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Successive approximation and variational iteration method boundary value problem with retarded argument

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Abstract

In this paper, conclusions obtained from this two method are compared with by using successive approximations and variational iteration method for approximate solution of a second-order linear differential equation with retarded argument.

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Key words and phrases: Fredholm-Volterra integral equations, Successive approximation method, Variational iteration method.

1 Introduction

Integral equations are equations in which the unknown function appears inside a definite integral. They are closely related to differential equations. Initial value problems and boundary value problems for ordinary and partial differential equations can be written as integral equations and some integral equations can be written as initial or boundary value problems for differential equations. Problems that can be cast in both forms are generally more familiar as differential equations, owing to the larger collection of analytical procedures for solving differential equations.

Many applications are best modeled with integral equations, but most of these problems require a lengthy derivation. A relatively simple example is the model for population dynamics, with birth and death rates that depend on age.

Integral equations are also important in theory and numerical analysis of differential equations [13].

One of methods used in obtaining analytical solution for the boundary value problems is the integral equation method [17]-[15]. Thanks to this method, it is acquired an integral equation that is equivalent to the boundary value problem (1) and the solution of the integral equation is known as equivalent to the solution of the boundary value problem. As equivalent integral equation is used usually a

Fredholm equation in the classical theory, we obtain a Fredholm-Volterra integral equation different from classical theory for the problem (1) and the approximation solutions is obtained by using this equation by means of Green function.

The Fredholm operator included in the equivalent integral equation is an operator with a degenerated kernel. We applied the ordinary successive approximation and variational iteration method for problem (1).

In this study, these methods can be applied for the boundary value problems with retarded argument. The problem (1) has been studied for $\tau(t) = 0$ in [14], $\tau(t) =$ constant in [11]. Furthermore, problem (1) has been studied with varied boundary conditions [3]. We investigated the solution of boundary value problem (1) for arbitrary continuous function $\tau(t)$.

In addition, in this paper, we used also variational iteration method (VIM) to find the approximate solution of boundary value problems with retarded argument. The variational iteration method is a new method for solving linear and nonlinear problems and was introduced by a Chinese mathematician, He J. H.([7]-[8]-[9]). He modified the general Lagrange multiplier method [10] and constructed an iterative sequence of functions which converges to the exact solution. In most linear problem the Lagrange multiplier, the approximate solution turns into the exact solution and is available with just iterations [18].

2 Statement of the Problem

In this section, we will give a boundary value problem with retarded argument as follows:

$$(1) \qquad \begin{gathered} x''(t) + a(t)\,x(t-\tau(t)) = f(t), \quad (0 < t < T) \\ x(t) = \varphi(t), \ \ (\lambda_0 \le t \le 0), \ x(T) = x_T \end{gathered}$$

where $a(t) \ge 0, f(t) \ge 0, \tau(t) \ge 0$,$(0 \le t \le T)$ and $\varphi(t), (\lambda_0 \le t \le 0)$ are known previously continuous functions.

On the one hand we find an equivalent integral equation in order to apply for successive approximation method, on the other hand it is not necessary to an equivalent integral equation for variational iteration method.

2.1 An equivalent integral equation

In the problem (1), when we take as $\lambda(t) = t - \tau(t)$ then $t \in [0, T]$ is a point located at the left side of T such that conditions $\lambda(t_0) = 0$ and $\lambda(t) \le 0, 0 \le t \le t_0$ are satisfied, where $\lambda_0 = \min_{0 \le t \le t_0} \lambda(t)$. We suppose that $\lambda(t)$ is a nondecreasing function in the interval $[t_0, T]$ and the equation $\lambda(t) = \sigma$ has a differentiable continuous solution $t = \gamma(\sigma)$ for arbitrary $\sigma \in [0, \lambda(t)]$. It can be seen that if $x(t)$ is a solution of the boundary value problem (1) then $x(t)$ is also the solution of the equation

$$(2) \ \ x(t) = h^*(t) + \frac{t}{T}\int_0^T (T-s)\,a(s)\,x(s-\tau(s))\,ds - \int_0^t (t-s)\,a(s)\,x(s-\tau(s))\,ds$$

Here,

$$h^{*}(t)=\varphi(0)+\left(x_{T}-\varphi(0)\right)\frac{t}{T}-\frac{t}{T}\int_{0}^{T}(T-s)\,f(s)\,ds-\int_{0}^{t}(t-s)\,f(s)\,ds$$

Let $\sigma=s-\tau(s)$. Therefore equation (2) can be written as follows:

$$x(t)=H(t)+\frac{t}{T}\int_{0}^{\lambda(T)}[T-\gamma(\sigma)]\,a(\gamma(\sigma))\,x(\sigma)\,\gamma'(\sigma)\,d\sigma - \int_{0}^{\lambda(t)}[t-\gamma(\sigma)]\,a(\gamma(\sigma))\,x(\sigma)\,\gamma'(\sigma)\,d\sigma \tag{3}$$

where

$$H(t)=h^{*}(t)+\frac{t}{T}\int_{\lambda_0}^{0}[T-\gamma(\sigma)]\,a(\gamma(\sigma))\,x(\sigma)\,\gamma'(\sigma)\,d\sigma - \int_{\lambda_0}^{0}[t-\gamma(\sigma)]\,a(\gamma(\sigma))\,x(\sigma)\,\gamma'(\sigma)\,d\sigma. \tag{4}$$

Let

$$\begin{aligned} K(\sigma) &= [T-\gamma(\sigma)]\,a(\gamma(\sigma))\,\gamma'(\sigma), \\ K(t,\sigma) &= -[t-\gamma(\sigma)]\,a(\gamma(\sigma))\,\gamma'(\sigma). \end{aligned}$$

Therefore we write

$$x(t)=H(t)+\frac{t}{T}\int_{0}^{\lambda(T)}K(\sigma)\,x(\sigma)\,d\sigma+\int_{0}^{\lambda(t)}K(t,\sigma)\,x(\sigma)\,d\sigma \tag{5}$$

or

$$x(t)=H(t)+\frac{t}{T}F_{\lambda}x+V_{\lambda}x \tag{6}$$

where,

$$F_{\lambda}x\equiv\int_{0}^{\lambda(T)}K(\sigma)\,x(\sigma)\,d\sigma$$

is the Fredholm operator,

$$V_{\lambda}x\equiv\int_{0}^{\lambda(t)}K(t,\sigma)\,x(\sigma)\,d\sigma$$

is the Volterra operator. Equation (6) is called Fredholm-Volterra integral equation and equivalent to the problem (1).

