b_n(j+1)}{a_n} \right) - f(x) \right|
\]

\[
\leq \left| f \left( \frac{b_n(j+1)}{a_n} \right) - f(x) \right| \leq \omega_1 \left( f, \frac{b_n}{a_n} \right).
\]

**Corollary 1** Let \( f : [0, \infty) \to [0, \infty) \) be a bounded, nondecreasing function and \( g : (0, \infty) \to [0, \infty), g(x) = \frac{f(x)}{x} \) be a nonincreasing function, then

\[
\left| S_n^{(M)}(f)(x) - f(x) \right| \leq \omega_1 \left( f, \frac{b_n}{a_n} \right) \text{ for all } x \in [0, \infty).
\]

**Proof.** Since \( f \) is a nondecreasing function, we have

\[
S_n^{(M)}(f)(x) = \sum_{k \geq j} f_{k,n,j}(x), \text{ for all } x \in \left[ \frac{b_nj}{a_n}, \frac{b_n(j+1)}{a_n} \right].
\]

Let \( x \in \left[ \frac{b_nj}{a_n}, \frac{b_n(j+1)}{a_n} \right], k, j \in \{0, 1, 2, \ldots\} \) and \( k \geq j \). Then

\[
f_{k+1,n,j}(x) = \frac{j!}{(k+1)!} \cdot \left( \frac{a_n}{b_n} \right)^{k-j+1} \cdot f \left( \frac{b_n}{a_n} \right)
\]

\[
= \left( \frac{a_n}{b_n} \right)^{k-j} \cdot \frac{j!}{(k+1)!} \cdot \left( \frac{a_n}{b_n} \right)^{k-j} \cdot f \left( \frac{b_n}{a_n} \right)
\]

Since \( g \) is a nonincreasing function, we have

\[
f \left( \frac{b_n}{a_n} \right) = \left( \frac{b_n}{a_n} \right)^{k-j} \cdot \frac{j!}{(k+1)!} \cdot \frac{b_n}{a_n} \leq \left( \frac{b_n}{a_n} \right)^{k-j+1} \cdot \frac{j!}{(k+1)!} \cdot \frac{b_n}{a_n}
\]

\[
\leq \left( \frac{b_n}{a_n} \right)^{k+1} \cdot \frac{j!}{(k+1)!} \cdot \frac{b_n}{a_n}
\]

\[
\leq \left( \frac{b_n}{a_n} \right)^{k+1} \cdot \frac{j!}{(k+1)!} \cdot \frac{b_n}{a_n}
\]

\[
= \left( \frac{b_n}{a_n} \right)^{k+1} \cdot \frac{j!}{(k+1)!} \cdot \frac{b_n}{a_n}
\]

Therefore \( k \geq j + 1 \) we get \( f_{k,n,j}(x) \geq f_{k+1,n,j}(x) \). Hence we have \( f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \ldots \geq f_{n,n,j}(x) \geq \ldots \) which is

\[
S_n^{(M)}(f)(x) = \max \{ f_{j,n,j}(x), f_{j+1,n,j}(x) \} \text{ for all } x \in \left[ \frac{b_nj}{a_n}, \frac{b_n(j+1)}{a_n} \right].
\]

Thus from Lemma 1 we obtain

\[
\left| S_n^{(M)}(f)(x) - f(x) \right| \leq \omega_1 \left( f, \frac{b_n}{a_n} \right) \text{ for all } x \in [0, \infty).
\]
Lemma 2 \((9)\) If the function \(f : [0, \infty) \to [0, \infty)\) is concave, then the function \(g : (0, \infty) \to [0, \infty), g(x) = \frac{f(x)}{x}\) is nonincreasing.

Proof. Let \(x, y \in (0, \infty)\) be with \(x \leq y\). Then
\[
f(x) = f\left(\frac{x}{y} y + \frac{y - x}{y} 0\right) \geq \frac{x}{y} f(y) + \frac{y - x}{y} f(0) \geq \frac{x}{y} f(y),
\]
which implies \(\frac{f(x)}{x} \geq \frac{f(y)}{y}\).

Corollary 2 Let \(f : [0, \infty) \to [0, \infty)\) be a bounded, nondecreasing concave function, then \(\left| S_n^{(M)} (f)(x) - f(x) \right| \leq \omega_1 \left( f, \frac{b_n}{a_n} \right)\) for all \(x \in [0, \infty)\).

Proof. Since \(f : [0, \infty) \to [0, \infty)\) is a concave function, from Lemma 2 we have the function \(g : (0, \infty) \to [0, \infty), g(x) = \frac{f(x)}{x}\) is nonincreasing. Thus from Corollary 1 we get the desired result.

3 Shape Preserving Properties

In this section, we concern with some shape preserving properties of max-product type generalized Szász operators.

As in previous section for any \(k, j \in \{0, 1, 2, \ldots\}\) consider the functions \(f_{k,n,j} : \left[\frac{b_n}{a_n}, \frac{b_n(j+1)}{a_n}\right] \to \mathbb{R}\) defined by
\[
f_{k,n,j}(x) = m_{k,n,j}(x).f\left(\frac{b_n k}{a_n}\right) = c_{n,k}(x).f\left(\frac{b_n k}{a_n}\right) = j! \left(\frac{a_n x}{b_n}\right)^{k-j}.f\left(\frac{b_n k}{a_n}\right).
\]

Lemma 3 Let \(f : \{0, \infty) \to \mathbb{R}_+\) be a nondecreasing function. Then for all \(k, j \in \{0, 1, 2, \ldots\}, k \leq j\) and \(x \in \left[\frac{b_n}{a_n}, \frac{b_n(j+1)}{a_n}\right]\) we have \(f_{k,n,j}(x) \geq f_{k-1,n,j}(x)\).

Proof. Since for \(k \leq j\) we get
\[
\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} = \frac{a_n x}{b_n k} \geq \frac{a_n b_n j}{b_n k a_n} \geq 1,
\]
which implies
\[
m_{k,n,j}(x) \geq m_{k-1,n,j}(x).
\]
Because \(f\) is a nondecreasing function we get
\[
m_{k,n,j}(x).f\left(\frac{b_n k}{a_n}\right) \geq m_{k-1,n,j}(x).f\left(\frac{b_n (k-1)}{a_n}\right)
\]
which is desired result.
Corollary 3 Let \( f : [0, \infty) \to \mathbb{R}_+ \) be a nonincreasing function. Then for all \( k, j \in \{0, 1, 2, \ldots \}, k \geq j \) and \( x \in \left[ \frac{b_{n,j}}{a_n}, \frac{b_{n,(j+1)}}{a_n} \right] \) we have \( f_{k,n,j}(x) \geq f_{k+1,n,j}(x) \).

Proof. Let \( k \geq j \). Since the function \( g(x) = \frac{1}{x} \) is nonincreasing on \( \left[ \frac{b_{n,j}}{a_n}, \frac{b_{n,(j+1)}}{a_n} \right] \), it follows

\[
\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} = \frac{b_n (k+1)}{a_n} \frac{1}{x} \geq \frac{b_n (k+1)}{a_n} \frac{a_n}{b_n (j+1)} = \frac{k+1}{j+1} \geq 1,
\]

which implies

\[ m_{k,n,j}(x) \geq m_{k+1,n,j}(x). \]

Because \( f \) is a nonincreasing function we get

\[ m_{k,n,j}(x) \cdot f \left( \frac{b_n k}{a_n} \right) \geq m_{k+1,n,j}(x) \cdot f \left( \frac{b_n (k+1)}{a_n} \right) \]

which is desired result.

Theorem 1 Let \( f : [0, \infty) \to \mathbb{R}_+ \) be a nondecreasing and bounded function. Then \( S_n^{(M)}(f)(x) \) is nondecreasing and bounded on \([0, \infty)\).

Proof. If \( f : [0, \infty) \to \mathbb{R}_+ \) is a bounded function we know that \( S_n^{(M)}(f)(x) \) is continuous and bounded on \([0, \infty)\). Therefore it is sufficient to show that \( S_n^{(M)}(f)(x) \) is nondecreasing on each subinterval of \([0, \infty)\) when \( f \) is a nonincreasing function.

Let \( j \in \{0, 1, 2, \ldots \} \) and \( x \in \left[ \frac{b_{n,j}}{a_n}, \frac{b_{n,(j+1)}}{a_n} \right] \) and \( f \) is a nondecreasing function. From Lemma 3, we can write \( f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq \ldots \geq f_{0,n,j}(x) \) and so

\[ S_n^{(M)}(f)(x) = \lim_{k \to \infty} f_{k,n,j}(x) \text{ for all } x \in \left[ \frac{b_{n,j}}{a_n}, \frac{b_{n,(j+1)}}{a_n} \right]. \]

For \( k \geq j \), since the function \( f_{k,n,j}(x) \) is nondecreasing and \( S_n^{(M)}(f)(x) \) can be written as the supremum of nondecreasing functions, then \( S_n^{(M)}(f)(x) \) is nondecreasing.

Corollary 4 If \( f : [0, \infty) \to \mathbb{R}_+ \) is a nonincreasing function then \( S_n^{(M)}(f)(x) \) is nonincreasing.

Proof. Because \( f \) is nondecreasing and positive \( f \) is bounded on \([0, \infty)\). Since \( S_n^{(M)}(f)(x) \) is continuous and bounded on \([0, \infty)\), it is sufficient to show that \( S_n^{(M)}(f)(x) \) is nonincreasing on each subinterval of \([0, \infty)\).

Let \( j \in \{0, 1, 2, \ldots \} \) and \( x \in \left[ \frac{b_{n,j}}{a_n}, \frac{b_{n,(j+1)}}{a_n} \right] \). Since \( f \) is nonincreasing, from Corollary 3, we get \( f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \ldots \geq f_{n,n,j}(x) \geq \ldots \) and so

\[ S_n^{(M)}(f)(x) = \lim_{k \to j} f_{k,n,j}(x), \text{ for all } x \in \left[ \frac{b_{n,j}}{a_n}, \frac{b_{n,(j+1)}}{a_n} \right]. \]
For \( k \leq j \) since the function \( f_{k,n,j}(x) \) is nonincreasing and \( S_n^{(M)}(f)(x) \) can be written as the supremum of nonincreasing functions, \( S_n^{(M)}(f)(x) \) is nonincreasing.

**Remark 1** ([9], [13]) A continuous function \( f \) is quasiconvex on the bounded interval \([0,a]\) if there exists a point \( c \in [0,a] \) such that \( f \) is nonincreasing on \([0,c]\) and nondecreasing on \([c,a]\). The quasiconvexity of \( f \) on \([0,\infty)\) means that quasiconvexity of \( f \) any bounded interval \([0,a]\), with arbitrary large \( a > 0 \). The class of nondecreasing functions, the class of nonincreasing functions and the class of convex functions on \([0,\infty)\) are included by the class of quasiconvex functions on \([0,\infty)\).

**Corollary 5** If \( f : [0,\infty) \rightarrow \mathbb{R}_+ \) is a continuous, bounded and quasiconvex function then for all \( n \in \mathbb{N} \), \( S_n^{(M)}(f)(x) \) is quasiconvex on \([0,\infty)\).

**Proof.** If \( f \) is nonincreasing or nondecreasing function then by the Corollary 4 or by the Theorem 1, for all \( n \in \mathbb{N} \), \( S_n^{(M)}(f)(x) \) is nonincreasing or nondecreasing on \([0,\infty)\).

Now suppose that there exists a point \( c \in (0,\infty) \) such that \( f \) is nonincreasing on \([0,c]\) and nondecreasing on \([c,\infty)\). The functions \( F,G : [0,\infty) \rightarrow \mathbb{R}_+ \) are defined by \( F(x) = f(x) \) for all \( x \in [0,c] \), \( F(x) = f(c) \) for all \( x \in [c,\infty) \) and \( G(x) = f(x) \) for all \( x \in [0,c] \), \( G(x) = f(x) \) for all \( x \in [c,\infty) \).

It is obvious that \( F \) is nonincreasing and continuous on \([0,\infty)\), \( G \) is nondecreasing and continuous on \([0,\infty)\) and \( f(x) = \max\{F(x),G(x)\} \), for all \( x \in [0,\infty) \).

In addition, since \( S_n^{(M)}(f)(x) \) is pseudo-linear, we can write for all \( x \in [0,\infty) \)

\[
S_n^{(M)}(f)(x) = \max \left\{ S_n^{(M)}(F)(x), S_n^{(M)}(G)(x) \right\}.
\]

Hence by Corollary 4 and Theorem 1, \( S_n^{(M)}(F)(x) \) is nonincreasing and continuous on \([0,\infty)\), \( S_n^{(M)}(G)(x) \) is nondecreasing and continuous on \([0,\infty)\).

Now, we have two cases: 1) \( S_n^{(M)}(F)(x) \) and \( S_n^{(M)}(G)(x) \) don’t intersect each other, 2) \( S_n^{(M)}(F)(x) \) and \( S_n^{(M)}(G)(x) \) intersect each other.

Case 1) For all \( x \in [0,\infty) \) since \( \max \left\{ S_n^{(M)}(F)(x), S_n^{(M)}(G)(x) \right\} = S_n^{(M)}(F)(x) \) or \( \max \left\{ S_n^{(M)}(F)(x), S_n^{(M)}(G)(x) \right\} = S_n^{(M)}(G)(x) \), by using Remark 1 we get \( S_n^{(M)}(f)(x) \) is quasiconvex on \([0,\infty)\).

Case 2) If \( S_n^{(M)}(F)(x) \) and \( S_n^{(M)}(G)(x) \) intersect each other then by Remark 1, there exists a point \( c \in [0,\infty) \) such that \( S_n^{(M)}(f)(x) \) is nonincreasing on \([0,c]\) and nondecreasing on \([c,\infty)\) which implies that \( S_n^{(M)}(f)(x) \) is quasiconvex on \([0,\infty)\).

**Theorem 2** Let \( f : [0,\infty) \rightarrow [0,\infty) \) be an arbitrary function. Then \( S_n^{(M)}(f)(x) \) is convex on \( \left[ \frac{b_j}{a_n}, \frac{b_{j+1}}{a_n} \right] \subset [0,\infty) \), \( j = 0, 1, 2, \ldots \).
Proof. For any \( j \in \{0, 1, 2, \ldots \} \) and \( x \in \left[ \frac{b_0-j}{a_n}, \frac{b_0(j+1)}{a_n} \right] \) since we can write
\[
S_n^M(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x)
\]
it is enough to show that for any fixed \( j \), \( f_{k,n,j}(x) \) is convex on \( \left[ \frac{b_0-j}{a_n}, \frac{b_0(j+1)}{a_n} \right] \).

Since \( f \geq 0 \), \((a_n)\) and \((b_n)\) are sequences of positive real numbers and \( f_{k,n,j}(x) = \binom{k}{j} \left( \frac{a_n}{b_n} \right)^{k-j} \left( \frac{b_n}{a_n} \right)^j \) it is sufficient to show that the functions \( g_{k,j}(x) = x^{k-j} \) are convex on \( \left[ \frac{b_0-j}{a_n}, \frac{b_0(j+1)}{a_n} \right] \). For \( k = j \), \( g_{j,j} = 1 \) is a constant function and it is convex. For \( k = j + 1 \), \( g_{j+1,j}(x) = x \) is convex. For \( k = j - 1 \), \( g_{j-1,j}(x) = \frac{x}{j} \) and \( g_{j-1,j}''(x) = \frac{2}{x^3} > 0 \) for all \( x \in \left[ \frac{b_0-j}{a_n}, \frac{b_0(j+1)}{a_n} \right] \) and it is convex. For \( k > j + 2 \), \( g_{k,j}'(x) = (k-j)(k-j-1)x^{k-j-2} > 0 \) for all \( x \in \left[ \frac{b_0-j}{a_n}, \frac{b_0(j+1)}{a_n} \right] \) and it is also convex. For \( k < j - 2 \), \( g_{k,j}(x) = (k-j)(k-j-1)x^{k-j-2} > 0 \) for all \( x \in \left[ \frac{b_0-j}{a_n}, \frac{b_0(j+1)}{a_n} \right] \) and it is also convex. Since \( S_n^M(f)(x) \) can be written as the supremum of convex functions, it is convex on \( \left[ \frac{b_0-j}{a_n}, \frac{b_0(j+1)}{a_n} \right] \).

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New integral inequalities for twice differentiable functions

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Abstract

In this paper we establish some new general integral inequalities for twice differentiable functions. Then we apply these inequalities to obtain some inequalities for special means of real numbers. Finally, some new general quadrature rules of trapezoidal type are provided.

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1 Introduction

In 1938, A. Ostrowski [11] proved the following integral inequality

\[ \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - a + b}{b-a} \right)^2 \right] (b-a) \| f' \|_\infty, \]

provided \( f \) is differentiable and \( \| f' \|_\infty = \sup_{t \in (a,b)} |f'(t)| < \infty. \)

The constant \( \frac{1}{4} \) is sharp in the sense that it cannot be replaced by a smaller constant.

In the last years, many authors have concentrated their efforts in generalising (1) and have applied the obtained results in different fields, including Numerical Integration, Probability Theory and Statistics, Information Theory, etc.

For results and generalizations concerning Ostrowski’s integral inequality see [1]-[10], [12], [13] and the references therein.

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In recent years a number of authors have considered an error analysis for some known and some new quadrature formulas.

In this article, we first give a general integral identity for twice derivatives functions. Then we apply this identity to obtain our results using functions whose twice derivatives in absolute value are bounded and/or convex on \([a, b]\), we obtained new integral inequalities of trapezoidal type which are better than some known. Finally, we gave some applications for special means of real numbers and some numerical quadrature rules.

## 2 Main results

In order to prove our main results, we need the following new Lemmas:

**Lemma 1** Let \(f_1, f_2, f_3 : [0, 1] \to \mathbb{R}\) be three functions defined by

\[
    f_1(t) = |1 - nt + nt^2|, \quad f_2(t) = t|1 - nt + nt^2|, \quad f_3(t) = (1 - t)|1 - nt + nt^2|,
\]

with \(n \in (0, +\infty)\). Then we have:

\[
\begin{align*}
    (i) & \quad \int_0^1 f_1(t) \, dt = v(n); \\
    (ii) & \quad \int_0^1 f_2(t) \, dt = \int_0^1 f_3(t) \, dt = \frac{1}{2} v(n),
\end{align*}
\]

where

\[
    v(n) = \begin{cases} 
        \frac{6 - n}{6}, & n \in (0, 4] \\
        \frac{6n - n^2 + 2(n - 4)\sqrt{n^2 - 4n}}{6n}, & n \in (4, +\infty). 
    \end{cases}
\]

**Proof.** (i) If \(n \in (0, 4]\), then we find

\[
    |1 - nt + nt^2| = 1 - nt + nt^2
\]

for all \(t \in [0, 1]\) and

\[
    \int_0^1 f_1(t) \, dt = \left( t - nt^2 + \frac{nt^3}{3} \right) \bigg|_0^1 = \frac{6 - n}{6}.
\]

If \(n \in (4, +\infty)\), then we obtain

\[
    |1 - nt + nt^2| = \begin{cases} 
        1 - nt + nt^2, & t \in [0, t_1] \cup [t_2, 1] \\
        -1 + nt - nt^2, & t \in (t_1, t_2),
    \end{cases}
\]

where \(t_1 = \frac{n - \sqrt{n^2 - 4n}}{2n}\) and \(t_2 = \frac{n + \sqrt{n^2 - 4n}}{2n}\).
Thus, we can write
\[
\int_0^1 f_1(t) dt = \left( t - n \frac{t^2}{2} + n \frac{t^3}{3} \right)\bigg|_0^{t_1} - \left( t - n \frac{t^2}{2} + n \frac{t^3}{3} \right)\bigg|_1^{t_2} + \left( t - n \frac{t^2}{2} + n \frac{t^3}{3} \right)\bigg|_1^{t_2} = \frac{6n - n^2 + 2(n - 4)\sqrt{n^2 - 4n}}{6n}.
\]

(ii) Using the change of the variable \( u = 1 - t \) for \( t \in [0, 1] \), we find
\[
\int_0^1 f_2(t) dt = \int_0^1 f_3(t) dt.
\]

A simple computation shows that
\[
\int_0^1 f_2(t) dt = \frac{1}{2}v(n).
\]

**Lemma 2** Let \( g : (0, +\infty) \to \mathbb{R} \) be a mapping defined by
\[
g(x) = \frac{v(x)}{x}
\]

Then we have
\[
\min_{x \in (0, +\infty)} g(x) = g\left(\frac{16}{3}\right) = \frac{1}{16}.
\]

**Proof.** \( g \) is continuous function on \( (0, +\infty) \) and is differentiable on \( (0, 4) \cup (4, +\infty) \).

Note that
\[
g'(x) = \begin{cases} 
-\frac{1}{x^2}, & x \in (0, 4) \\
\frac{2\sqrt{x^2 - 4} - x}{x^3}, & x \in (4, +\infty)
\end{cases}
\]

and \( g'(x) < 0 \) for all \( x \in (0, 4) \cup \left(4, \frac{16}{3}\right) \), \( g'\left(\frac{16}{3}\right) = 0 \) and \( g'(x) > 0 \) for all \( x \in \left(\frac{16}{3}, +\infty\right) \). Finally, we get the desired result.

**Lemma 3** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a twice differentiable function on \( I^0 \) with \( f'' \in L_1[a, b] \), where \( a, b \in I^0 \), \( a < b \), then
\[
\frac{1}{b - a} \int_a^b f(u) du - \frac{1}{2(b - a)} [(x - a)(f(a) + f(x)) + (b - x)(f(x) + f(b))] + \frac{1}{2n(b - a)} [(x - a)^2(f'(x) - f'(a)) + (b - x)^2(f'(b) - f'(x))]
\]
If we take $x = a$ or $x = b$ in Lemma 3, then (5) reduces to

\begin{equation}
\frac{1}{b-a} \int_a^b f(u)du - \frac{f(a) + f(b)}{2} + \frac{(b-a)(f'(b) - f'(a))}{2n}
\end{equation}

for all $n \in (0, +\infty)$.

Remark 1: If we take $x = a$ or $x = b$ in Lemma 3, then (5) reduces to

\begin{equation}
\frac{1}{b-a} \int_a^b f(u)du - \frac{f(a) + f(b)}{2} + \frac{(b-a)(f'(b) - f'(a))}{2n}
\end{equation}

for all $n \in (0, +\infty)$.
Remark 2 If we take \( x = \frac{a + b}{2} \) in Lemma 3, then (5) reduces to

\[
\frac{1}{b-a} \int_{a}^{b} f(u) du - \frac{1}{2(b-a)} [(x-a)(f(a) + f(x)) + (b-x)(f(x) + f(b))] 
+ \frac{1}{2n(b-a)} [(x-a)^2(f'(x) - f'(a)) + (b-x)^2(f'(b) - f'(x))] 
\leq \frac{1}{2} g(n) \frac{(x-a)^3 + (b-x)^3}{b-a} \|f''\|_{\infty}
\]

for all \( x \in [a, b] \), where \( \|f''\|_{\infty} = \sup_{u \in [a,b]} |f''(u)| < \infty \).

Proof. From Lemma 3 and using the properties of modulus, we have

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(u) du - \frac{1}{2(b-a)} [(x-a)(f(a) + f(x)) + (b-x)(f(x) + f(b))] 
+ \frac{1}{2n(b-a)} [(x-a)^2(f'(x) - f'(a)) + (b-x)^2(f'(b) - f'(x))] \right|
\leq \frac{\|f''\|_{\infty}}{2n(b-a)} [(x-a)^3 + (b-x)^3] \int_{0}^{1} |1 - nt + nt^2| dt.
\]

From Lemma 1 and the above inequality we obtain the desired inequality.

Corollary 1 Under the assumptions of Theorem 1, for \( n = \frac{16}{3} \) we have

\[
\frac{1}{b-a} \int_{a}^{b} f(u) du - \frac{1}{2(b-a)} [(x-a)(f(a) + f(x)) + (b-x)(f(x) + f(b))] 
+ \frac{3}{32(b-a)} [(x-a)^2(f'(x) - f'(a)) + (b-x)^2(f'(b) - f'(x))] 
\leq \frac{1}{32} \cdot \frac{(x-a)^3 + (b-x)^3}{b-a} \|f''\|_{\infty}
\]

for all \( x \in [a, b] \).
Theorem 2 Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a twice differentiable function on \( I^0 \) such that \( f'' \in L_1[a, b] \), where \( a, b \in I, \ a < b \).

If \( |f''| \) is convex on \([a, b] \), then the following inequality holds

\[
\text{(13)} \quad \left| \frac{1}{b - a} \int_a^b f(u)du - \frac{1}{2(b - a)}[(x - a)(f(a) + f(x)) + (b - x)(f(x) + f(b))] 
+ \frac{1}{2n(b - a)}[(x - a)^2(f'(x) - f'(a)) + (b - x)^2(f'(b) - f'(x))] \right| 
\leq \frac{1}{4(b - a)}g(n)[(x - a)^3(|f''(a)| + |f''(x)|) + (b - x)^3(|f''(x)| + |f''(b)|)]
\]

for all \( x \in [a, b] \).

Proof. From Lemma 3, using the properties of modulus, by the convexity of \( |f''| \), we arrive at

\[
\left| \frac{1}{b - a} \int_a^b f(u)du - \frac{1}{2(b - a)}[(x - a)(f(a) + f(x)) + (b - x)(f(x) + f(b))] 
+ \frac{1}{2n(b - a)}[(x - a)^2(f'(x) - f'(a)) + (b - x)^2(f'(b) - f'(x))] \right| 
\leq \frac{1}{2n(b - a)} \left\{ (x - a)^3 \left[ |f''(a)| \int_0^1 t|1 - nt + nt^2|dt 
+ |f''(x)| \int_0^1 (1 - t)|1 - nt + nt^2|dt \right] 
+ (b - x)^3 \left[ |f''(x)| \int_0^1 (1 - t)|1 - nt + nt^2|dt 
+ |f''(b)| \int_0^1 t|1 - nt + nt^2|dt \right] \right\}.
\]

Applying Lemma 2 in the above inequality we obtain (13).

Corollary 2 Under the assumptions of Theorem 2, for \( n = \frac{16}{3} \), we deduce

\[
\text{(14)} \quad \left| \frac{1}{b - a} \int_a^b f(u)du - \frac{1}{2(b - a)}[(x - a)(f(a) + f(x)) + (b - x)(f(x) + f(b))] 
+ \frac{3}{32(b - a)}[(x - a)^2(f'(x) - f'(a)) + (b - x)^2(f'(b) - f'(x))] \right| 
\leq \frac{1}{64} \cdot \frac{(x - a)^3(|f''(a)| + |f''(x)|) + (b - x)^3(|f''(x)| + |f''(b)|)}{b - a}
\]

for all \( x \in [a, b] \).
Corollary 3 Under the assumptions of Theorem 1, for \( x = a \) or \( x = b \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u) \, du - \frac{f(a) + f(b)}{2} + \frac{(b-a)(f'(b) - f'(a))}{2n} \right| \leq \frac{1}{2} g(n)(b-a)^2 \| f'' \|_\infty.
\]

Corollary 4 Under the assumptions of Theorem 1, for \( x = a \) or \( x = b \) and \( n = 4 \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u) \, du - \frac{f(a) + f(b)}{2} + \frac{(b-a)(f'(b) - f'(a))}{8} \right| \leq \frac{1}{24} (b-a)^2 \| f'' \|_\infty.
\]

(see [3], [13]).

Corollary 5 Under the assumptions of Theorem 1, for \( x = a \) or \( x = b \) and \( n = \frac{16}{3} \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u) \, du - \frac{f(a) + f(b)}{2} - \frac{3(b-a)}{32} (f'(b) - f'(a)) \right| \leq \frac{1}{32} (b-a)^2 \| f'' \|_\infty.
\]

Corollary 6 Under the assumptions of Theorem 1, for \( x = a \) or \( x = b \) and \( f'(a) = f'(b) \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u) \, du - \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{2} g(n)(b-a)^2 \| f'' \|_\infty.
\]

Corollary 7 Under the assumptions of Theorem 1, for \( x = a \) or \( x = b \), \( f'(a) = f'(b) \) and \( n = 4 \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u) \, du - \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{24} (b-a)^2 \| f'' \|_\infty.
\]

Corollary 8 Under the assumptions of Theorem 1, for \( x = a \) or \( x = b \), \( f'(a) = f'(b) \) and \( n = \frac{16}{3} \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u) \, du - \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{32} (b-a)^2 \| f'' \|_\infty.
\]
Corollary 9 \textit{Under the assumptions of Theorem 1, for } x = \frac{a + b}{2}, \text{ we have}

\begin{equation}
\left| \frac{1}{b-a} \int_a^b f(u) du - \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f \left( \frac{a + b}{2} \right) \right] + \frac{b-a}{8n} [f'(b) - f'(a)] \right| \\
\leq \frac{1}{8} g(n)(b-a)^2 \|f''\|_{\infty}.
\end{equation}

Corollary 10 \textit{Under the assumptions of Theorem 1, for } x = \frac{a + b}{2} \text{ and } n = 4, \text{ we have}

\begin{equation}
\left| \frac{1}{b-a} \int_a^b f(u) du - \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f \left( \frac{a + b}{2} \right) \right] + \frac{b-a}{32} [f'(b) - f'(a)] \right| \\
\leq \frac{1}{96} (b-a)^2 \|f''\|_{\infty}.
\end{equation}

Corollary 11 \textit{Under the assumptions of Theorem 1, for } x = \frac{a + b}{2} \text{ and } n = \frac{16}{3}, \text{ we have}

\begin{equation}
\left| \frac{1}{b-a} \int_a^b f(u) du - \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f \left( \frac{a + b}{2} \right) \right] + \frac{3(b-a)}{128} [f'(b) - f'(a)] \right| \\
\leq \frac{1}{128} (b-a)^2 \|f''\|_{\infty}.
\end{equation}

Corollary 12 \textit{Under the assumptions of Theorem 2, for } x = a \text{ or } x = b, \text{ we have}

\begin{equation}
\left| \frac{1}{b-a} \int_a^b f(u) du - \frac{f(a) + f(b)}{2} + \frac{(b-a)(f'(b) - f'(a))}{2n} \right| \\
\leq \frac{1}{4} g(n)(b-a)^2 (|f''(a)| + |f''(b)|).
\end{equation}

Corollary 13 \textit{Under the assumptions of Theorem 2, for } x = a \text{ or } x = b \text{ and } n = 4, \text{ we have}

\begin{equation}
\left| \frac{1}{b-a} \int_a^b f(u) du - \frac{f(a) + f(b)}{2} + \frac{(b-a)(f'(b) - f'(a))}{8} \right| \\
\leq \frac{1}{48} (b-a)^2 (|f''(a)| + |f''(b)|).
\end{equation}

(see [13]).
Corollary 14 Under the assumptions of Theorem 2, for \( x = a \) or \( x = b \) and \( n = \frac{16}{3} \), we have

\[
\frac{1}{b-a} \int_a^b f(u) du - \frac{f(a) + f(b)}{2} + \frac{3(b-a)}{32} (f'(b) - f'(a)) \leq \frac{1}{64} (b-a)^2 (|f''(a)| + |f''(b)|).
\]

Corollary 15 Under the assumptions of Theorem 2, for \( x = a \) or \( x = b \) and \( f'(a) = f'(b) \), we have

\[
\frac{1}{b-a} \int_a^b f(u) du - \frac{f(a) + f(b)}{2} \leq \frac{1}{g(n)} (b-a)^2 (|f''(a)| + |f''(b)|).
\]

(see [13]).