2.2 The ordinary successive approximation method

In this section we know the existence and the uniqueness of solution for the problem (1). However, we will also know that the solution of the problem (1) converges to the solution of the ordinary successive approximations,

$$x_n(t) = h(t) + \int_0^{\lambda(T)} G(t, \gamma(\sigma))\, a(\gamma(\sigma))\, \gamma^{'}(\sigma)\, x_{n-1}(\sigma)\, d\sigma,\ n = (1, 2, ..) \tag{7}$$

for the arbitrary continuous function $x_0(t)$, $(0 \leq t \leq T)$ where

$$G(t, s) = \begin{cases} \frac{(T-t)s}{T}, & 0 \leq s \leq t \\ \frac{(T-s)t}{T}, & t \leq s \leq T \end{cases}$$

is Green function. The function $G(t, s)$, $(0 \leq s \leq t)$ is positive, symmetric and continuous. It is hold following conditions for this function.

$$|G(t, s)| \leq \frac{T}{4}, \quad \int_0^T |G(\tau, s)|\, d\tau \leq \frac{T^2}{8}$$

We use the Fredholm integral equation in order to show the existence and the uniqueness of solution for the problem (1).

Lemma 1 *We assume that E is a Banach space and $A : E \to E$ a contraction mapping. Then the equation*

$$x = A(x) \tag{8}$$

has a unique solution x^* and the ordinary approximations $\{x_n\}$ which are defined by

$$x_n = A(x_{n-1}),\ n = (1, 2, ..) \tag{9}$$

and these approximations converge to x^* where the first approximation is $x_0 \in E$.

Theorem 1 *Suppose that $a = a(t)$ is a continuous function in the interval $(0 \leq t \leq T)$ and*

$$l = \frac{T^2}{8} \|a\| < 1$$

Therefore problem (1) has a unique solution and the approximation (7) converge to the solution of the problem (1) and the speed of the convergence is determined by

$$x_n - x \leq l^n \leq x_0 - x.$$

3 Variational Iteration Method

We consider the following term with retarded to explain the basic concepts of Variational iteration method:

$$Lu(t) + N[u(t), u(\xi(t))] = f(t) \tag{10}$$

Here, L linear operator, N nonlinear operator, $\xi(t)$ term with retarded and $f(t)$ nonhomogeneous term.

General Lagrange multiplier method is claimed by (10). He converted general Lagrange multiplier method to correction function written as follows:

$$u_{k+1}(t) = u_k(t) + \int_0^t \lambda(s)\left[Lu_k(s) + N(\tilde{u}_k(t), \tilde{u}_k(\xi(t))) - f(s)\right] ds \tag{11}$$

Here λ general Lagrange multiplier, k order of approximation and $\tilde{u}_k$ constrained variation, that is $\delta u_k = 0$(7),(8),(9).

Lagrange multipliers can be obtained exactly and easily for the linear problems. But it is not easy to get them nonlinear problems. In VIM (variational iteration method), $\tilde{u}_k$ nonlinear term are considered as constrained variation, a concept benefited from variational theory enabling to determination Lagrange multiplier.

In this study, $\tilde{u}_k(\xi(t))$ .Lagrange multiplier can be stated easily this assumption. So using the equality,

$$u(t) = \lim_{t\to\infty} u_k(t)$$

it can be arrived to the approximate solution successfully.

4 Findings

Example 1 *Let us consider the boundary value problem:*

$$\left.\begin{array}{c} x''(t) + tx\left(t - \frac{1}{2}\sqrt{t}\right) = 2t^3 - 2t^{5/2} - \frac{1}{2}t^2 - \frac{1}{2}t^{3/2} + 4 \\ x(t) = 0, \quad (-1/16 \le t \le 0), \quad x(1) = 1 \end{array}\right\} \tag{12}$$

This equation can be written as the Fredholm-Volterra integral equation

$$\begin{aligned} x_n(t) &= -0.8742063492t + 2t^2 - 0.05714285714t^{7/2} - 0.044166666667t^4 \\ &-0.1269841270t^{9/2} + 0.1t^5 + \frac{t}{8}\int_0^{1/2}\left[3 + 4\sigma - 16\sigma^2 + \frac{3 + 28\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right] x_{n-1}(\sigma)\, d\sigma \\ &-\frac{1}{16}\int_0^{t-\sqrt{t}/2}\left[(4t-1) + (16t-12)\sigma - 16\sigma^2 + \frac{(4t-1) + (48t-20)\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right] x_{n-1}(\sigma)\, d\sigma \end{aligned} \tag{13}$$

Let

$$h(t) = -0.8742063492t + 2t^2 - 0.05714285714t^{7/2} - 0.044166666667t^4$$

$$-0.1269841270t^{9/2}+0.1t^{5},$$

$$K(\sigma)=\frac{1}{8}\left(3+4\sigma-16\sigma^{2}+\frac{3+28\sigma-80\sigma^{2}}{\sqrt{1+16\sigma}}\right),$$

$$K(t,\sigma)=-\frac{1}{16}\left((4t-1)+(16t-12)\sigma-16\sigma^{2}+\frac{(4t-1)+(48t-20)\sigma-80\sigma^{2}}{\sqrt{1+16\sigma}}\right)$$

and

$$F_{\lambda}^{T}x \equiv \int_{0}^{1/2} K(\sigma)\,x(\sigma)\,d\sigma,$$

$$V_{\lambda}^{T}x \equiv \int_{0}^{t-\sqrt{t}/2} K(t,\sigma)\,x(\sigma)\,d\sigma.$$

Therefore, the integral equation (13) can be written as

$$x(t)=h(t)+tF_{\lambda}^{T}x+V_{\lambda}^{T}x$$

and this equation is equivalent to problem (12). Some values of the solution of this equation are obtained by using the ordinary successive approximations of order second which are given in Table 1, where the first approximation is $x_0(t) = -0.8742063492t$. *Now, to solve the problem (12) with variational itertion method, we consider the correction functional written in the form of*

$$x_{k+1}(t)=x_{k}(t)+\int_{0}^{t}\lambda(s)\left[x_{k}^{''}(s)+t\tilde{x}(s-\tau(s))-f(s)\right]ds \tag{14}$$

to solve the equation (12), where λ *general Lagrange multiplier,* $\tilde{x}(s-\tau(s))$ *constraint variation, i.e.* $\delta\tilde{x}(s-\tau(s))=0$ *and*

$$f(s)=2s^{3}-2s^{5/2}-\frac{1}{2}s^{2}-\frac{1}{2}s^{3/2}+4 \tag{15}$$

$$\delta x_{k+1}(t)=\delta x_{k}(t)+\delta\int_{0}^{t}\lambda(s)\left[x_{k}^{''}(s)\right]ds \tag{16}$$