Corollary 16 Under the assumptions of Theorem 2, for \( x = a \) or \( x = b \), \( f'(a) = f'(b) \) and \( n = 4 \), we have

\[
\frac{1}{b-a} \int_a^b f(u) du - \frac{f(a) + f(b)}{2} \leq \frac{1}{48} (b-a)^2 (|f''(a)| + |f''(b)|)
\]

Corollary 17 Under the assumptions of Theorem 2, for \( x = a \) or \( x = b \), \( f'(a) = f'(b) \) and \( n = \frac{16}{3} \), we have

\[
\frac{1}{b-a} \int_a^b f(u) du - \frac{f(a) + f(b)}{2} \leq \frac{1}{64} (b-a)^2 (|f''(a)| + |f''(b)|).
\]

Corollary 18 Under the assumptions of Theorem 2, for \( x = \frac{a+b}{2} \), we have

\[
\frac{1}{b-a} \int_a^b f(u) du - \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f\left( \frac{a+b}{2} \right) \right] + \frac{b-a}{8n} [f'(b) - f'(a)] \leq \frac{1}{16} g(n) (b-a)^2 (|f''(a)| + |f''(b)|).
\]
Corollary 19 Under the assumptions of Theorem 2, for \( x = \frac{a+b}{2} \) and \( n = 4 \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} \left[ f(a) + f(b) + f \left( \frac{a+b}{2} \right) \right] + \frac{b-a}{32} [f'(b) - f'(a)] \right| \leq \frac{1}{192} (b-a)^2 (|f''(a)| + |f''(b)|).
\]

Corollary 20 Under the assumptions of Theorem 2, for \( x = \frac{a+b}{2} \) and \( n = \frac{16}{3} \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} \left[ f(a) + f(b) + f \left( \frac{a+b}{2} \right) \right] + \frac{3(b-a)}{128} [f'(b) - f'(a)] \right| \leq \frac{1}{256} (b-a)^2 (|f''(a)| + |f''(b)|).
\]

Corollary 21 Under the assumptions of Theorem 2, for \( x = \frac{a+b}{2} \) and \( f'(a) = f'(b) \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} \left[ f(a) + f(b) + f \left( \frac{a+b}{2} \right) \right] \right| \leq \frac{1}{16} g(n)(b-a)^2 (|f''(a)| + |f''(b)|).
\]

Corollary 22 Under the assumptions of Theorem 2, for \( x = \frac{a+b}{2} \), \( f'(a) = f'(b) \) and \( n = 4 \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} \left[ f(a) + f(b) + f \left( \frac{a+b}{2} \right) \right] \right| \leq \frac{1}{192} (b-a)^2 (|f''(a)| + |f''(b)|).
\]

Corollary 23 Under the assumptions of Theorem 2, for \( x = \frac{a+b}{2} \), \( f'(a) = f'(b) \) and \( n = \frac{16}{3} \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} \left[ f(a) + f(b) + f \left( \frac{a+b}{2} \right) \right] \right| \leq \frac{1}{256} (b-a)^2 (|f''(a)| + |f''(b)|).
\]
3 Applications for special means

Recall the following means:

(a) The arithmetic mean

\[ A = A(a, b) := \frac{a + b}{2}, \quad a, b \geq 0; \]

(b) The geometric mean

\[ G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0; \]

(c) The harmonic mean

\[ H = H(a, b) := \frac{2ab}{a + b}, \quad a, b > 0; \]

(d) The logarithmic mean

\[ L = L(a, b) := \begin{cases} 
  a & \text{if } a = b \\
  \frac{b - a}{\ln b - \ln a} & \text{if } a \neq b
\end{cases}, \quad a, b > 0; \]

(e) The identric mean

\[ I = I(a, b) := \begin{cases} 
  a & \text{if } a = b \\
  \frac{1}{e} \left( \frac{b^a}{a^b} \right)^{\frac{1}{b-a}} & \text{if } a \neq b
\end{cases}, \quad a, b > 0; \]

(f) The \( p \)-logarithmic mean

\[ L_p = L_p(a, b) := \begin{cases} 
  a & \text{if } a = b \\
  \frac{b^{p+1} - a^{p+1}}{(p + 1)(b - a)} & \text{if } a \neq b
\end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}, \quad a, b > 0. \]

It is also known that \( L_p \) is monotonically nondecreasing in \( p \in \mathbb{R} \) with \( L_{-1} := L \) and \( L_0 := I \). The following simple relationship are known in the literature

\[ H \leq G \leq L \leq I \leq A. \]

Now, using the results of Section 2, some new inequalities are derived for the above means.

**Proposition 1** Let \( p \geq 2 \) and \( 0 < a < b \). Then we have the inequality

\[
\left| L_p^p(a, b) - A(a^p, b^p) + \frac{p(p - 1)}{2n} (b - a)^2 L_{p-2}^p(a, b) \right| \\
\leq \frac{p(p - 1)}{2} g(n) (b - a)^2 A(a^{p-2}, b^{p-2}).
\]
Proof. The assertion follows from (24) applied for \( f(x) = x^p, x \in [a, b] \).

**Proposition 2** Let \( 0 < a < b \). Then we have the inequality

\[
\left| L^{-1}(a, b) - H^{-1}(a, b) - \frac{(b - a)^2 A(a, b)}{nG^4(a, b)} \right| \leq g(n)(b - a)^2 A(a^{-3}, b^{-3}).
\]

Proof. The assertion follows from (24) applied for \( f(x) = 1, x \in [a, b] \).

**Proposition 3** Let \( 0 < a < b \). Then we have the inequality

\[
\left| \ln I(a, b) + \ln G(a, b) - \frac{(b - a)^2}{2nG^2(a, b)} \right| \leq \frac{1}{2} g(n)(b - a)^2 A(a^{-2}, b^{-2}).
\]

Proof. The assertion follows from (24) applied for \( f(x) = -\ln x, x \in [a, b] \).

**Proposition 4** Let \( 0 < a < b \) and \( p \geq 2 \). Then we have the inequality

\[
\left| L_p(a, b) - A(A(a, b)) + \frac{p(p - 1)}{8n} (b - a)^2 L_{p-2}^p(a, b) \right|
\leq \frac{p(p - 1)}{8} g(n)(b - a)^2 A(a^{p-2}, b^{p-2}).
\]

Proof. The assertion follows from (30) applied for \( f(x) = x^p, x \in [a, b] \).

**Proposition 5** Let \( 0 < a < b \). Then we have the inequality

\[
\left| L^{-1}(a, b) - A(H^{-1}(a, b), A^{-1}(a, b)) + \frac{(b - a)^2 A(a, b)}{4nG^4(a, b)} \right|
\leq \frac{1}{4} g(n)(b - a)^2 A(a^{-3}, b^{-3}).
\]

Proof. The assertion follows from (30) applied for \( f(x) = \frac{1}{x}, x \in [a, b] \).

**Proposition 6** Let \( 0 < a < b \). Then we have the inequality

\[
\left| \ln I(a, b) + \ln G(G(a, b), A(a, b)) + \frac{(b - a)^2}{8nG^2(a, b)} \right|
\leq \frac{1}{8} g(n)(b - a)^2 A(a^{-2}, b^{-2}).
\]

Proof. The assertion follows from (30) applied for \( f(x) = -\ln x, x \in [a, b] \).
4 Applications for composite quadrature formula

Let $\Delta$ be a division $a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b$ of the interval $[a, b]$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ a sequence of intermediate points, $\xi_i \in [x_{i-1}, x_i]$, $i = 1, m$. Then the following results hold:

**Theorem 3** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I$ such that $f'' \in L_\infty[a, b]$, where $a, b \in I$, $a < b$. Then we have

$$\int_a^b f(u) \, du = A(f, f', \Delta, \xi) + R(f, f', \Delta, \xi)$$

where

$$A(f, f', \Delta, \xi) := \frac{1}{2} \sum_{i=1}^{m} [(\xi_i - x_{i-1})(f(x_{i-1}) + f(\xi_i)) + (x_i - \xi_i)(f(\xi_i) + f(x_i))]$$

$$- \frac{1}{2n} \sum_{i=1}^{m} [(\xi_i - x_{i-1})^2(f'(\xi_i) - f'(x_{i-1})) + (x_i - \xi_i)^2(f'(x_i) - f'(\xi_i))].$$

The remainder $R(f, f', \Delta, \xi)$ satisfies the estimation:

$$|R(f, f', \Delta, \xi)| \leq \frac{1}{2} g(n)\|f''\|_{\infty} \sum_{i=1}^{m} [(\xi_i - x_{i-1})^3 + (x_i - \xi_i)^3],$$

for any choice $\xi$ of the intermediate points.

**Proof.** Apply Theorem 1 on the interval $[x_{i-1}, x_i]$, $i = 1, m$ to get

$$\left| \int_{x_{i-1}}^{x_i} f(u) \, du - \frac{1}{2} [(\xi_i - x_{i-1})(f(x_{i-1}) + f(\xi_i)) + (x_i - \xi_i)(f(\xi_i) + f(x_i))]$$

$$+ \frac{1}{2n} [(\xi_i - x_{i-1})^2(f'(\xi_i) - f'(x_{i-1})) + (x_i - \xi_i)^2(f'(x_i) - f'(\xi_i))] \right|$$

$$\leq \frac{1}{2} g(n)\|f''\|_{\infty} [(\xi_i - x_{i-1})^3 + (x_i - \xi_i)^3].$$

Summing the above inequalities over $i$ from 1 to $m$ and using the generalized triangle inequality, we get the desired estimation.

**Corollary 24** The following perturbed trapezoid rule holds

$$\int_a^b f(u) \, du = T(f, f', \Delta) + R_T(f, f', \Delta)$$

where

$$T(f, f', \Delta) := \sum_{i=1}^{m} \frac{h_i}{2}[f(x_{i-1}) + f(x_i)] - \sum_{i=1}^{m} \frac{h_i^2}{2n} [f'(x_i) - f'(x_{i-1})]$$
The following twice repeated trapezoidal rule holds: Let

\[ |R_T(f, f', \Delta)| \leq \frac{1}{2} g(n) \| f'' \|_{\infty} \sum_{i=1}^{m} h_i^3. \]

**Corollary 25** The following twice repeated trapezoidal rule holds:

\[ \int_a^b f(u)du = T_r(f, f', \Delta) + R_r(f, f', \Delta) \]

where

\[ T_r(f, f', \Delta) := \sum_{i=1}^{m} \frac{h_i}{2} \left[ \frac{f(x_{i-1}) + f(x_i)}{2} + f \left( \frac{x_{i-1} + x_i}{2} \right) \right] \]

and the remainder term \( R_r(f, f', \Delta) \) satisfies the estimation

\[ |R_r(f, f', \Delta)| \leq \frac{1}{8} g(n) \| f'' \|_{\infty} \sum_{i=1}^{m} h_i^3. \]

**Theorem 4** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a twice differentiable function on \( I \) such that \( f'' \in L_1[a, b] \), where \( a, b \in I \), \( a < b \). If \( |f''| \) is convex on \([a, b]\) then we have

\[ \int_a^b f(u)du = A(f, f', \Delta, \xi) + R_1(f, f', \Delta, \xi) \]

where the remainder \( R_1(f, f', \Delta, \xi) \) satisfies estimation

\[ |R_1(f, f', \Delta, \xi)| \leq \frac{1}{4} g(n) \sum_{i=1}^{m} \left( (\xi_i - x_{i-1})^3 (|f''(x_{i-1})| + |f''(\xi_i)|) \right. \]

\[ + (x_i - \xi_i)^3 (|f''(\xi_i)| + |f''(x_i)|) \]

for any choice \( \xi \) of the intermediate points.

**Proof.** Applying Theorem 2 on the interval \([x_{i-1}, x_i], i = 1, m\), we find

\[ \left| \int_{x_{i-1}}^{x_i} f(u)du - \frac{1}{2}((\xi_i - x_{i-1})(f(x_{i-1}) + f(\xi_i)) + (x_i - \xi_i)(f(\xi_i) + f(x_i))) \right. \]

\[ + \frac{1}{2m}((\xi_i - x_{i-1})^2(f'(\xi_i) - f'(x_{i-1})) + (x_i - \xi_i)^2(f'(x_i) - f'(\xi_i))) \]

\[ \leq \frac{1}{4} g(n)((\xi_i - x_{i-1})^3 (|f''(x_{i-1})| + |f''(\xi_i)|) + (x_i - \xi_i)^3 (|f''(\xi_i)| + |f''(x_i)|)). \]

Summing the above inequalities over \( i \) from 1 to \( m \) and using the generalized triangle inequality, we get the desired estimation (37).
Corollary 26 The following perturbed trapezoid rule holds
\[ \int_{a}^{b} f(u) du = T(f, f', \Delta) + R_1^T(f, f', \Delta) \]
where the remainder term \( R_1^T(f, f', \Delta) \) satisfies the estimation
\[ |R_1^T(f, f', \Delta)| \leq \frac{1}{4} g(n) \sum_{i=1}^{m} h_i^3 (|f''(x_i)| + |f''(x_{i-1})|). \]

Corollary 27 The following twice repeated trapezoidal rule holds
\[ \int_{a}^{b} f(u) du = T_r(f, f', \Delta) + R_1^r(f, f', \Delta) \]
where the remainder term \( R_1^r(f, f', \Delta) \) satisfies the estimation
\[ |R_1^r(f, f', \Delta)| \leq \frac{1}{16} g(n) \sum_{i=1}^{m} h_i^3 (|f''(x_i)| + |f''(x_{i-1})|). \]

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Interpreting modal logics using labeled graphs

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Abstract

Modal logic is an extension of the logic of predicates and propositions which includes operators that express modality. In modal logic we deal with truth and falsehood in different possible worlds, as well as in the real world. In this paper we construct a labeled graph associated to a transition system, as a starting point in analyzing modal logics. We define the concepts of inclusion, isomorphism, “modal equivalence” and equivalence between graphs.

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1 Transition systems

Transition systems are concepts used in computer science. They consist of states and transitions among them. The set of states can be countable or uncountable, and so can the set of transitions.

Definition 1 Formally, a transition system $ST$ is a triple $(S, A, \rightarrow)$, where $S$ is a set of states, $A$ a set of actions and $\rightarrow \subseteq S \times A \times S$ is the transition relation.

We can consider the transition system $ST$, a tuple $(S, A, \rightarrow, P, L)$ where $S$ is a set of states, $A$ a set of actions, $\rightarrow \subseteq S \times A \times S$ is a transition relation, $P$ a set of atomic propositions and $L : S \rightarrow 2^P$ a label function.

To a transition system we can associate a set of atomic propositions which depend on the properties taken into account. Thus we can obtain a variety of choices which a logical analysis is able to predict. From the point of view of transition mechanisms, the choice is arbitrary.

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2 Labeled graphs

**Definition 2** A labeled graph is a tuple \( LG = (S, E, T, f) \) where \( S \) is a finite set of elements representing the vertices of \( LG \), \( E \) is a set of elements used to label the edges of the graph, \( T \) is a set of binary relations on \( S \) and \( f : E \to T \) a surjective function.

**Remark.** In the graphic representation of this structure, the vertices are drawn as boxes which contain their names. An edge from \( x_i \in S \) to \( x_j \in S \) is labeled by \( a \in E \) if and only if \((x_i, x_j) \in f(a)\). Note that the

![Figure 1: Graphic representation of a labeled graph](image)

**Example.** Consider

- \( S = \{x_1, x_2, x_3, x_4\} \);
- \( L = \{a, b, c\} \);
- \( T = \{\rho_1, \rho_2, \rho_3\} \);
- \( \rho_1 = \{(x_1, x_4), (x_3, x_4)\} \);
- \( \rho_2 = \{(x_2, x_1), (x_2, x_3)\} \);
- \( \rho_3 = \{(x_4, x_2)\} \);
- \( f(a) = \rho_1 \);
- \( f(b) = \rho_2 \);
- \( f(c) = \rho_3 \).
3 Labeled graphs and transition systems

We notice the following:

1. A labeled graph can be interpreted as a transition system.
2. The states of $ST$ are the vertices of the graph.
3. The actions of $ST$ can be associated with the labels of the graph.

We define the labeled graph associated to a transition system, and denote this by $LG(ST)$, to be a tuple $(S, E, T, f)$, where $S$ is the set of states, $E$ the set of actions, $T$ the set of atomic propositions and $f : E \rightarrow T$ a surjective function.

Let $LG_1(ST) = (S_1, E_1, T_1, f_1)$ and $LG_2(ST) = (S_2, E_2, T_2, f_2)$ two labeled graphs associated to a transition system. Consider the function

$$g : S_1 \rightarrow S_2$$

and define

$$\overline{g} : 2^{S_1 \times S_1} \rightarrow 2^{S_2 \times S_2}$$

by

$$\overline{g}(\emptyset) = \emptyset$$

and

$$\overline{g}(R) = \{ (x_1, x_2) \in S_2 \times S_2 : \exists (a_1, a_2) \in R \text{ with } g(a_i) = x_i \text{ for } i = 1, 2 \}.$$ 

**Definition 3** We say that $LG_1$ is included in $LG_2$, and denote this by $LG_1 \subseteq LG_2$, if the following conditions hold:

1. $E_1 \subseteq E_2$;
2. there is an injective function $g : S_1 \rightarrow S_2$ for which

$$\overline{g}(f_1(a)) \subseteq f_2(a) \quad \forall a \in E_1.$$

**Proposition 1** The inclusion relation previously defined is reflexive and transitive.

**Proof.** It is clear that $LG \subseteq LG$, therefore the relation is reflexive.

We assume now $LG_1 \subseteq LG_2$ and $LG_2 \subseteq LG_3$ and prove that $LG_1 \subseteq LG_3$. Since $LG_1 \subseteq LG_2$, there is an injective function $g_1 : S_1 \rightarrow S_2$ so that

$$\overline{g_1}(f_1(a)) \subseteq f_2(a) \quad \forall a \in E_1.$$ 

In a similar way, there is an injective function $g_2 : S_2 \rightarrow S_3$ so that

$$\overline{g_2}(f_2(a)) \subseteq f_3(a) \quad \forall a \in E_2.$$ 

Clearly, from $E_1 \subseteq E_2$ and $E_2 \subseteq E_3$ we get $E_1 \subseteq E_3$. It means that we can define an injective function $g_3 : S_1 \rightarrow S_3$ by $g_3 = g_2 \circ g_1$ such that

$$g_3(g_1(a)) \subseteq g_3(f_2(a)) \quad \forall a \in E_1.$$

But $g_3(f_2(a)) \subseteq f_3(a) \Rightarrow g_3(g_1(f_1(a))) \subseteq f_3(a)$, therefore

$$g_3(f_1(a)) \subseteq f_3(a) \quad \forall a \in L_1,$$

hence $LG_1 \subseteq LG_3$.

**Definition 4** If $LG_1 \subseteq LG_2$ and $LG_2 \subseteq LG_1$ we say that $LG_1$ and $LG_2$ are isomorphic and denote this by $LG_1 \sim LG_2$.

**Proposition 2** The relation $\sim$ is reflexive, symmetric and transitive.

**Proof.** We prove that $\sim$ is reflexive, symmetric and transitive.

1) Obviously $LG_1 \sim LG_1$.

2) If $LG_1 \sim LG_2$ then clearly $LG_2 \sim LG_1$.

3) Let $LG_1 \sim LG_2$ and $LG_2 \sim LG_3$. Then we obtain $LG_1 \subseteq LG_2$ and $LG_2 \subseteq LG_1$, and also $LG_2 \subseteq LG_3$ and $LG_3 \subseteq LG_2$. But then it follows that $LG_1 \subseteq LG_3$ and $LG_3 \subseteq LG_1$ and the claim is proved.

**Definition 5** Let $s \in S$. Define the relation $\models$ in the following way:

1. $s \models p$ if and only if $p$ is true in the state $s$.

2. $s \models \rho \land \psi$ if and only if $s \models \rho$ and $s \models \psi$.

3. $s \models \neg \rho$ if and only if $s \not\models \rho$.

The most famous theories about modal logic are based on the model built by Saul Kripke which, in a restricted sense, refers to necessity and possibility.

The semantics of modal logics consists of a non-empty set $G$, whose elements are called possible worlds, a binary relation $R$ between the elements of $G$ called accessibility relation and a labeling function which describes every situation. Modal logic makes use of the modal operators $\Box$ (necessary) and $\Diamond$ (possible).

The truth of a modal formula $\varphi$ for a state $w$ in $LG(ST)$, denoted by $(LG) w \models \varphi$ is defined as follows.

**Definition 6**

1. $w \models p$ if $p$ is true in the state $w$.

2. $w \models \neg p$ if and only if $w \not\models p$.

3. $w \models p \land q$ if and only if $w \models p$ and $w \models q$.

4. $w \models \Box p$ if and only if $u \models p$ for any element $u \in G$ such that $wRu$.

5. $w \models \Diamond p$ if and only if there is $u \in G$ such that $wRu$ and $u \models p$. 

Figure 2: Example of a labeled graph associated to a transition system

- \( S = \{x_1, x_2, x_3, x_4\} \);
- \( L = \{p, q, r\} \);
- \( T = \{\rho_1, \rho_2, \rho_3\} \);
- \( \rho_1 = \{(x_3, x_4)\} \);
- \( \rho_2 = \{(x_2, x_1), (x_2, x_3), (x_1, x_4)\} \);
- \( \rho_3 = \{(x_2, x_4)\} \).

**Definition 7** We define a function \( h : T \times S \rightarrow \{t, f\} \) (with \( t \) and \( f \) truth values “true” and “false”) which assigns the truth value to the sentence \( p \) at the state \( x \), denoted by \( h(p, x) \).

**Remark.** In the preceding example we have \( h(q, x_4) = h(r, x_4) = h(p, x_4) = h(x_3, q) = h(x_1, q) = T \), the rest of the sentences being false.

We can interpret the basic formulas of the modal language in terms of the states of an \( LG(ST) \):

- \( LG(ST) \), \( x_2 \vDash \Box q : x_1 \) is the only state accessible from \( x_2 \) and \( q \) is true in \( x_1 \).
- \( LG(ST) \), \( x_1 \vDash \Diamond q : \) there is a state accessible \( x_4 \) from \( x_1 \), where \( q \) is true.
- \( LG(ST) \), \( x_1 \vDash \Diamond \Box q : x_1 \) is accessible from \( x_2 \) and \( q \) is true in all states accessible from \( x_1 \).
- \( LG(ST) \), \( x_4 \vDash \Box \bot q : \) there are no accessible states from \( x_4 \) so that any formula starting with \( \Box \) is true. Also, all formulas starting with \( \Diamond \) are false.
Definition 8  Let $LG(ST) = (S, E, T, f)$ be a labeled graph associated to a transition system. A set $A \subseteq S$ is defined in $LG(ST)$ if $A = \{ w \in S : (LG) w \models \varphi \}$ for some modal formulas $\varphi$.

![A second example of a labeled graph associated to a transition system](image)

Figure 3: A second example of a labeled graph associated to a transition system

Example.

- $x_4$ is defined by $\Box \bot$ ($x_4$ is the only final state);
- $x_1$ is defined by $\Diamond \Box \bot \land \Box \bot$ ($x_1$ can be a final state);
- $x_2$ is defined by $\Diamond (\Diamond \bot \land \Box \bot)$ ($x_2$ is the only state from which $x_1$ can be reached);
- $x_3$ is defined by $\Diamond \bot$.

Note that all subsets of $S = \{x_1, x_2, x_3, x_4\}$ are defined. However, troubles might appear if the states do not have distinct modal formulas.

Definition 9  Let $LG_1(S_1, E_1, T_1, f_1)$ and $LG_2(S_2, E_2, T_2, f_2)$ be two labeled graphs associated to a transition system. We say that $(LG_1) w_1$ and $(LG_2) w_2$ are “modally equivalent” (see also [3]) if, for all modal formulas $\varphi$, $(LG_1) w_1 \models \varphi$ if and only if $(LG_2) w_2 \models \varphi$. We denote this by $(LG_1) \approx (LG_2)$.

Definition 10  Let $LG_1(ST)$ and $LG_2(ST)$ be two labeled graphs associated to a transition system. A subset $S \subseteq S_1 \times S_2$ is called a ”bisimulation” (see [2]) if, for any $w_1 \in S_1$ and $w_2 \in S_2$ with $w_1, w_2 \in S$, we have

1. $h(p, w_1) = h(p, w_2)$ for all $p \in T$;
2. if $w_1 E_1 v_1$ then there exists $v_2 \in S_2$ such that $w_2 E_2 v_2$ and $v_1 S v_2$;
3. if $w_2 E_2 v_2$ then there exists $v_1 \in S_1$ such that $w_1 E_1 v_1$ and $v_1 S v_2$.

We denote $LG_1(ST) \leftrightarrow LG_2(ST)$ if there is a bisimulation $v$ relative to $w_1$ with $w_2$.

Proposition 3 The relation " $\leftrightarrow$ " is an equivalence relation.

Proof. Clearly $LG_1(ST) \leftrightarrow LG_1(ST)$. Then, if $LG_1(ST) \leftrightarrow LG_2(ST)$, clearly we also have $LG_2(ST) \leftrightarrow LG_1(ST)$. Finally, $LG_1(ST) \leftrightarrow LG_2(ST)$ implies that

\begin{equation}
(1) \quad h(p, w_1) = h(p, w_2) \quad \forall p \in T \text{ and } w_1 \in S_1, w_2 \in S_2.
\end{equation}

Similarly, $LG_2(ST) \leftrightarrow LG_3(ST)$ implies that

\begin{equation}
(2) \quad h(q, w_2) = h(q, w_3) \quad \forall q \in T \text{ and } w_2 \in S_2, w_3 \in S_3.
\end{equation}

Equalities (1) and (2) imply now that, when $p = q$ is arbitrary, $h(p, w_1) = h(p, w_3)$ for all $(w_1, w_3) \in S_1 \times S_3$, hence the first part of the definition is satisfied. To verify the second part, if $w_1 E_1 v_1$ then there exists $v_2 \in S_2$ so that $w_2 E_2 v_2$ and $v_1 S v_2$ where $S \subseteq S_1 \times S_2$. Since $w_2 E_2 v_2$ then there exists $v_3 \in S_3$ so that $w_3 E_3 v_3$ and $v_2 S' v_3$ where $S' \subseteq S_2 \times S_3$. Thus, if $w_1 E_1 v_1$ then there is $v_3 \in S_3$ such that $w_3 E_3 v_3$ and $v_1 S'' v_3$ where $S'' \subseteq S_1 \times S_3$ and, consequently, the second part of the definition is checked. One can similarly deal with the third part and conclude the proof.

4 Conclusions

In a broad sense, modal logic covers a family of logics with a series of norms and a variety of different symbols. The most famous theories regarding modal logic are based on the model built by Saul Kripke. If in the transition system we include a supplementary labeling function for the states, we obtain a Kripke-type structure. Representing such structures through labeled graphs allows for a graphical interpretation which proves useful in the theoretical analysis of their properties.

References


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Approximation by blending type operators based on Szász-Lupaş basis functions

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Abstract

In this study, a certain bivariate summation integral type operators based on Lupas-Szász functions are introduced and investigated the degree of approximation. In terms of partial and total modulus of continuity and K-functional. Furtermore, the operators extended to Bögel continuous functions by the means of Generalized Boolean Approach.

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1 Introduction

Inspired by Lupaş’s paper [25], in [5] Agratini investigated some convergence properties of the following operators

\[ L_n(\ell; x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f \left( \frac{k}{n} \right), \]

where \( f \in C[0, \infty) \), and \( C[0, \infty) \) denotes the space of all real valued continuous functions on \([0, \infty)\).

In [3] Agratini introduced Kantorovich variant of the operators (1) and obtained the smoothness properties in terms of modulus of continuity. Also, by using probabilistic methods, he investigated rate of convergence of the Kantorovich variant of the operators (1) for functions of bounded variation.

Erençin and Taşdelen introduced the following generalization of the operators (1)

\[ L_n^* (\ell; x) = 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} f \left( \frac{k}{b_n} \right), x \geq 0 \]

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where \((a_n), (b_n)\) are increasing and unbounded sequences of positive numbers such that
\[
\frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right), \lim_{n \to \infty} \frac{1}{b_n} = 0
\]
and studied their weighted approximation properties by means of weighted approximation \([16]\), later they gave the rate of convergence for the Kantorovich type version of the operators \(L^*_n\) by means of the modulus of continuity, for local Lipschitz class and Peetre’s \(K\)-functional \([17]\).

In \([23]\), for integrable functions, we studied the approximation properties of certain summation-integral type operators based on Lupaș-Szász functions as follows:

\[
D_{a_n,b_n} (f; x) = \frac{a_n}{b_n} \sum_{k=0}^{\infty} l_{n,k} (x) \int_0^\infty P_{n,k} (u) f (u) \, du,
\]
where \(p_{n,k} (x) = e^{-\frac{a_n x}{b_n}} \frac{\binom{a_n x}{k}}{k!}, l_{n,k} (x) = \frac{\binom{a_n x}{b_n x}}{2^k k!}\) and \((a_n), (b_n)\) are real number sequences such that \(\lim_{n \to \infty} a_n = \infty, \lim_{n \to \infty} b_n = \infty, \lim_{n \to \infty} \frac{b_n}{a_n} = 0, \frac{b_n}{a_n} \leq 1\) and \(x \in [0, \infty)\).

Inspired by the above works, we can introduce the bivariate case of the summation integral type operators based on Lupaș-Szász functions as follows:

\[
D_{n,m}^* (f; x, y) = \frac{a_n c_m}{b_n d_m} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} j_{n,m}^{k,j} (x,y) \int_0^\infty \int_0^\infty p_{n,m}^{k,j} (u,v) f (u,v) \, du \, dv
\]
where \(f\) is Lebesgue integrable function, \(p_{n,m}^{k,j} (x,y) = p_{n,k} (x) p_{m,j} (y)\) is a tensorial product of \(p_{n,k} (x) = e^{-\frac{a_n x}{b_n}} \frac{\binom{a_n x}{k}}{k!}\) and \(p_{m,j} (y) = e^{-\frac{c_m y}{d_m}} \frac{\binom{c_m y}{j}}{j!}\), \(n, m \in \mathbb{N}\).