It is obtained stationary conditions from (16) as follows,

$$\begin{aligned}
\delta x_{k}(t) &: \quad 1-\lambda^{'}(t)\,|_{s=t}=0,\\
\delta x_{k}(t) &: \quad \lambda(t)\,|_{s=t}=0,\\
\delta x_{k}(t) &: \quad \lambda^{''}(t)\,|_{s=t}=0.
\end{aligned}$$

General Lagrange multiplier is found from these equations,

$$\lambda\left(s\right)=s-t. \tag{17}$$

We get iteration formula

$$x_{k+1}\left(t\right)=x_{k}\left(t\right)+\int_{0}^{t}\left(s-t\right)\left[x_{k}^{\shortparallel}\left(s\right)+tx_{k}\left(s-\sqrt{s}/2\right)-f\left(s\right)\right]ds \tag{18}$$

writing Lagrange multiplier in (14).It can be reached to the approximate solution to desired order using appropriate initial function chosen. Here we choose $x_0 = -0.2t^2 - 0.9t$ *and we solved the iteration formula to second order. So, we have*

$$\begin{aligned}&x_2\left(t\right)=2t^2-0.9t+0.2998184382t^4-0.0215550132t^5-0.1396825397t^{9/2}\\&-0.2171428572t^{7/2}-0.3076157001t^6+0.01613127795t^9-0.005300539828t^{11/2}\\&-0.008897370511t^{13/2}+0.003134507231t^{10}-0.004320066827t^{17/2}\\&+0.01938208824t^{15/2}-0.01222748951t^{19/2}-0.01274651145t^8+0.2742694031t^7.\end{aligned}$$

(1)

t_i	$x\left(t_i\right)$	$x_2^1\left(t_i\right)$	$x_2^2\left(t_i\right)$	$\varepsilon^1\left(t_i\right)$	$\varepsilon^2\left(t_i\right)$
0.0	*0.000000*	*-0.000135*	*0.000000*	*0.000135*	*0.000000*
0.2	*-0.120000*	*-0.097001*	*-0.120392*	*0.022998*	*0.000392*
0.4	*-0.080000*	*-0.037704*	*-0.084735*	*0.042295*	*0.004735*
0.6	*0.120000*	*0.166049*	*0.101490*	*0.046049*	*0.018509*
1.0	*1.000000*	*1.007731*	*0.885868*	*0.007731*	*0.114131*

Table 1. *Values at some point in the interval* $[0,1]$ *.*

$x\left(t_i\right)$ *is exact solution,* $x_2^1\left(t_i\right)$ *is the ordinary successive approximations of order second for problem (1).* $x_2^2\left(t_i\right)$ *is approximations of order second of variational iteration method for problem (1).* $\varepsilon^1\left(t_i\right)$ *is error value of the ordinary successive approximations of order second.* $\varepsilon^2\left(t_i\right)$ *is error value of approximations of order second of variational iteration method. In 1, it has chosen as*

$$\begin{aligned}\varepsilon^1\left(t_i\right)&=\left|\,x\left(t_i\right)-x_2^1\left(t_i\right)\right|,\ i=1,2,3,4,5,\\\varepsilon^2\left(t_i\right)&=\left|\,x\left(t_i\right)-x_2^2\left(t_i\right)\right|eps,\ i=1,2,3,4,5.\end{aligned}$$

5 Conclusion

The fundamental aim of this paper has been to apply convenient two approximation method to the solution of differential equation with retarded argument. To find solution of boundary value problem (1), it is obtained an integral equation equivalent to the boundary value problem (1). In this paper, we obtained a Fredholm-Volterra

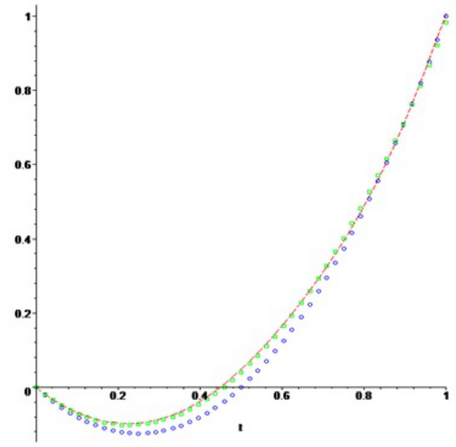

Figure 1: The convergence of $U_{n,\alpha}(f;x)$ to $f(x)$

integral equation for the problem (1). After we obtain integral equation, we used ordinary successive approximation method. However, we used variational iteration method over boundary value problem (1). With these methods, we find successive approximate solutions for problem (1). From 1, it is seen that values calculated for problem (1) are coincide with exact solution. Our aim has been compare with solutions obtained by carrying out ordinary successive approximation and variational iteration method to problem (1). From 1, it can be seen easily that variational iteration method gives good conclusion respect to ordinary successive approximation method. The computations associated with the example mentioned above were carried out by using Maple.

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On some Bernstein-type operators

Adriana Rusu

Abstract

The aim of this paper is to present an alternative proof for the theorems given by Z. Finta in [2], which states that there exists no sequence L_n of generalized Bernstein-type operators that have e_i and e_j as fixed points for any $n = j, j+1, \ldots$, but there exist an infinity of such operators that have e_i and e_j as fixed points, where j is fixed.

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1 Introduction

We consider the following sequence of operators

$$L_n : C[0,1] \to C[0,1], (L_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x)\lambda_{n,k}(f) \tag{1}$$

where $p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}$, and $\lambda_{n,k}$, $k = \overline{0,n}$, $n \in \mathbb{N}$ are positive linear functionals defined on $C[0,1]$. The operators given by (1) are Bernstein-type operators. For $\lambda_{n,k}(f) = f\left(\dfrac{k}{n}\right)$ we get the classical Bernstein operators:

$$B_n : C[0,1] \to C[0,1], \qquad B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right).$$

Approximating functions of type (1) operators was studied in [3], by I. Gavrea and D. H. Mache.

In [2], Z. Finta proved the existence of type (1) operators, that have as fixed points the monomials e_i, $e_i(x) = x^i$, $x \in [0,1]$, $i = \overline{0,1}$.

For $\lambda_{n,k}(f) = f\left(\left(\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}\right)^{\frac{1}{j}}\right)$, $k=\overline{0,n}$, $1 \le j \le n$, $n \in \mathbb{N}$, from (1) we get the following operator

$$U_n(f)(x) = \sum_{k=0}^{n} p_{n,k} f\left(\left(\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}\right)^{\frac{1}{j}}\right) \tag{2}$$

The U_n operators, $n \in \mathbb{N}$ were introduced in [1], Proposition 11.

In [1] it was shown that

$$U_n(e_0) = e_0$$
$$U_n(e_j) = e_j$$
$$\lim_{n\to\infty} \|U_n f - f\| = 0, \quad f \in C[0,1].$$

Z. Finta proved the following results in [2].

Theorem 1 *Let $i, j \in \{1, 2, \dots\}$, $i < j$ be given. There exists no sequence $\{L_n\}$ of type (1) operators such that e_i and e_j are fixed points for L_n for any $n = j, j+1, \dots$.*

Theorem 2 *There is an infinity of L_n type (1) operators sequences such that*

$$\lim_{n\to\infty} \|L_n f - f\| = 0, \quad f \in C[0,1]$$

and L_n has e_i and e_j as fixed points, where j is fixed.

The previous theorem was proved by Z. Finta in [2] using a Hahn-Banach type theorem. The aim of this paper is to give another proof for Theorem 1 and Theorem 2.

2 Main Results

The following is an alternative proof of Theorem 1.

We assume that for $i < j$ we have

$$(L_n e_i)(x) = x^i \tag{3}$$

$$(L_n e_j)(x) = x^j \tag{4}$$

where $n \ge j$.

Relation (3) can be written as follows:

$$\sum_{k=0}^{n} p_{n,k}(x)\lambda_{n,k}(e_i) = x^i$$

from where we deduce that

$$\lambda_{n,0}(e_i) = \lambda_{n,1}(e_i) = \ldots = \lambda_{n,i-1}(e_i) = 0$$

(5) $$\lambda_{n,i} = \frac{i!}{n(n-1)\ldots(n-i+1)}$$

From (4) we get

(6) $$\lambda_{n,0}(e_j) = \lambda_{n,1}(e_j) = \ldots = \lambda_{n,j-1}(e_j) = 0$$

Using the Cauchy-Buniakowsky-Schwarz inequality for positive linear functionals we have

$$\lambda_{n,i}(e_i) = \lambda_{n,i}(e_{i-1}^{\frac{1}{2}} \cdot e_{i+1}^{\frac{1}{2}}) \leq \sqrt{\lambda_{n,i}(e_{i-1}) \cdot \lambda_{n,i}(e_i + 1)}$$

or

(7) $$\lambda_{n,i}^2(e_i) \leq \lambda_{n,i}(e_{i-1}) \cdot \lambda_{n,i}(e_{i+1}).$$

In the same way, we have

(8) $$\begin{gathered} \lambda_{n,i}^2(e_{i+1}) \leq \lambda_{n,i}(e_i) \cdot \lambda_{n,i}(e_{i+2}) \\ \ldots \\ \lambda_{n,i}^2(e_{j-2}) \leq \lambda_{n,i}(e_{j-3}) \cdot \lambda_{n,i}(e_{j-1}) \end{gathered}$$

From (7) and (8) we get:

(9) $$\lambda_{n,i}^{2^{j-i}}(e_i) \leq \lambda_{n,i}^{2^{j-i-1}}(e_{i-1}) \cdot \lambda_{n,i}^{2^{j-i-2}}(e_i) \cdot \ldots \cdot \lambda_{n,i}^2(e_{i-3}) \cdot \lambda_{n,i}(e_{j-1})$$

From (6) and (9) we get

$$\lambda_{n,i}(e_i) = 0,$$

which contradicts (5).

In the following theorem, we emphasize a type (1) class operators that have 1 and e_j as fixed points.

Theorem 3 *Let $(a_n)_{n\in\mathbb{N}}$, $a_n \in (0,1)$ be a sequence of positive numbers such that $\lim\limits_{n\to\infty} a_n = 0$. The sequence of operators defined by*

(10) $$\begin{aligned} L_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \Bigg[&(1-a_n) f\left(\left(\frac{k(k-1)\cdots(k-j+1)}{n(n-1)\cdots(n-j+1)}\right)^{\frac{1}{j}}\right) \\ &+ a_n\left(1 - \frac{k(k-1)\cdot(k-j+1)}{n(n-1)\cdots(n-j+1)}\right) f(0) + a_n \frac{k(k-1)\cdots(k-j+1)}{n(n-1)\cdots(n-j+1)} f(1) \Bigg] \end{aligned}$$

has the following properties:

$$\begin{aligned} L_n e_0 &= e_0 \\ L_n e_j &= e_j \end{aligned} \tag{11}$$

and

$$\lim_{n\to\infty} \|L_n f - f\| = 0, \qquad f \in C[0,1]. \tag{12}$$

Proof. We notice that the operator $(L_n f)(x)$ can be written in the following way

$$(L_n f)(x) = (1-a_n)(U_n f)(x) + a_n[(1-x^j)f(0) + x^j f(1)] \tag{13}$$

The relation (13) results from (2) and from the following equality

$$x^j = \sum_{k=0}^{n} \frac{k(k-1)\cdots(k-j+1)}{n(n-1)\cdots(n-j+1)} p_{n,k}(x)$$

Because $U_n e_0 = e_0$ and $U_n e_j = e_j$, from (12) we have

$$L_n e_o = e_0$$

$$L_n e_j = e_j$$

To prove (12), we see that

$$\|L_n f - f\| \le \|U_n f - f\| + 2a_n \|f\|$$

from where the proof of the theorem results.

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Preserving properties and estimation of coefficients for functions that belong to a subclass of analytic functions $M^*(\alpha, \beta, \gamma, A, \lambda)$

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Abstract

In this paper we present preserving properties and estimation of coefficients for functions that belong to a subclass of analytic functions $M^*(\alpha, \beta, \gamma, A, \lambda)$.

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Key words and phrases: Alexander type integral operator, Bernardi type integral operator, $I_{c+\delta}$ integral operator, L_a type operator.