The bivariate operators \((4)\) are defined on \(C (J^2), J^2 = J \times J\) with \(J = [0, \infty)\), \(C (J^2)\) denotes the space of all real valued continuous bivariate functions \(f\) defined on the \(J^2\). \(C_B (J^2)\) is the space of all real valued continuous and bounded bivariate functions \(f\) defined on \(J^2\), \(C_B (J^2)\) is endowed with the norm \(\|f\|_{C_B (J^2)} = \sup_{(x,y) \in J^2} |f (x,y)|\).

In the present study, we give some basic convergence properties of the operators \(D_{n,m}^* (4)\) and investigate the degree of approximation in terms of the complete and partial modulus of continuity of these operators. We obtain the order of
convergence with using Petree-K functional. Later, we introduce the GBS (Generalized Boolean Sum) operators of the operators (4) and investigate some convergence properties on the space of the Bögel continuous functions.

2 Approximation properties

In order to examine the approximation properties, we give some results using the two-dimensional test functions $e_{i,j}(t, s) = t^i s^j$, $(i, j = 0, 1, 2)$. By simple calculations we have the following lemma:

**Lemma 1** Let $D_{n,m}^*$ be the bivariate linear positive operators defined by (4). For all $m, n \in \mathbb{N}$, $D_{n,m}^*$ satisfy the following results:

\[
D_{n,m}^*(e_{0,0}; x, y) = 1,
\]

\[
D_{n,m}^*(e_{1,0}; x, y) = x + \frac{b_n}{a_n},
\]

\[
D_{n,m}^*(e_{0,1}; x, y) = y + \frac{d_m}{c_m},
\]

\[
D_{n,m}^*(e_{2,0}; x, y) = x^2 + 5 \frac{b_n}{a_n} x + 2 \left( \frac{b_n}{a_n} \right)^2,
\]

\[
D_{n,m}^*(e_{0,2}; x, y) = y^2 + 5 \frac{d_m}{c_m} y + 2 \left( \frac{d_m}{c_m} \right)^2,
\]

\[
D_{n,m}^*(e_{3,0}; x, y) = x^3 + 12 \frac{b_n}{a_n} x^2 + 29 \left( \frac{b_n}{a_n} \right)^2 x + 6 \left( \frac{b_n}{a_n} \right)^3,
\]

\[
D_{n,m}^*(e_{0,3}; x, y) = y^3 + 12 \frac{d_m}{c_m} y^2 + 29 \left( \frac{d_m}{c_m} \right)^2 y + 6 \left( \frac{d_m}{c_m} \right)^3
\]

and

\[
D_{n,m}^*(e_{4,0}; x, y) = x^4 + 22 \frac{b_n}{a_n} x^3 + 131 \left( \frac{b_n}{a_n} \right)^2 x^2 + 206 \left( \frac{b_n}{a_n} \right)^3 x + 24 \left( \frac{b_n}{a_n} \right)^4,
\]

\[
D_{n,m}^*(e_{0,4}; x, y) = y^4 + 22 \frac{d_m}{c_m} y^3 + 131 \left( \frac{d_m}{c_m} \right)^2 y^2 + 206 \left( \frac{d_m}{c_m} \right)^3 y + 24 \left( \frac{d_m}{c_m} \right)^4.
\]

**Proof.** Proof is clearly by definition of the operators $D_{n,m}^*$ and Lemma 2 in [23].

**Remark 1** For $n, m \in \mathbb{N}$ and for all $(x, y) \in J^2$, using Remark 2 in [23] we have

\[
D_{n,m}^*(t - x; x, y) = \frac{b_n}{a_n},
\]

\[
D_{n,m}^*(s - y; x, y) = \frac{d_m}{c_m}.
\]
\( D_{n,m}^* \left( (t-x)^2; x, y \right) = 3 \frac{b_n}{a_n} x + 2 \left( \frac{b_n}{a_n} \right)^2, \quad (7) \)

\( D_{n,m}^* \left( (s-y)^2; x, y \right) = 3 \frac{d_m}{c_m} y + 2 \left( \frac{d_m}{c_m} \right)^2. \quad (8) \)

Taking into account the condition (7) and by Lemma 1 we can write

\[
D_{n,m}^* \left( (e_{1,0} - x)^2; x, y \right) = O \left( \frac{b_n}{a_n} \right) (x + 1),
\]

\[
D_{n,m}^* \left( (e_{1,0} - x)^4; x, y \right) = O \left( \frac{b_n}{a_n} \right) (x^2 + x + 1),
\]

and as similar we get

\[
D_{n,m}^* \left( (e_{0,1} - y)^2; x, y \right) = O \left( \frac{d_m}{c_m} \right) (y + 1),
\]

\[
D_{n,m}^* \left( (e_{0,1} - y)^4; x, y \right) = O \left( \frac{d_m}{c_m} \right) (y^2 + y + 1).
\]

### 2.1 Direct Estimates

We start with the following theorem.

**Theorem 1** Let \( \{ D_{n,m}^* \} \) are positive linear operators given (4), we have,

\[
\lim_{n,m \to \infty} \| D_{n,m}^* (f) - f \| = 0,
\]

where, for all \( f \in C(J^2) \), \( D_{n,m}^* \) operators are uniformly convergent to \( f \) on the compact subintervals of \( J^2 \).

**Proof.** Taking into account the conditions of Lemma 1 and for all \((x, y) \in [0, a] \times [0, a] \subset J^2, a > 0\), we can easily see that

\[
\lim_{n,m \to \infty} D_{n,m}^* (e_{i,j}; x, y) = e_{i,j}(x, y), i, j = 0, 1, 2
\]

By using the universal Korovkin-type theorem in [31], the theorem is proved.

Now, we give the convergence properties about modulus of continuity.
2.2 Rate of approximation

Next the degree of approximation of the operators $D_{n,m}^*$ given by (4) will be established in the space $C_B(J^2)$. Then, the complete modulus of continuity for bivariate case is defined as follows:

$\omega(f; \delta) = \sup \{|f(t,s) - f(x,y)| : \sqrt{(t-x)^2 + (s-y)^2} \leq \delta\}$

for every $(t,s), (x,y) \in J^2$. Further, the partial moduli of continuity with respect to $x$ and $y$ is defined as

$\omega^1(f; \delta) = \sup \left\{ |f(x_1, y) - f(x_2, y)| : y \in J \text{ and } |x_1 - x_2| \leq \delta \right\}$,

$\omega^2(f; \delta) = \sup \left\{ |f(x, y_1) - f(x, y_2)| : x \in J \text{ and } |y_1 - y_2| \leq \delta \right\}$.

It is obvious that they satisfy the main features of the usual modulus of continuity [1].

Now, we can give the rate of convergence of the sequence of the operators $\{D_{n,m}^*\}$ to the function $f \in C_B(J^2)$.

**Theorem 2** Let $\{D_{n,m}^*\}$ are positive linear operators given (4), the following inequalities

$|D_{n,m}^*(f; x, y) - f(x,y)| \leq 2\omega(f; \delta_{n,m})$

$|D_{n,m}^*(f; x, y) - f(x,y)| \leq 2[\omega^1(f; \delta_{n}) + \omega^2(f; \delta_{m})]$

hold. Here $\omega$ is the complete modulus of continuity, $\omega^1$ and $\omega^2$ are the partial moduli of continuity with respect to $x$ and $y$ and

$\delta_{n,m}(x, y) = \left\{ \frac{3x}{a_n} + 2 \left( \frac{b_n}{a_n} \right)^2 + 3y \left( \frac{d_m}{c_m} \right)^2 \right\}^{1/2}$,

$\delta_n(x) = \left\{ \frac{3x}{a_n} + 2 \left( \frac{b_n}{a_n} \right)^2 \right\}^{1/2}$ and $\delta_m(y) = \left\{ 3y \left( \frac{d_m}{c_m} \right)^2 + 2 \left( \frac{d_m}{c_m} \right)^2 \right\}^{1/2}$, $(x,y) \in J^2$.

**Proof.** Using the definition of 9, we can write

$|D_{n,m}^*(f; x, y) - f(x,y)| \leq D_{n,m}^* \left( |f(t,s) - f(x,y)| : x, y \right)$

$\leq D_{n,m}^* \left( \omega \left( f; \sqrt{(t-x)^2 + (s-y)^2} \right) ; x, y \right)$

$\leq \omega(f; \delta) \left[ 1 + \frac{1}{2} D_{n,m}^* \left( \sqrt{(t-x)^2 + (s-y)^2} ; x, y \right) \right]$.

Applying the Cauchy-Schwarz inequality, we have

$|D_{n,m}^*(f; x, y) - f(x,y)| \leq \omega(f; \delta) \left[ 1 + \frac{1}{2} \left\{ D_{n,m}^* \left( (t-x)^2 + (s-y)^2 ; x \right) \right\}^{1/2} \right]$.

$\leq \omega(f; \delta) \left[ 1 + \frac{1}{2} \left\{ D_{n,m}^* \left( (t-x)^2 ; x \right) + D_{n,m}^* \left( (s-y)^2 ; y \right) \right\}^{1/2} \right]$.

$\leq \omega(f; \delta) \left[ 1 + \frac{1}{2} \left\{ 3x \left( \frac{b_n}{a_n} \right)^2 + 3y \left( \frac{d_m}{c_m} \right)^2 + 2 \left( \frac{d_m}{c_m} \right)^2 \right\}^{1/2} \right]$.
Taking \( \delta := \delta_{n,m}(x,y) = \left\{ 3x \frac{b_n}{a_n} + 2 \left( \frac{b_n}{a_n} \right)^2 + 3y \frac{d_m}{c_m} + 2 \left( \frac{d_m}{c_m} \right)^2 \right\}^{1/2} \), we obtain the desired result.

Now, using the well-known properties of 10, 11 and using Cauchy-Schwarz inequality we can write

\[
\begin{align*}
|D_{n,m}^* (f; x, y) - f (x, y)| &\leq D_{n,m}^* (|f (t, s) - f (x, y)| ; x, y) \\
&\leq D_{n,m}^* (|f (t, s) - f (t, y) | ; x, y) + D_{n,m}^* (|f (t, y) - f (x, y)| ; x, y) \\
&\leq D_{n,m}^* \left( 1 + \frac{1}{\delta_m} |s - y| \omega^2 (f; \delta_m) ; x, y \right) \\
&\quad + D_{n,m}^* \left( 1 + \frac{1}{\delta_n} |t - x| \omega^1 (f; \delta_n) ; x, y \right) \\
&\leq \omega^2 (f; \delta_m) \left( 1 + \frac{1}{\delta_m} (D_{n,m}^* ((s - y)^2) ; x, y) \right)^{1/2} \\
&\quad + \omega^1 (f; \delta_n) \left( 1 + \frac{1}{\delta_n} (D_{n,m}^* ((t - x)^2) ; x, y) \right)^{1/2} \\
&= A_1 + A_2
\end{align*}
\]

Taking \( \delta_m = \delta_n (y) = \left\{ 3y \frac{d_m}{c_m} + 2 \left( \frac{d_m}{c_m} \right)^2 \right\}^{1/2} \), from the definition of 11 we obtain \( A_1 = 2\omega^2 (f; \delta_m) \). Similarly using the definition of 10 we get \( A_2 = 2\omega^1 (f; \delta_n) \) with \( \delta_n = \delta_n (x) = \left\{ 3x \frac{b_n}{a_n} + 2 \left( \frac{b_n}{a_n} \right)^2 \right\}^{1/2} \). Hence, theorem is proved.

Now, we find the order of approximation of the \( D_{n,m}^* \) operators to \( f (x, y) \in C_B \left( J^2 \right) \) by Petree’s \( K \)-functional. Priorly, we recall some notations.

Let \( C_B^2 \left( J^2 \right) \) is the space of all differentiable function \( f \in C_B \left( J^2 \right) \) such that \( \frac{\partial^i f}{\partial y^i} \) for \( i = 1, 2 \) belong to \( C_B \left( J^2 \right) \). The norm on the space \( C_B^2 \left( J^2 \right) \), is defined as

\[
\| f \|_{C_B^2 (J^2)} = \| f \|_{C_B (J^2)} + \sum_{i=1}^{2} \left( \left\| \frac{\partial^i f}{\partial x^i} \right\|_{C_B (J^2)} + \left\| \frac{\partial^i f}{\partial y^i} \right\|_{C_B (J^2)} \right).
\]

The Petree’s \( K \)-functional of function \( f \in C \left( J^2 \right) \) is defined as:

\[
(12) \quad \mathcal{K}(f, \delta) = \inf_{g \in C_B^2 (J^2)} \{ \| f - g \|_{C_B (J^2)} + \delta \| g \|_{C_B^2 (J^2)} \}, \quad (\delta > 0).
\]

It is also known that

\[
(13) \quad \mathcal{K}(f, \delta) \leq M \left\{ \omega_2(f, \sqrt{\delta}) + \min (1, \delta) \| f \|_{C_B (J^2)} \right\}
\]

holds for \( \delta > 0 \) (see also [14]). Here the constant \( M \) in the above inequality is independent of \( \delta \) and \( f \) and \( \omega_2(f, \sqrt{\delta}) \) is the second order modulus of continuity.
Theorem 3 For the function \( f \in C_B(J^2) \), the following inequality
\[
\begin{align*}
|D^*_{n,m}(f; x, y) - f(x, y)| & \leq 4K(f; M_{n,m}(x, y)) + \omega(f; \delta_{n,m}(x, y)) \\
& \leq M \left\{ \omega_2(f, \sqrt{M_{n,m}(x, y)}) + \min(1, M_{n,m}(x, y)) \left\| f \right\|_{C_B(J^2)} \right\} \\
& + \omega(f; \delta_{n,m}(x, y))
\end{align*}
\]
holds where \( \delta^2_{n,m}(x, y) = \left( \frac{b_n}{a_n} \right)^2 + \left( \frac{d_m}{c_m} \right)^2 \) and \( M \) is constant independent from \( f \).

Proof. We define the auxiliary operators as follows:
\[
\begin{align*}
\tilde{D}^*_{n,m}(f; x, y) &= D^*_{n,m}(f; x, y) + f(x, y) - f \left( \frac{a_n x + b_n}{a_n}, \frac{c_m y + d_m}{c_m} \right).
\end{align*}
\]

Then using Lemma 2, we have
\[
\tilde{D}^*_{n,m}((t - x); x, y) = 0, \tilde{D}^*_{n,m}((s - y); x, y) = 0
\]
Let \( g \in C_B(J^2) \) and \( (t, s) \in J^2 \). Using the Taylor’s theorem we get
\[
\begin{align*}
g(t, s) - g(x, y) &= g(t, y) - g(x, y) + g(t, s) - (t, y) \\
& = (t - x) \frac{\partial g(x, y)}{\partial x} + \int_{x}^{t} (t - u) \frac{\partial^2 g(u, y)}{\partial u^2} du \\
& + (s - y) \frac{\partial g(x, y)}{\partial y} + \int_{y}^{s} (s - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv.
\end{align*}
\]

Operating \( \tilde{D}^*_{n,m} \) on both sides of the equality (16), we have
\[
\begin{align*}
\tilde{D}^*_{n,m}(g; x, y) - g(x, y) &= D^*_{n,m} \left( \int_{x}^{t} (t - u) \frac{\partial^2 g(u, y)}{\partial u^2} du; x, y \right) - \int_{x}^{t} \left( \frac{a_n x + b_n}{a_n} - u \right) \frac{\partial^2 g(u, y)}{\partial u^2} du \\
& + D^*_{n,m} \left( \int_{y}^{s} (s - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv; x, y \right) - \int_{y}^{s} \left( \frac{c_m y + d_m}{c_m} - v \right) \frac{\partial^2 g(x, v)}{\partial v^2} dv.
\end{align*}
\]

Hence,
\[
\begin{align*}
\left| \tilde{D}^*_{n,m}(g; x, y) - g(x, y) \right| & \leq D^*_{n,m} \left( \left| \int_{x}^{t} (t - u) \frac{\partial^2 g(u, y)}{\partial u^2} du \right| ; x \right) + \left| \int_{x}^{t} \left( \frac{a_n x + b_n}{a_n} - u \right) \frac{\partial^2 g(u, y)}{\partial u^2} du \right| \\
& + D^*_{n,m} \left( \left| \int_{y}^{s} (s - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv \right| ; x \right) + \left| \int_{y}^{s} \left( \frac{c_m y + d_m}{c_m} - v \right) \frac{\partial^2 g(x, v)}{\partial v^2} dv \right|
\end{align*}
\]
Taking the infimum on the right hand side over all \( g \)

Now, from equation (17), we have

Thus we get for \( f \in C_B(J^2) \), we find

By considering Lemma 2

Thus we get for \( f \in C_B(J^2) \), we find

Now, from equation (17), we have

Taking the infimum on the right hand side over all \( g \in C_B^2(J^2) \) and using (13) we obtain

Hence, the proof is completed.
3 Approximation in the space of Bögel continuous functions

In this section, we give a generalization of the operators (4) for the \( B \)-continuous (Bögel Continuous) functions. For this, we introduce the GBS (Generalized Boolean Sum) operators associated with the operators (4) and investigate some of its smoothness properties.

The definitions of \( B \)-continuity and \( B \)-differentiability were introduced by Karl Bögel in ([12], [13]). After, the well-known Korovkin theorem is improved for \( B \)-continuous functions in [6] and [7].

The approximation properties of the bivariate Bernstein type operators and corresponding GBS operators were investigated in [9], [10], [26], [27] and [28].

Let us give some basic definitions and notations which will be used in this study (for more information [12], [13]).

Let \( I_1 \) and \( I_2 \) be compact real intervals and let \( D = I_1 \times I_2 \). A function \( f : D \rightarrow \mathbb{R} \) is called a \( B \)-continuous (Bögel continuous) at a point \( (x_0, y_0) \in D \) if
\[
\lim_{(x,y) \to (x_0,y_0)} \Delta_{xy} f [x_0, y_0 ; x, y] = 0, \quad \text{for any } (x, y) \in D, \quad \text{with } \Delta_{xy} f [x_0, y_0 ; x, y] = f (x, y) - f (x, y_0) - f (x_0, y) + f (x_0, y_0).
\]

We define the GBS operators associated with \( D^*_{n,m} (f ; x, y) \) as follows:
\[
GD^*_{n,m} (f ; x, y) := D^*_{n,m} (f ; x, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{k^j}{k!} \left( \int \int p_{n,m}^k (u, v) \left[ f (u, y) + f (x, v) - f (u, v) \right] du dv \right) \Delta_{xy} f [x_0, y_0 ; x, y]
\]
where the operator \( GD^*_{n,m} \) is well-defined from the space \( C_b (I_{cd}) \) on itself and \( f \in C_b (I_{cd}) \). It is clear that \( GD^*_{n,m} \) is a linear positive operator and reproduces linear functions. \( C_b (I_{cd}) \) denotes the space of all \( B \)-continuous functions on \( I_{cd} = [0, c] \times [0, d] \) ([4] p.380).

We now give the rate of convergence of the sequences of the operators (18) to \( f \in C_b (I_{cd}) \) using the modulus of continuity in Bögel sense. We begin by recalling the definition of Bögel (mixed) modulus of smoothness of \( f \in C_b (I_{cd}) \). The Bögel (mixed) modulus of smoothness of \( f \in C_b (I_{cd}) \) is defined as
\[
\omega_B (f ; \delta_1, \delta_2) = \sup \left\{ \int \Delta_{xy} f [t, s ; x, y] : |t - s| < \delta_1, |y - s| < \delta_2 \right\},
\]
for all \( (x, y), (t, s) \in I_{cd} \) and for any \( (\delta_1, \delta_2) \in (0, \infty) \times (0, \infty) \) with \( \omega_B : [0, \infty) \times [0, \infty) \to \mathbb{R} \) [8]. This modulus has similar properties with usual modulus of continuity. For example, if \( f \in C_b (I_{cd}) \) then \( f \) is uniform \( B \)-continuous on \( I_{cd} \) then
\[
\lim_{n,m \to \infty} \omega_B (f ; \delta_n, \delta_m) = 0.
\]

**Theorem 4** For every \( f \in C_b (I_{cd}) \), in each \( (x, y) \in I_{cd} \), the operators (18) satisfies the following inequality
\[
|GD^*_{n,m} (f ; x, y) - f (x, y)| \leq 4\omega_B (f ; \delta_n, \delta_m)
\]
where \( \delta_n = \sqrt{3\frac{b_n}{a_n}x + \left(\frac{b_n}{a_n}\right)^2} \), \( \delta_m = \sqrt{3\frac{c_m}{d_m}y + \left(\frac{c_m}{d_m}\right)^2} \).

**Proof.** From the definition of \( \omega_B(f;\delta_n,\delta_m) \) and by the elementary inequality

\[
\omega_B(f;\lambda_n\delta_n,\lambda_m\delta_m) \leq (1 + \lambda_n)(1 + \lambda_m)\omega_B(f;\delta_n,\delta_m);\lambda_n,\lambda_m > 0
\]

we get,

\[
|\Delta_{(x,y)}f[t,s;x,y]| \leq \omega_B(f;|x-t|,|y-s|) \leq \left(1 + \frac{|x-t|}{\delta_n}\right)\left(1 + \frac{|y-s|}{\delta_m}\right)\omega_B(f;\delta_n,\delta_m)
\]

for every \((x,y),(t,s) \in I_{cd}\) and for any \(\delta_n,\delta_m > 0\). Using the definition of \(\Delta_{(x,y)}f[t,s;x,y]\), we may write

\[
f(x,s) + f(t,y) - f(t,s) = f(x,y) - \Delta_{(x,y)}f[t,s;x,y] .
\]

When we apply the \(GD_{n,m}^*(f;x,y)\) operator to this equality we get

\[
GD_{n,m}^*(f;x,y) = f(x,y)D_{n,m}(e_{0,0};x,y) - D_{n,m}(\Delta_{(x,y)}f[t,s;x,y];x,y) .
\]

Since \(D_{n,m}(e_{0,0};x,y) = 1\), taking into account inequality (19), using the linearity of the \(D_{n,m}^*\) operators and using Cauchy-Schwarz inequality we have,

\[
|GD_{n,m}^*(f;x,y) - f(x,y)| \leq D_{n,m}^*(|\Delta_{(x,y)}f[t,s;x,y]|;x,y) \leq D_{n,m}^*(|\Delta_{(x,y)}f[t,s;x,y]|;x,y)
\]

\[
\leq D_{n,m}^*(e_{0,0};x,y) + \delta_n^{-1}\sqrt{D_{n,m}^*(e_{1,0}-x)^2;x,y)}
\]

\[
+\delta_m^{-1}\sqrt{D_{n,m}^*(e_{0,1}-y)^2;x,y)}
\]

\[
\delta_n^{-1}\delta_m^{-1}\sqrt{D_{n,m}^*(e_{1,0}-x)^2;x,y)}D_{n,m}^*(e_{0,1}-y)^2;x,y)}\omega_B(f;\delta_n,\delta_m) .
\]

From Lemma 2, taking

\[
\delta_n = \sqrt{3\frac{b_n}{a_n}x + \left(\frac{b_n}{a_n}\right)^2} ,
\]

\[
\delta_m = \sqrt{3\frac{c_m}{d_m}y + \left(\frac{c_m}{d_m}\right)^2} ,
\]

we reach the desired inequality (18).

Now, we study the degree of approximation for the operators \(GD_{n,m}^*(f;x,y)\) by means of the Lipschitz class for \(B\)-continuous functions.

For \(f \in C_b(I_{cd})\), we define the Lipschitz class \(Lip_M(\lambda,\mu)\) with \(\lambda,\mu \in (0,1]\) as follows:

\[
Lip_M(\lambda,\mu) = \left\{ f \in C_b(I_{cd}) : |\Delta_{(x,y)}f[t,s;x,y]| \leq M |t-x|^{\lambda} |s-y|^{\mu} \right\}
\]

for \((t,s),(x,y) \in I_{cd}, M > 0 ([4] p.382).
Theorem 5 Let $f \in \text{Lip}_M (\lambda, \mu)$, then we have

$$|GD_{n,m}^* (f; x, y) - f (x, y)| \leq M \delta_n^{\lambda/2} \delta_m^{\mu/2},$$

where $\delta_n = \sqrt{3 \frac{b_n}{a_n} x + \left( \frac{b_n}{a_n} \right)^2}, \delta_m = \sqrt{3 \frac{c_m}{d_m} y + \left( \frac{c_m}{d_m} \right)^2}$ are the same as previous theorem and $\lambda, \mu \in (0, 1], (x, y) \in I_{cd}$.

Proof. By the definition of the $GD_{n,m}^*$ operators and by linearity of the $D_{n,m}^*$ operators, we may write

$$GD_{n,m}^* (f; x, y) = D_{n,m}^* (f(x, s) + f(t, y) - f(t; s); x, y)$$

$$= D_{n,m}^* (f(x; y) - \Delta_{(x,y)} f[t; s; x; y]; x, y).$$

And, by using the hypothesis, we have

$$|GD_{n,m}^* (f; x, y) - f (x, y)| \leq D_{n,m}^* (|\Delta_{(x,y)} f[t; s; x; y]|; x, y)$$

$$\leq MD_{n,m}^* \left( |t - x|^\lambda |s - y|^\mu; x, y \right)$$

Applying the H"older’s inequality with $p_1 = 2/\lambda, q_1 = 2/(2 - \lambda)$ and $p_2 = 2/\mu, q_1 = 2/(2 - \mu)$, we get

$$|GD_{n,m}^* (f; x, y) - f (x, y)| \leq MD_{n,m}^* \left( |t - x|^\lambda x, y \right) D_{n,m}^* \left( |s - y|^\mu; x, y \right)$$

Now we give some illustrations about convergence of the operators to a certain function by means of graphics.

Example 1 In Figure 1, for $n = 5, n = 10, n = 50$ the convergence of $GD_{n,m}^* (f; x, y)$ (respectively green, yellow and pink) to $f (x, y) = (x - 1)^2 + (y - 1)$ (blue) is illustrated with $a_n = n, b_n = \ln \sqrt{n + 10}, c_m = m, d_m = \ln \sqrt{m + 10}$.

Example 2 In Figure 2, for $n = 5, n = 10, n = 50$ the convergence of $GD_{n,m}^* (f; x, y)$ (respectively green, yellow and pink) to $f (x, y) = (x - 1)^2 (y - 1)$ (blue) is illustrated for the sequences are $a_n = n + 1, b_n = \sqrt{n + 10}, c_m = m + 1, d_m = \sqrt{m + 10}$. 
Figure 1: The convergence $GD_{n,m}^* (f; x, y)$ to $f(x, y) = (x - 1)^2 + (y - 1)$ for $a_n = n$, $b_n = \ln \sqrt{n + 10}$, and $c_m = m_d_m = \ln \sqrt{m + 10}$.

Figure 2: The convergence $GD_{n,m}^* (f; x, y)$ to $f(x, y) = (x - 1)^2 (y - 1)$ for $a_n = n$, $b_n = \ln \sqrt{n + 10}$, and $c_m = m_d_m = \ln \sqrt{m + 10}$.

References


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On some second order moduli of smoothness

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Abstract

The paper contains the equivalence of two moduli of smoothness with a corresponding generalized $K$-functional of order two. Also estimates of the degree of approximation of functions by using general positive linear operators in terms of this $K$-functional are given.

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1 Introduction and auxiliary results

Quantitative estimates for the approximation of functions in $C^1[a,b]$ by positive linear operators in terms of the first modulus of the derivative can be found in [1], [5], [2] and in many other papers. In the paper [6] (see also [7]), R. Păltănea defined two second order moduli of continuity that extend the first modulus of the derivative for the class of arbitrary functions. For $f : [a,b] \rightarrow \mathbb{R}$ and $t > 0$, these are

$$
\omega_2^d(f, t) = t \sup \left\{ \left| \frac{f(x + t_1)}{t_1} - \frac{f(x + t_2)}{t_2} - \frac{f(y + t_2)}{t_2} - f(y) \right| : t_1 > 0, t_2 > 0, x, x + t_1, y, y + t_2 \in [a, b], \max \{x + t_1, y + t_2\} - \min \{x, y\} \leq t \right\};
$$

(1)

and

$$
\omega_2^e(f, t) = t \sup \left\{ \left| \frac{f(x + t_1 + t_2) - f(x + t_1) - f(x + t_2) + f(x)}{t_1} \right| : t_1 > 0, t_2 > 0, x, x + t_1, y, y + t_2 \in [a, b], \max \{x + t_1, y + t_2\} - \min \{x, y\} \leq t \right\};
$$

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(2) \[ t_1 > 0, t_2 > 0, t_1 + t_2 \leq t, x, x + t_1 + t_2 \in [a, b] \}.

In these formulas the supremum is accepted to be infinite. Note that if \( f \) is a Lipschitz function then \( \omega_2^f(f, t) \) and \( \omega_2^f(f, t) \) are finite for all \( t > 0 \). The name second order moduli of continuity for \( \omega_2^f(f, t) \) and \( \omega_2^f(f, t) \) is justified by

\[
\omega_2^f(f + l, t) = \omega_2^f(f, t), \quad \omega_2^f(f + l, t) = \omega_2^f(f, t),
\]

for \( f : [a, b] \to \mathbb{R}, l \) linear function and \( t > 0 \).

In \[6\], \[7\], the following results are given

**Theorem A.** We have

(3) \[ \omega_2^f(f, t) = \omega_2^f(f, t), \quad (\forall) f \in B[a, b], \quad (\forall) t > 0. \]

**Theorem B.** We have

(4) \[ \omega_2^f(f, t) = t \omega_2^f(f', t), \quad (\forall) f \in C^1[a, b], \quad (\forall) t > 0. \]

In \[7\] estimates with optimal constants of the degree of approximation of functions by using general positive linear operators in terms of moduli \( \omega_2^f(f, t) \) and \( \omega_2^f(f, t) \) are given. In this paper we consider a generalized \( K \)-functional equivalent to these moduli and establish general estimates of the degree of approximation by positive linear operators with this \( K \)-functional.

Recall that a function \( f : [a, b] \to \mathbb{R} \) is said to be Lipschitz continuous if

\[ (\exists) M > 0, \quad (\forall) x, y \in [a, b] : |f(x) - f(y)| \leq M |x - y|. \]

The class of all Lipschitz continuous functions on \([a, b]\) denoted by \( \text{Lip}[a, b] \) is a seminormed space, seminormed by

(5) \[ |f|_{\text{Lip}[a, b]} = \sup_{x,y \in [a, b]} \frac{|f(x) - f(y)|}{|x - y|}. \]

Note that

\[ |f|_{\text{Lip}[a, b]} = \sup_{h \in (0, b-a]} \frac{\|\Delta_h(f)\|_{[a+\frac{h}{2}, b-\frac{h}{2}]}}{h}, \]

\( \Delta_h f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2}) \) being the central difference and \( \|\cdot\| \) the sup-norm.