1 Introduction

Let S denote the class of normalised analytic univalent function f defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

for $z \in U = \{z : |z| < 1\}$.

Let T denote the subclass of S consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{2}$$

Definition 1 *[2] Let I_A be a Alexander type integral operator defined as:*

$$I_A : A \to A,\ I_A(F) = f,\ \textit{where}$$

$$f(z) = \int_0^z \frac{F(t)}{t} dt. \tag{3}$$

Definition 2 *[1] Let I_a be a Bernardi type integral operator defined as:*

$$I_a : A \to A,\ I_a(F) = f,\ a = 1, 2, 3, \ldots,\ \textit{where}$$

$$f(z) = \tfrac{a+1}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{4}$$

Definition 3 *[1] Let L_a be a generalization of the previously integral operator defined as:*

$$L_a : A \to A,\ L_a(F) = f,\ a \in \mathbb{C}, \mathrm{Re\,a} \geq 0,\ \textit{where}$$

$$f(z) = \tfrac{a+1}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{5}$$

Definition 4 *[2] Let $I_{c+\delta}$ be the integral operator defined as: $I_{c+\delta} : A \to A,\ \ 0 < u \leq 1\,,\ 1 \leq \delta < \infty\,,\ 0 < c < \infty\,,$*

$$f(z) = I_{c+\delta}(F)(z) = (c+\delta) \int_0^1 u^{c+\delta-2} F(uz) du. \tag{6}$$

Remark 1 *[2] For $\delta = 1$ and c=1,2,..., from the integral operator $I_{c+\delta}$ we obtain the Bernardi integral operator defined by (4).*

Definition 5 *[2] Let $F \in A$, $F(z) = z + b_2 z^2 + \cdots + b_n z^n + \ldots$, and $a \in \mathbb{R}^*$. We define the integral operator $L : A \to A$ by*

$$f(z) = L(F)(z) = \frac{1+a}{z^a} \int_0^z F(t)(t^{a-1} + t^{a+1}) dt. \tag{7}$$

2 Preliminary results

Further, we define the class $M(\alpha, \beta, \gamma, A, \lambda)$ as follows:

Definition 6 *[3] A function f given by (1) is said to be a member of the class $M(\alpha, \beta, \gamma, A, \lambda)$ if it satisfies*

$$\left| \frac{zf'(z) - f(z)}{\alpha z f'(z) - Af(z) - (1-\lambda)(1-A)\gamma f(z)} \right| < \beta,$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $-1 \leq A < 1$, $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$ for all $z \in U$.

Let us write

$$M^*(\alpha, \beta, \gamma, A, \lambda) = T \cap M(\alpha, \beta, \gamma, A, \lambda). \tag{8}$$

Theorem 1 *[3] If $f \in S$ satisfies*

(9)
$$\sum_{n=2}^{\infty}(n-1+\beta(n\alpha-A-(1-\lambda)(1-A)\gamma))|a_n| \leq \beta(\alpha-A-(1-\lambda)(1-A)\gamma),$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $-1 \leq A < 1$, $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, then $f \in M(\alpha,\beta,\gamma,A,\lambda)$.

Theorem 2 *[3] Let the function f be defined by (1) and let $f \in T$. Then $f \in M^*(\alpha,\beta,\gamma,A,\lambda)$, if and only if (9) is satisfied. The result (9) is sharp.*

Corollary 1 *[3] Let the function f be defined by (1) and let $f \in M^*(\alpha,\beta,\gamma,A,\lambda)$, then*

(10)
$$a_n \leq \frac{\beta(\alpha-A-(1-\lambda)(1-A)\gamma)}{(n-1+\beta(n\alpha-A-(1-\lambda)(1-A)\gamma))},\ n \geq 2.$$

3 Main results

Theorem 3 *The Alexander type integral operator defined by (3) preserves the class $M^*(\alpha,\beta,\gamma,A,\lambda)$, that is: If $F \in M^*(\alpha,\beta,\gamma,A,\lambda)$, then $f(z) = I_AF(z) \in M^*(\alpha,\beta,\gamma,A,\lambda)$, for $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$.*

Proof. Let $F \subset T$, $F(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0$. Then

$$f(z) = I_AF(z) = \int_0^z \frac{F(t)}{t}dt$$

$$= \int_0^z \frac{1}{t}(t - \sum_{n=2}^{\infty} a_n\, t^n)dt$$

$$= z - \sum_{n=2}^{\infty} \frac{a_n}{n} z^n = z - \sum_{n=2}^{\infty} c_n z^n,\ \text{with}$$

$c_n = \frac{a_n}{n} \geq 0$, $n \geq 2$. It follows that $f \in T$. We have now to prove that $f \in M^*(\alpha,\beta,\gamma,A,\lambda)$. Using Theorem 2 we need to prove that:

(11)
$$\sum_{n=2}^{\infty}(n-1+\beta(n\alpha-A-(1-\lambda)(1-A)\gamma))|c_n| \leq \beta(\alpha-A-(1-\lambda)(1-A)\gamma),$$

for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $-1 \leq A < 1$, $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$. This means:

(12)
$$\sum_{n=2}^{\infty}(n-1+\beta(n\alpha-A-(1-\lambda)(1-A)\gamma))\frac{|a_n|}{n} \leq \beta(\alpha-A-(1-\lambda)(1-A)\gamma).$$

But we have $\frac{|a_n|}{n} \leq |a_n|,\ for\ n \geq 2,$ and by using (9) and (12), we observe that inequality (11) is fulfilled.This means that $f \in M^*(\alpha,\beta,\gamma,A,\lambda)$.

Theorem 4 *The integral operator $I_{c+\delta}$ defined by (6) preserves the class $M^*(\alpha,\beta,\gamma,A,\lambda)$, that is: If $F \in M^*(\alpha,\beta,\gamma,A,\lambda)$, then $f(z) = I_{c+\delta}(F)(z) \in M^*(\alpha,\beta,\gamma,A,\lambda)$, for $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$.*

Proof. Let $F \in M^*(\alpha,\beta,\gamma,A,\lambda)$, $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$.

We have, from Theorem 2:

(13)
$$\sum_{n=2}^{\infty}(n-1+\beta(n\alpha - A - (1-\lambda)(1-A)\gamma))|a_n| \leq \beta(\alpha - A - (1-\lambda)(1-A)\gamma).$$

From (6) we obtain $f(z) = I_{c+\delta}(F)(z) = z - \sum_{n=2}^{\infty} \frac{c+\delta}{c+n+\delta-1} a_n z^n$, *where* $0 < c < \infty$, $1 \leq \delta < \infty$. We also remark that for $0 < c < \infty$, $n \geq 2$ and $1 \leq \delta < \infty$, we have

(14)
$$0 < \frac{c+\delta}{c+n+\delta-1} < 1.$$

Thus $f \in T$ and by using Theorem 2 we have only to prove that.

(15)
$$\sum_{n=2}^{\infty}(n-1+\beta(n\alpha - A - (1-\lambda)(1-A)\gamma))\frac{c+\delta}{c+n+\delta-1}|a_n| \leq \beta(\alpha - A - (1-\lambda)(1-A)\gamma).$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $-1 \leq A < 1$, $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$ and $0 < c < \infty$ and $1 \leq \delta < \infty$, $n \geq 2$.

By using the relation (14) we have

$$\frac{c+\delta}{c+n+\delta-1} \cdot |a_n| < |a_n|,$$

for $0 < c < \infty$, $n \geq 2$, $1 \leq \delta < \infty$, and thus from (15) we conclude that the condition (13) take place and thus the proof it is complete.

The following theorem is proved similarly (see Remark 1):

Theorem 5 *The Bernardi type integral operator defined by (4) preserves the class $M^*(\alpha,\beta,\gamma,A,\lambda)$, that is: If $F \in M^*(\alpha,\beta,\gamma,A,\lambda)$, then $f(z) = I_a F(z) \in M^*(\alpha,\beta,\gamma,A,\lambda)$, for $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$.*

Theorem 6 *Let $F \in M^*(\alpha,\beta,\gamma,A,\lambda)$ with $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $-1 \leq A < 1$, $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$. For $f(z) = L_a(F)(z)$, $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, $z \in U$, where the integral operator L_a it is defined by (5), we have:*

$$|a_n| \leq \left| \frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n-1+\beta(n\alpha - A - (1-\lambda)(1-A)\gamma))} \cdot \frac{a+1}{a+n} \right|, \ n \geq 2.$$

Proof. For $f = L_a(F)(z)$ with $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$ and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ we have

$$a_n = b_n \cdot \frac{a+1}{a+n},$$

where $a \in \mathbb{C}$, $\mathrm{Re\,a} \geq 0$, $\mathrm{n} \geq 2$.
The coefficient bounds for the functions belonging to the class $M^*(\alpha, \beta, \gamma, A, \lambda)$ are

$$b_n \leq \frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n-1+\beta(n\alpha - A - (1-\lambda)(1-A)\gamma))},\ n \geq 2.$$

For $n \geq 2$ we obtain

$$|a_n| = |b_n| \cdot \left|\frac{a+1}{a+n}\right| \leq$$

$$\leq \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n-1+\beta(n\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+n}\right| =$$

$$= \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n-1+\beta(n\alpha - A - (1-\lambda)(1-A)\gamma))} \cdot \frac{a+1}{a+n}\right|.$$

Hence the theorem is proved.

Theorem 7 *Let* $F \in M^*(\alpha, \beta, \gamma, A, \lambda)$ *with* $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $-1 \leq A < 1$, $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$. *For* $f(z) = L(F)(z)$, $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$, $z \in U$, *where the integral operator* L *it is defined by (7), we have:*

$$|a_2| \leq \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(1+\beta(2\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+2}\right|,$$

$$|a_3| \leq \left[\left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(2+\beta(3\alpha - A - (1-\lambda)(1-A)\gamma))}\right| + 1\right] \cdot \left|\frac{a+1}{a+3}\right|,$$

$$|a_n| \leq \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n-1+\beta(n\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+n}\right|$$

$$+ \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n-3+\beta((n-2)\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+n}\right|.$$

Proof. For $f = L(F)(z)$ with $F(z) = z - \sum_{n=2}^{\infty} b_n z^n$ and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ we have:

$$a_2 = b_2 \cdot \frac{a+1}{a+2},$$

$$a_3 = (b_3 + 1) \cdot \frac{a+1}{a+3},$$

$$a_n = (b_n + b_{n-2}) \cdot \frac{a+1}{a+n},$$

where $a \in \mathbb{R}^*,\ n \geq 4$.
The coefficient bounds for the functions belonging to the class $M^*(\alpha, \beta, \gamma, A, \lambda)$ are:

$$b_n \leq \frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n - 1 + \beta(n\alpha - A - (1-\lambda)(1-A)\gamma))},\ n \geq 2.$$

For $n \geq 4$ we obtain:

$$|a_n| = |b_n + b_{n-2}| \cdot \left|\frac{a+1}{a+n}\right|$$

$$\leq (|b_n| + |b_{n-2}|) \cdot \left|\frac{a+1}{a+n}\right|$$

$$\leq \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n - 1 + \beta(n\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+n}\right|$$

$$+ \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n - 3 + \beta((n-2)\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+n}\right|.$$

$$|a_n| \leq \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n - 1 + \beta(n\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+n}\right|$$

$$+ \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(n - 3 + \beta((n-2)\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+n}\right|.$$

For $n = 2$ we have:

$$|a_2| = |b_2| \cdot \left|\frac{a+1}{a+2}\right|$$

$$\leq \left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(1 + \beta(2\alpha - A - (1-\lambda)(1-A)\gamma))}\right| \cdot \left|\frac{a+1}{a+2}\right|.$$

Similarly for $n = 3$ we have:

$$|a_3| \leq \left[\left|\frac{\beta(\alpha - A - (1-\lambda)(1-A)\gamma)}{(2 + \beta(3\alpha - A - (1-\lambda)(1-A)\gamma))}\right| + 1\right] \cdot \left|\frac{a+1}{a+3}\right|.$$

Hence the theorem is proved.

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Simultaneous approximation Baskakov Durrmeyer Kantorovich operators

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Abstract

To investigate approximation properties of integrable functions, Durrmeyer and Kantorovich modification of approximation operators are main tool. To combine these modification for Baskakov operators, we recently introduced a generalization of Baskakov operators in [6]. In this paper, we investigate pointwise convergence of derivatives of these operators by the means of Voronovskaya type asymptotic formula. This formula is presented in quantitative form in terms of a modulus of smoothness, which allows us describe the rate of pointwise convergence and the upper bound for the error of approximation simultaneously.