**Lemma 1.** For \( f \in \text{Lip}[a, b] \) we have

(6) \[ |f|_{\text{Lip}[a, b]} = \sup_{h \in (0, t]} \frac{\|\Delta_h(f)\|_{[a+\frac{h}{2}, b-\frac{h}{2}]}}{h}, \quad (\forall) t \in (0, b-a]. \]
Proof. We denote by
\[ \alpha_f(t) = \sup_{h \in (0,t]} \frac{\|\Delta_h(f)\|_{[a+\frac{h}{2},b-\frac{h}{2}][t,1]}}{h}, \quad t \in (0,b-a). \]

Since \( \alpha_f(t) \) is an increasing positive function on \( (0,b-a) \), there exists the limit
\[ \lim_{t \to 0^+} \alpha_f(t) = \alpha_f(0^+) \leq \|f\|_{\text{Lip}[a,b]}. \]

On the other hand, for \( n \in \mathbb{N} \), \( h \in (0,b-a) \) and \( x \in [a+\frac{h}{2},b-\frac{h}{2}] \), with the notation \( x_i = x - \frac{h}{2} + i\frac{h}{n} \), \( i = 0, n \), we have
\[ \frac{|\Delta_h(f)(x)|}{h} \leq \frac{1}{h} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \leq \frac{n}{h} \frac{\|\Delta_h f\|_{[a+\frac{h}{2},b-\frac{h}{2}]}}{n} \leq \alpha_f\left( \frac{h}{n} \right) \]
whence \( |f|_{\text{Lip}[a,b]} \leq \lim_{t \to 0^+} \alpha_f(t) \).

Below it is given a useful property of \( \omega^2_2 \):

Lemma 2. Let \( f \in \text{Lip}[a,b] \). Then:
\[ \omega^2_2(f,nt) \leq n^2 \omega^2_2(f,t), \quad (\forall)n \in \mathbb{N} \]

Proof. Let \( t_1 > 0, t_2 > 0, t_1 + t_2 \leq nt, x, x + t_1 + t_2 \in [a,b] \). We have:
\[
\begin{align*}
& f(x + t_1 + t_2) - f(x + t_1) - f(x + t_2) + f(x) \\
& = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ f\left(x + \frac{it_1}{n} + \frac{jt_2}{n}\right) - f\left(x + \frac{(i-1)t_1}{n} + \frac{jt_2}{n}\right) \\
& \quad - f\left(x + \frac{it_1}{n} + \frac{(j-1)t_2}{n}\right) + f\left(x + \frac{(i-1)t_1}{n} + \frac{(j-1)t_2}{n}\right) \right].
\end{align*}
\]
Then
\[
\frac{|f(x + t_1 + t_2) - f(x + t_1) - f(x + t_2) + f(x)|}{t_1} \leq \frac{1}{n} \cdot \frac{\omega^2_2(f,t)}{t} \cdot n^2 = n \cdot \frac{\omega^2_2(f,t)}{t}.
\]
This implies \( \frac{\omega^2_2(f,nt)}{nt} \leq n \cdot \frac{\omega^2_2(f,t)}{t} \) i.e. the inequality.

2 The \( K \)-functional \( K^d_2 \)

Definition 1. For \( f \in \text{Lip}[a,b] \) and \( t > 0 \) set
\[
K^d_2(f,t) = \inf_{g \in C^2[a,b]} \left\{ |f - g|_{\text{Lip}[a,b]} + t \|g''\| \right\}. \tag{7}
\]

Remark 1. If \( f \in C^1[a,b] \), then
\[
K^d_2(f,t) = \inf_{g \in C^2[a,b]} \left\{ \|f' - g'\| + t \|g''\| \right\} = tK_1(f',t), \tag{8}
\]
where \( K_1(g,h) = K_1(g,h,C[a,b],C^1[a,b]) \) is the usual \( K \)-functional.
Remark 2. It can be proved that
\[
K_2^d(f, t) = t \inf_{g \in C^1[a, b], g' \in \text{Lip}[a, b]} \left\{ |f - g|_{\text{Lip}[a, b]} + t |g'|_{\text{Lip}[a, b]} \right\}.
\]

Theorem 1. For all \( f \in \text{Lip}[a, b] \),
\[
\frac{1}{2} \omega_2^d(f, t) \leq K_2^d(f, t) \leq 4 \omega_2^d(f, t), \ t \in (0, b - a].
\]

Proof. Let \( t_1 > 0, t_2 > 0, x, x + t_1, y, y + t_2 \in [a, b] \) such that \( \max \{x + t_1, y + t_2\} - \min \{x, y\} \leq t \). Let \( g \in C^2[a, b] \).
\[
\begin{align*}
\left| \frac{f(x + t_1) - f(x)}{t_1} - \frac{f(y + t_2) - f(y)}{t_2} \right| & \leq \frac{|(f - g)(x + t_1) - (f - g)(x)|}{t_1} + \frac{|(f - g)(y + t_2) - (f - g)(y)|}{t_2} \\
& \quad + \frac{|g(x + t_1) - g(x)|}{t_1} - \frac{g(y + t_2) - g(y)}{t_2} \\
& \leq 2 |f - g|_{\text{Lip}[a, b]} + t \|g''\| \\
& \leq 2 \left( |f - g|_{\text{Lip}[a, b]} + t \|g''\| \right)
\end{align*}
\]
Since \( g \) is arbitrary it follows that
\[
\left| \frac{f(x + t_1) - f(x)}{t_1} - \frac{f(y + t_2) - f(y)}{t_2} \right| \leq 2 K_2^d(f, t) \frac{t}{t}
\]
from where the left inequality in (9).

To prove the right inequality we consider the following extension of function \( f \) to a larger interval as in [3], \( \tilde{f}_t : [a + \frac{t}{2}, b + \frac{t}{2}] \rightarrow \mathbb{R} \),
\[
\tilde{f}_t(x) = \begin{cases} 
\frac{f(x + \frac{t}{2}) - f(a + \frac{t}{2}) + f(a)}{2}, & x \in [a + \frac{t}{2}, a] ; \\
\frac{f(x)}{2}, & x \in [a, b] ; \\
\frac{f(x - \frac{t}{2}) - f(b - \frac{t}{2}) + f(b)}{2}, & x \in (b, b + \frac{t}{2}) . 
\end{cases}
\]

We define \( g_t : [a, b] \rightarrow \mathbb{R} \),
\[
g_t(x) = \frac{4}{t^2} \int_a^x \frac{dv}{t} \left[ \tilde{f}_t \left( u + v + \frac{t}{4} \right) - \tilde{f}_t \left( u + v - \frac{t}{4} \right) \right] du .
\]
Let \( h \in (0, t], x \in [a + \frac{h}{2}, b - \frac{h}{2}] \).
\[
\frac{|\Delta_h(f - g_t)(x)|}{h} = \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h} - \frac{g_t(x + \frac{h}{2}) - g_t(x - \frac{h}{2})}{h}
\]
On some second order moduli of smoothness

\[\begin{align*}
&= \left| \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h} - \frac{4}{t^2} \int_{x - \frac{h}{2}}^{x + \frac{h}{2}} dv \int_{-\frac{t}{4}}^{\frac{t}{4}} \left[ \tilde{f}_t(u + v + \frac{t}{4}) - \tilde{f}_t(u + v - \frac{t}{4}) \right] du \right| \\
&\leq \frac{2}{th} \int_{x - \frac{h}{2}}^{x + \frac{h}{2}} dv \int_{-\frac{t}{4}}^{\frac{t}{4}} \left| \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h} - \frac{\tilde{f}_t(u + v + \frac{t}{4}) - \tilde{f}_t(u + v - \frac{t}{4})}{\frac{t}{2}} \right| du \\
\end{align*}\]

We will evaluate the expression

\[E = \left| \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h} - \frac{\tilde{f}_t(u + v + \frac{t}{4}) - \tilde{f}_t(u + v - \frac{t}{4})}{\frac{t}{2}} \right| .\]

If \(u + v \in [a + \frac{t}{4}, b - \frac{t}{4}]\), then

\[E = \left| \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h} - \frac{f(u + v + \frac{t}{4}) - f(u + v - \frac{t}{4})}{\frac{t}{2}} \right| \]

\[\leq \frac{\omega_2^d(f, 2t)}{2t} \leq 2 \frac{\omega_2^d(f, t)}{t} .\]

We used that \(\max \{x + \frac{h}{2}, u + v + \frac{t}{4}\} - \min \{x - \frac{h}{2}, u + v - \frac{t}{4}\} \leq \frac{3}{2}t < 2t\) and Lemma 2.

If \(b - \frac{t}{4} \leq u + v \leq b + \frac{t}{4}\), then

\[E = \left| \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h} - \frac{f(b) - f(b - \frac{t}{2})}{\frac{t}{2}} \right| \]

\[\leq \frac{\omega_2^d(f, 2t)}{2t} \leq 2 \frac{\omega_2^d(f, t)}{t} .\]

If \(a - \frac{t}{4} \leq u + v < a + \frac{t}{4}\), then

\[E = \left| \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h} - \frac{f(a + \frac{t}{2}) - f(a)}{\frac{t}{2}} \right| \]

\[\leq \frac{\omega_2^d(f, 2t)}{2t} \leq 2 \frac{\omega_2^d(f, t)}{t} .\]

Therefore \(\|\Delta_h(f - g_t)(x)\|_h \leq 2 \frac{\omega_2^d(f, t)}{t}\), whence

\[|f - g_t|_{Lip[a, b]} = \sup_{h \in (0, t]} \frac{\|\Delta_h(f - g_t)\|_{[a + \frac{3}{2}t, b - \frac{3}{2}t]}}{h} \leq 2 \frac{\omega_2^d(f, t)}{t} .\]

The second derivative of function \(g_t\) is

\[(12) \quad g''_t(x) = \frac{4}{t^2} \left[ \tilde{f}_t(x + \frac{t}{2}) - 2 \tilde{f}_t(x) + \tilde{f}_t(x - \frac{t}{2}) \right] .\]
If \( x \in \left[ a + \frac{t}{2}, b - \frac{t}{2} \right] \), then
\[
\left| t \left| g''_t(x) \right| = \frac{4}{t} \left| f \left( x + \frac{t}{2} \right) - 2f(x) + f \left( x - \frac{t}{2} \right) \right| \leq 2 \frac{\omega_2^d(f, t)}{t}.
\]

If \( x \in \left[ a, a + \frac{t}{2} \right) \), then
\[
\left| t \left| g''_t(x) \right| = \frac{4}{t} \left| f \left( x + \frac{t}{2} \right) - f(x) - f \left( a + \frac{t}{2} \right) + f(a) \right| \leq 2 \frac{\omega_2^d(f, t)}{t}.
\]

If \( x \in (b - \frac{t}{2}, b] \), then
\[
\left| t \left| g''_t(x) \right| = \frac{4}{t} \left| f \left( b - \frac{t}{2} \right) - f(x) - f \left( b - \frac{t}{2} \right) + f(x - \frac{t}{2}) \right| \leq 2 \frac{\omega_2^d(f, t)}{t}.
\]

So \( t \left| g''_t \right| \leq 2 \frac{\omega_2^d(f, t)}{t} \) and \( |f - g_t|_{\text{Lip}[a,b]} \leq 2 \frac{\omega_2^d(f, t)}{t} \) which leads to
\[
K_2^d(f, t) \leq t \left( |f - g_t|_{\text{Lip}[a,b]} + t \left| g''_t \right| \right) \leq 4 \omega_2^d(f, t).
\]

### 3 General estimates with the K-functional \( K_2^d \)

**Theorem 2.** Let \( L : C[a, b] \rightarrow C[a, b] \) a positive linear operator such that \( Le_0 = e_0, Le_1 = e_1 \) and \( f \in \text{Lip}[a, b] \). Then \( (\forall) x \in [a, b], (\forall) t > 0 \)

\[
(13) \quad |L(f, x) - f(x)| \leq \max \left\{ L \left( |e_1 - xe_0|, x \right), \frac{L \left( (e_1 - xe_0)^2, x \right)}{2t} \right\} \frac{K_2^d(f, t)}{t}.
\]

Conversely, if \( (\exists) A, B \geq 0 \) such that

\[
(14) \quad |L(f, x) - f(x)| \leq \max \left\{ AL \left( |e_1 - xe_0|, x \right), B \frac{L \left( (e_1 - xe_0)^2, x \right)}{t} \right\} \frac{K_2^d(f, t)}{t}
\]

holds for all positive linear operator, any \( f \in \text{Lip}[a, b] \), any \( x \in [a, b] \) and any \( t > 0 \) then \( B \geq \frac{1}{2} \) and \( A \geq 1 \).

**Proof.** Let \( g \in C^2[a, b] \) be. We have
\[
g(y) - g(x) = g'(x)(y - x) + \frac{1}{2} g''(\xi) (y - x)^2
\]
with \( \xi \) between \( x \) and \( y \). Therefore

\[
|L(g - g(x)e_0)| = |L(g - g(x)e_0 - g'(x)(e_1 - x e_0), x)| \\
\leq L\left(|g - g(x)e_0 - g'(x)(e_1 - x e_0)|, x\right) \leq \frac{1}{2} \|g''\| L\left((e_1 - x e_0)^2, x\right)
\]

and

\[
|L(f, x) - f(x)| \leq |L(f - g - (f - g)(x)e_0, x)| + |L(g - g(x)e_0, x)| \\
\leq |f - g|_{Lip[a,b]} L\left(|e_1 - x e_0|, x\right) + \frac{L\left((e_1 - x e_0)^2, x\right)}{2t} \cdot t \|g''\| \\
\leq \max \left\{ L\left(|e_1 - x e_0|, x\right), \frac{L\left((e_1 - x e_0)^2, x\right)}{2t} \right\} \cdot \left(|f - g|_{Lip[a,b]} + t \|g''\|\right).
\]

Since \( g \) was arbitrary it follows that

\[
|L(f, x) - f(x)| \leq \max \left\{ L\left(|e_1 - x e_0|, x\right), \frac{L\left((e_1 - x e_0)^2, x\right)}{2t} \right\} \cdot \frac{K_2^d(f, t)}{t}.
\]

We prove now the converse part. We consider positive linear operator \( L \) defined by

\[
L(f, x) = (1 - x) f(0) + x f(1), \ f \in C[0, 1].
\]

For \( f = e_2 \) we have \( L(f, x) = x, \ L\left(|e_1 - x e_0|, x\right) = 2x(1 - x), \ L\left((e_1 - x e_0)^2, x\right) = x(1 - x), \ K_2^d(f, t) \leq t^2 \|f''\| = 2t^2 \) and from (14) it follows \( 1 \leq \max\{4At, 2B\} \).

Passing to the limit \( t \to 0 \) we obtain \( B \geq \frac{1}{2} \).

We consider positive linear operator \( L \) defined by

\[
L(f, x) = \frac{1}{2} [(1 - x) f(0) + x f(1) + f(x)], \ f \in C[0, 1].
\]

For \( f = |e_1 - \frac{1}{2} e_0| \) we have \( L(f, x) = x(1-x)+\frac{1}{2} |x - \frac{1}{2}|, \ L\left(|e_1 - x e_0|, x\right) = x(1-x), \ L\left((e_1 - x e_0)^2, x\right) = \frac{x(1-x)}{2}, \ K_2^d(f, t) \leq t \|f\|_{Lip[0,1]} = t \) and from (14), for \( x = \frac{1}{2} \) it follows \( \frac{1}{4} \leq \max\{\frac{1}{4} A, \frac{B}{8t}\} \). Passing to the limit \( t \to \infty \) we obtain \( A \geq 1 \).