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Key words and phrases: Baskakov Durrmeyer Operators, Baskakov Kantorovich Operatorss, Simultaneous approximation.

1 Introduction

Very recently, in [14], Stan defined a new operator using the structural properties of Durrmeyer and Kantorovich methods for classical Bernstein operators and called Bernstein Durrmeyer Kantorovich operator which is defined by

$$\widetilde{K}_n(f;x) = (n+3)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_0^1\binom{n}{k}t^k(1-t)^{n-k}f(t)\,dt,$$

where $x \in [0,1]$ and $n \in \mathbb{N}$. Thus an approximation process on a bounded interval representing again an integral form in sense of Bernstein operator was obtained. Also some approximation properties of mentioned operator was examined in the continuos functions space and Lebesgue spaces.

Later in [6], to extend Stan's construction to unbounded interval, authors have constructed a new sequence of integral type operators which contain characteristic properties of Baskakov Durrmeyer and Baskakov Kantorovich operators that is

$$\widetilde{B}_n(f;x) = (n-1)\sum_{k=0}^{\infty}p_{n+2,k}(x)\int_0^{\infty}f(u)\,p_{n,k+1}(u)\,du, \qquad (1)$$

where $p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-(n+k)}$, $x \in [0,\infty)$ and $n \in \mathbb{N}$.

These operators are called Baskakov Durrmeyer Kantorovich operators (BDK). It is shown that these operators reproduce constants as well as linear functions. Also, the alternate form of these operators in terms of hypergeometric series were given. For more details of Baskakov Durrmeyer and Baskakov Kantorovich operators and some generalizations we can refer the readers to [1, 3, 5, 13] and references therein.

The aim of the paper is to study a Voronovskaya type asymptotic formula for derivatives of functions by the corresponding order of derivatives of operators. Also we give an error estimate in simultaneous approximation by the BDK operators.

Throughout the paper we consider the functions belong to class of all Lebesgue measurable functions f on $[0,\infty)$, that is,

$$\mathcal{H} \equiv \left\{ f : \int_0^\infty \frac{|f(t)|}{(1+t)^n} dt < \infty, \text{ for some } n \in \mathbb{N} \right\}$$

with the norm $\|\cdot\|_{C_\alpha}$ given by $\|f\|_{C_\alpha} = \sup\limits_{t\in[0,\infty)} \frac{|f(t)|}{t^\alpha}$.

Considering this space, several researchers studied simultaneous approximation properties of some other operators. In this direction we refer [2], [7], [12], [10] and references therein.

2 Representations and moments

In the sequel, we shall need the following results. First we give a collection of some properties of the kernel functions $p_{n,k}$ which can be easily derived from the definition of the operators B_n.

Lemma 1 *[11] For every $n \in N$, $n > 1$, $k \in \mathbb{N} \cup \{0\}$, $x \in [0,\infty)$ we have*

1. $\sum\limits_{k=0}^{\infty} p_{n,k}(x) = 1$
2. $\int\limits_0^\infty p_{n,k}(t)dt = \dfrac{1}{n-1}$
3. $\dfrac{k}{n} p_{n,k}(x) = x p_{n,k-1}(x)$
4. $\varphi(x)^2 \dfrac{d}{dx} p_{n,k}(x) = (k-nx) p_{n,k}(x)$
5. $n\left[p_{n+1,k-1}(x) - p_{n+1,k}(x)\right] = \dfrac{d}{dx} p_{n,k}(x)$.

Lemma 2 *Let f be an r times differentiable function on $[0,\infty)$ such that $f^{(r-1)}(u) = \mathcal{O}(u^\alpha)$ for some $\alpha > 0$ as $u \to \infty$. For $r = 0, 1, 2, \ldots$ and $n > \alpha + r$, we have*

$$\frac{d^r}{dx^r} \widetilde{B}_n(f;x) = (n-1)\,\beta(n,r) \sum_{k=0}^{\infty} p_{n+r+2,k}(x) \int_0^\infty p_{n-r,k+r+1}(u)\, f^{(r)}(u)\, du,$$

where $\beta(n,r) = \prod_{j=0}^{r-1} \frac{(n+j+2)}{(n-j-1)}$ *for* $\beta(n,0) = 1$.

Lemma 3 *Let* $r, s \in \mathbb{N} \cup \{0\}$ *and* $n > r$, *if we define the functions* $\widetilde{B}_{r,n,s}(x)$ *as follows:*

$$\widetilde{B}_{r,n,s} = (n-r-1)\sum_{k=0}^{\infty} p_{n+r+2}(x)\int_0^{\infty} p_{n-r,k+r+1}(t)(t-x)^s dt,$$

then the recurrence relation holds:

$$\begin{aligned}[n-(r+s+2)]\widetilde{B}_{r,n,s+1} &= \varphi^2(x)[\widetilde{B}'_{r,n,s}(x) + 2s\widetilde{B}_{r,n,s-1}] \\ &\quad + (s+r+2)(1+2x)\widetilde{B}_{r,n,s}(x),\end{aligned} \tag{2}$$

where $\varphi(x) = \sqrt{x(1+x)}$ *and* $n > (r+s+2)$. *Consequently,*

$$\widetilde{B}_{r,n,0} = 1$$

$$\widetilde{B}_{r,n,1} = \frac{(r+2)(1+2x)}{n-(r+2)}$$

$$\widetilde{B}_{r,n,2} = \frac{\varphi^2(x)2n + (1+2x)^2(r+2)(r+3)}{[n-(r+2)]\,[n-(r+3)]}.$$

For all $x \in [0,\infty)$, $\widetilde{B}_{r,n,s}(x) = \mathcal{O}\left((n+2)^{-\lfloor \frac{s+1}{2} \rfloor}\right)$, *where* $\lfloor \alpha \rfloor$ *denotes the integer part of* α.