**Corollary 1.** Let \( L : C[a,b] \to C[a,b] \) a positive linear operator such that \( Le_0 = e_0, \ Le_1 = e_1 \) and \( f \in Lip[a,b] \). Then

\[
|L(f, x) - f(x)| \leq 2K_2^d\left(f, \frac{1}{2} \sqrt{L\left((e_1 - x e_0)^2, x\right)} \right).
\]

**Proof.** Using \( L\left(|e_1 - x e_0|, x\right) \leq \sqrt{L\left((e_1 - x e_0)^2, x\right)} \), from (13) we obtain the estimates (15) for \( t = \frac{1}{2} \sqrt{L\left((e_1 - x e_0)^2, x\right)} \).
Remark 3. As a consequence we mention the following estimates for positive linear operators reproducing linear functions in terms of the least concave majorant of the first modulus of the derivative given in [4]:

\[
|L(f, x) - f(x)| \leq \frac{1}{2} \cdot \sqrt{L((e_1 - xe_0)^2, x)} \cdot \overline{\omega_1}(f', \sqrt{L((e_1 - xe_0)^2, x)}).
\]

Indeed, if \( f \in C^1[a, b] \) then using the relations (15), (8) and the well known representation \( K_1(g, h) = \frac{1}{2} \overline{\omega_1}(g, 2h) \) for \( g \in C[a, b] \) and \( h > 0 \), we have

\[
|L(f, x) - f(x)| \leq 2K_2^d\left(f, \frac{1}{2} \sqrt{L((e_1 - xe_0)^2, x)}\right)
= \sqrt{L((e_1 - xe_0)^2, x)} \cdot \overline{\omega_1}(f', \sqrt{L((e_1 - xe_0)^2, x)}).
\]

Corollary 2. Let \( F : C[a, b] \rightarrow \mathbb{R} \) be a positive linear functional and \( z \in [a, b] \) be such that \( F(e_0) = 1 \), \( F(e_1) = z \). Then for all \( f \in \text{Lip}[a, b] \) and all \( t > 0 \)

\[
|F(f) - f(z)| \leq \max\left\{ \frac{F((e_1 - ze_0)^2)}{2t}, \frac{F((e_1 - ze_0)^2)}{2t} \right\} \cdot K_2^d(f, t).
\]

Proof. Consider positive linear operator \( L : C[a, b] \rightarrow C[a, b] \), defined for \( f \in C[a, b] \) by \( L(f, z) = F(f) \) and \( L(f, x) = f(x) \), for \( x \in [a, b] \setminus \{ z \} \). It is clear that \( L \) satisfies conditions in Theorem 2. Then relation (17) follows directly from relation (13), for \( x = z \).

Theorem 3. Let \( L : C[a, b] \rightarrow C[a, b] \) be a positive linear operator. Then for all \( f \in \text{Lip}[a, b] \), \( x \in [a, b] \), \( s > 0 \) and \( t > 0 \)

\[
|L(f, x) - f(x)| \leq |f(x)| \cdot |L(e_0, x) - 1| + \left[ L(e_0, x) + \frac{|L(xe_0 - e_1, x)|}{s} \right] \omega(f, s)
+ \max\left\{ \frac{L(L(e_0, x)e_1 - L(e_1, x)e_0, x)}{L(e_0, x)}, \frac{L(L(e_0, x)e_1 - L(e_1, x)e_0, x)}{2tL(e_0, x)^2} \right\} \times
\]

\[
\frac{K_2^d(f, t)}{t}.
\]

Proof. Fix \( f \in \text{Lip}[a, b] \), \( x \in [a, b] \) and \( t > 0 \). Denote

\[
z = \frac{L(e_1, x)}{L(e_0, x)}.
\]
It follows that \( z \in [a, b] \). Define positive linear functional \( F : C[a, b] \rightarrow \mathbb{R} \),
\[
F(f) = \frac{L(f, x)}{L(e_0, x)}, \quad f \in C[a, b].
\]

Then we have \( F(e_0) = 1 \) and \( F(e_1) = z \). We can apply Corollary 2 and we obtain relation (17). But we have
\[
F(|e_1 - ze_0|) = 1 \\
F((e_1 - ze_0)^2) = \frac{1}{L(e_0, x)^3} L\left((L(e_0, x) - L(e_1, x) e_0)^2, x\right).
\]

We can write
\[
|L(f, x) - f(x)| \leq |L(f, x) - L(e_0, x)f(z)| + |f(x)| \cdot |L(e_0, x) - 1| \\
+ L(e_0, x)|f(z) - f(x)|.
\]

Now, using Corollary 2 and relations (19), (20) we obtain
\[
\begin{align*}
|L(f, x) - L(e_0, x)f(z)| &= L(e_0, x)|F(f) - f(z)| \\
\leq & \max \left\{ \frac{L(|L(e_0, x)e_1 - L(e_1, x)e_0|, x)}{L(e_0, x)} \cdot \frac{L\left((L(e_0, x)e_1 - L(e_1, x)e_0)^2, x\right)}{2tL(e_0, x)^2} \right\} \times \frac{K_2^3(f, t)}{t}.
\end{align*}
\]

The third term in (21) can be estimate as follows
\[
L(e_0, x)|f(z) - f(x)| \leq L(e_0, x)\omega_1(f, |z - x|) \leq L(e_0, x) \left(1 + \frac{|z - x|}{s}\right) \omega_1(f, s) \\
= \left(L(e_0, x) + \frac{|L(xe_0 - e_1, x)|}{s}\right) \omega_1(f, s).
\]

From relations above relation (18) follows immediatly.

References


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Interpolation operators on a square with one curved side

Alina Baboş

Abstract

We construct some Lagrange, Hermite and Birkhoff-type operators which interpolate a given function and some of its derivatives on the border of a square with one curved side. We also consider their product and Boolean sum operators. We study the interpolation properties and the degree of exactness of the constructed operators.

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1 Introduction

There have been constructed interpolation operators of Lagrange, Hermite and Birkhoff type on a triangle with all straight sides, starting with the paper [6] of R.E. Barnhill, G. Birkhoff and W.J. Gordon, and in many others papers (see, e.g., [3], [5], [7], [8], [15]). Then, were considered interpolation operators on triangles with curved sides (one, two or all curved sides), many of them in connection with their applications in computer aided geometric design and in finite element analysis (see, e.g., [1]-[4], [9], [10], [12], [16]-[19]).

In [13] the authors consider $D_h$ the square with one curved side having the vertices $V_1 = (0, 0), V_2 = (h, 0), V_3 = (h, h)$ and $V_4 = (0, h)$, three straight sides $\Gamma_1, \Gamma_2$, along the coordinate axes, $\Gamma_3$ parallel to axis $Ox$, and the curved side $\Gamma_4$ which is defined by the function $g$, such that $g(h) = g(0) = h$.

They construct and analyze Bernstein-type operators on the square with one and two curved side.

Let $F$ be a real-valued function defined on $D_h$ and $(0, y), (g(y), y)$, respectively, $(x, 0), (x, h)$ be the points in which the parallel lines to the coordinate axes, passing through the point $(x, y) \in D_h$, intersect the sides $\Gamma_2, \Gamma_4$, respectively $\Gamma_1$ and $\Gamma_3$.
2 Lagrange-type operators

Let $L_1$ and $L_2$ be the interpolation operators defined by

\[
(L_1 F)(x, y) = \frac{g(y) - x}{g(y)} F(0, y) + \frac{x}{g(y)} F(g(y), y),
\]

\[
(L_2 F)(x, y) = \frac{h - y}{h} F(x, 0) + \frac{y}{h} F(x, h).
\]

1) Both operators $L_1$ and $L_2$ interpolates the function $F$ along two sides of the square $D_h$:

\[
(L_1 F)(0, y) = F(0, y) \quad (L_1 F)(g(y), y) = F(g(y), y), \quad y \in [0, h]
\]

\[
(L_2 F)(x, 0) = F(x, 0) \quad (L_2 F)(x, h) = F(x, h) \quad x \in [0, h]
\]

2) The degree of exactness: $\text{dex}(L_i) = 1, \quad i = 1, 2.$
Let $P_{21}^L$ be the product of the operators $L_2$ and $L_1$, i.e., $P_{21} = L_2L_1$.

We have

\[
(P_{21}^L F)(x, y) = \frac{(h-x)(h-y)}{h^2} F(0, 0) + \frac{x(h-y)}{h^2} F(h, 0) + \frac{y(h-x)}{h^2} F(0, h) + \frac{xy}{h^2} F(h, h)
\]

(2)

1) The interpolation properties: $P_{21}^L F = F$, on the four vertices of the square $V_1, V_2, V_3$ and $V_4$.

Let $S_{21}^L$ be the Boolean sum of the operators $L_2$ and $L_1$, i.e., $S_{21}^L = L_2 \oplus L_1 = L_2 + L_1 - L_2L_1$.

We have

\[
(S_{21}^L(x, y) = \frac{h-y}{h} F(x, 0) + \frac{y}{h} F(x, h) + \frac{g(y)-x}{g(y)} F(0, y) + \frac{x}{g(y)} F(g(y), y) - \frac{(h-x)(h-y)}{h^2} F(0, 0) - \frac{x(h-y)}{h^2} F(h, 0) - \frac{y(h-x)}{h^2} F(0, h) - \frac{xy}{h^2} F(h, h)
\]

(3)

1) The interpolation properties: $S_{21}^L F = F$ on $\partial D_h$.
2) The degree of exactness: $dex(S_{21}^L) = 1$. 

Figure 3: The interpolation domain for $P_{21}$
3 Hermite-type operators

Suppose that the real valued function $F$ is defined on the square $D_h$ and it possesses the partial derivatives $F^{(1,0)}$ on the side $\Gamma_4$ and $F^{(0,1)}$ on $\Gamma_3$.

We consider the operators $H_1$ and $H_2$ defined by

$$
(H_1 F)(x, y) = \frac{(x - g(y))^2}{g^2(y)} F(0, y) + \frac{x[2g(y) - x]}{g^2(y)} F'(g(y), y) \\
+ \frac{x[x - g(y)]}{g(y)} F^{(1,0)}(g(y), y), \\
(H_2 F)(x, y) = \frac{(y - h)^2}{h^2} F(x, 0) + \frac{y(2h - y)}{h^2} F(x, h) \\
+ \frac{y(y - h)}{h} F^{(0,1)}(x, h)
$$

(4)

1) The interpolation properties:

$$
(H_1 F) = F, \text{ on } \Gamma_2 \cup \Gamma_4 \quad (H_1 F)^{(1,0)} = F^{(1,0)}, \text{ on } \Gamma_4 \\
(H_2 F) = F, \text{ on } \Gamma_1 \cup \Gamma_3 \quad (H_2 F)^{(0,1)} = F^{(0,1)}, \text{ on } \Gamma_3.
$$

![Figure 4: The interpolation domain for $H_1$ and $H_2$](image)

2) The degree of exactness: $\text{dex}(H_1) = \text{dex}(H_2) = 2$

The product of the operators $H_1$ and $H_2$ is given by
\[(F^H_{12}(x, y) = \frac{x - g(y)}{g^2(y)} F(0, y) + \frac{x[2g(y) - x]}{g^2(y)} F(g(y), y) + \frac{y(2h - y)}{h^2} F(0, 0) + \frac{y(2h - y)}{h^2} F(0, h) + \]
\[+ \frac{y(y - h)}{h} F^{(0,1)}(0, h), + \frac{y(y - h)}{h} F^{(0,1)}(g(y), h) \]
\[+ \frac{x[x - g(y)]}{g(y)} \left[ \frac{(y - h)^2}{h^2} F(0, 0) + \frac{y(2h - y)}{h^2} F(0, h) + \frac{y(2h - y)}{h^2} F(g(y), h) + \frac{y(y - h)}{h} F^{(0,1)}(g(y), h) \right] \]
\[+ \frac{x[x - g(y)]}{g(y)} \left[ \frac{(y - h)^2}{h^2} F(g(y), 0) + \frac{y(2h - y)}{h^2} F(g(y), 0) + \frac{y(y - h)}{h} F^{(0,1)}(g(y), h) \right] \]
\[+ \frac{x[x - g(y)]}{g(y)} \left[ \frac{(y - h)^2}{h^2} F^{(1,0)}(g(y), 0) + \frac{y(2h - y)}{h^2} F^{(1,0)}(g(y), h) + \frac{y(y - h)}{h} F^{(1,1)}(g(y), h) \right] \]

with the degree of exactness \(dx(F^H_{12}) = 2\)

The Boolean sum of the operators \(H_1\) and \(H_2\) is given by

\[(S^H_{12}(x, y) = \frac{x - g(y)}{g^2(y)} F(0, y) + \frac{x[2g(y) - x]}{g^2(y)} F(g(y), y) + \frac{y(2h - y)}{h^2} F(0, 0) + \frac{y(2h - y)}{h^2} F(0, h) + \]
\[+ \frac{y(y - h)}{h} F^{(0,1)}(0, h), + \frac{y(y - h)}{h} F^{(0,1)}(g(y), h) \]
\[+ \frac{x[x - g(y)]}{g(y)} \left[ \frac{(y - h)^2}{h^2} F(0, 0) + \frac{y(2h - y)}{h^2} F(0, h) + \frac{y(2h - y)}{h^2} F(g(y), h) + \frac{y(y - h)}{h} F^{(0,1)}(g(y), h) \right] \]
\[+ \frac{x[x - g(y)]}{g(y)} \left[ \frac{(y - h)^2}{h^2} F(g(y), 0) + \frac{y(2h - y)}{h^2} F(g(y), 0) + \frac{y(y - h)}{h} F^{(0,1)}(g(y), h) \right] \]
\[+ \frac{x[x - g(y)]}{g(y)} \left[ \frac{(y - h)^2}{h^2} F^{(1,0)}(g(y), 0) + \frac{y(2h - y)}{h^2} F^{(1,0)}(g(y), h) + \frac{y(y - h)}{h} F^{(1,1)}(g(y), h) \right] \]

with the degree of exactness \(dx(S^H_{12}) = 2\)

### 4 Birhoff-type operators

We suppose that the function \(F : D_h \rightarrow \mathbb{R}\) has the partial derivatives \(F^{(1,0)}\) and \(F^{(0,1)}\) on the sides \(\Gamma_4\) and respectively \(\Gamma_3\).

We consider the Birhoff-type operators \(B_1\) and \(B_2\) defined respectively by
\[(B_1 F)(x,y) = F(0,y) + x F^{(1,0)}(g(y),y),\]
\[(B_2 F)(x,y) = F(x,0) + y F^{(0,1)}(x,h),\]

1) The interpolation properties:

\[B_1 F = F \text{ on } \Gamma_2 \text{ and } (B_1 F)^{(1,0)} = F^{(1,0)} \text{ on } \Gamma_4,\]
\[B_2 F = F \text{ on } \Gamma_1 \text{ and } (B_2 F)^{(0,1)} = F^{(0,1)} \text{ on } \Gamma_3.\]

Figure 5: The interpolation domain for \(B_1\) and \(B_2\)

2) The degree of exactness: \(dex(B_1) = dex(B_2) = 1\)

References


Interpolation operators on a square with one curved side


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