Proof. Firstly, let us prove the (2). It is clear that

$$\begin{aligned}\varphi^2(x)\widetilde{B}'_{r,n,s}(x) &= (n-r-1)\sum_{k=0}^{\infty}\varphi^2(x)p'_{n+r+2,k}(x)\int_0^{\infty}(t-x)^s p_{n-r,k+r+1}(t)dt \\ &\quad -s\varphi^2(x)\widetilde{B}_{r,n,s-1}(x).\end{aligned}$$

Using Lemma 1-(4), we get

$$\begin{aligned}&\varphi^2(x)\widetilde{B}'_{r,n,s}(x) \\ = &\;(n-r-1)\sum_{k=0}^{\infty}[(k-(n+r+2)x]p_{n+r+2,k}(x)\int_0^{\infty}(t-x)^s p_{n-r,k+r+1}(t)dt \\ &-s\varphi^2(x)\widetilde{B}_{r,n,s-1}(x) \\ = &\;(n-r-1)\sum_{k=0}^{\infty}p_{n+r+2,k}(x)\int_0^{\infty}[k+r+1-[(n-r)t]-(r+1)(1+2x) \\ &+(n-r)(t-x)](t-x)^s p_{n-r,k+r+1}(t)dt - s\varphi^2(x)\widetilde{B}_{r,n,s-1}(x)\end{aligned}$$

$$\begin{aligned} &= (n-r-1)\sum_{k=0}^{\infty} p_{n+r+2,k}(x)\int_0^{\infty}[k+r+1-[(n-r)t](t-x)^s p_{n-r,k+r+1}(t)dt \\ &\quad -(n-r-1)\sum_{k=0}^{\infty} p_{n+r+2,k}(x)(r+1)(1+2x)\int_0^{\infty}(t-x)^s p_{n-r,k+r+1}(t)dt \\ &\quad +(n-r-1)\sum_{k=0}^{\infty} p_{n+r+2,k}(x)\int_0^{\infty}(n-r)(t-x)(t-x)^s p_{n-r,k+r+1}(t)dt \\ &\quad -s\varphi^2(x)\widetilde{B}_{r,n,s-1}(x)\end{aligned}$$

$$\begin{aligned} &= (n-r-1)\sum_{k=0}^{\infty} p_{n+r+2,k}(x)\int_0^{\infty}\varphi^2(t)p'_{n-r,k+r+1}(t)(t-x)^s dt \\ &\quad -(r+1)(1+2x)\widetilde{B}_{r,n,s}(x)+(n-r)\widetilde{B}_{r,n,s+1}-s\varphi^2(x)\widetilde{B}_{r,n,s-1}(x).\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}\varphi^2(x)\widetilde{B}'_{r,n,s}(x) &= \sum_{k=0}^{\infty} p_{n+r+2,k}(x)\int_0^{\infty} p_{n-r,k+r+1}(t)[-(1+2t)(t-x)^s - s\varphi^2(t)(t-x)^{s-1}]dt \\ &\quad -(r+1)(1+2x)\widetilde{B}_{r,n,s}(x)+(n-r)\widetilde{B}_{r,n,s+1}-s\varphi^2(x)\widetilde{B}_{r,n,s-1}(x). \qquad (3)\end{aligned}$$

Since

$$-s\varphi^2(t)-(1+2t)(t-x) = -s\varphi^2(x)-(s+1)(1+2x)(t-x)-(s+2)(t-x)^2,$$

we obtain from (3)

$$\begin{aligned}\varphi^2(x)\widetilde{B}'_{r,n,s}(x) &= -s\varphi^2(x)\widetilde{B}_{r,n,s-1}-(s+1)(1+2x)\widetilde{B}_{r,n,s}-(s+2)\widetilde{B}_{r,n,s+1} \\ &\quad -(r+1)(1+2x)\widetilde{B}_{r,n,s}(x)+(n-r)\widetilde{B}_{r,n,s+1}-s\varphi^2(x)\widetilde{B}_{r,n,s-1}(x),\end{aligned}$$

which is desired. The moments can be easily seen from the recurrence formula.

Lemma 4 *[2] For each $x \in (0,\infty)$ and $r \in \mathbb{N}\cup\{0\}$, there exist polynomials $q_{i,j,r}(x)$ in x independent of n and k such that*

$$(x(1+cx))^r \frac{d^r}{dx^r}p_{n,k}(x) = \sum_{2i+j\le r, i,j\ge 0}(n)^i((k-nx))^j q_{i,j,r}(x)\,p_{n,k}(x),$$

where $\varphi(x)=\sqrt{x(1+cx)}$, $p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$.

3 Main results

This section deals with the main results, we study Voronovskaya type asymptotic formula and give an error estimate in simultaneous approximation. Now we state our main results as follows:

Theorem 1 *Let $f \in \mathcal{H}$ be bounded on every finite subinterval of $[0,\infty)$ admitting a derivative of order $(r+2)$ at a fixed point $x \in (0,\infty)$. Let $f(u) = \mathcal{O}(u^{\alpha})$ as $u \to \infty$ for some $0 < \alpha$, then we have*

$$\lim_{n\to\infty} n\left\{\frac{(n-(r+1))}{(n-1)\beta(n,r)}(\widetilde{B}_n^{(r)}f)(x) - f^{(r)}(x)\right\} = (r+2)(1+2x)f^{(r+1)}(x) + \varphi^2(x) f^{(r+2)}(x).$$

Proof. By finite Taylor's expansion of f, we have

$$f(u) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(u-x)^i + \varepsilon(u,x)(u-x)^{r+2},$$

where $\varepsilon(u,x) \to 0$ as $u \to x$. Using Lemma 2 we can have

$$\begin{aligned} n\left\{\frac{(n-(r+1))}{(n-1)\beta(n,r)}(\widetilde{B}_n^{(r)}f)(x) - f^{(r)}(x)\right\} &= n(n-(r+1))\sum_{k=0}^{\infty} p_{n+r+2,k}(x) \\ &\quad \times \int_0^{\infty} p_{n-r,k+r+1}(u)\left[f^{(r)}(u) - f^{(r)}(x)\right] du \end{aligned}$$

$$\begin{aligned} &= n(n-(r+1))\sum_{k=0}^{\infty} p_{n+r+2,k}(x) \times \int_0^{\infty} p_{n-r,k+r+1}(u)\Big[f^{(r+1)}(x)(u-x) \\ &\quad + \frac{f^{(r+2)}(x)}{2!}(u-x)^2 + \frac{d^r}{du^r}\left[\varepsilon(u,x)(u-x)^{r+2}\right]\Big] du \end{aligned}$$

$$\begin{aligned} &= f^{(r+1)}(x)\, n\widetilde{B}_{r,n,1} + \frac{f^{(r+2)}(x)}{2!} n\widetilde{B}_{r,n,2} \\ &\quad + n(n-(r+1))\sum_{k=0}^{\infty} p_{n+r+2,k}(x)\int_0^{\infty} p_{n-r,k+r+1}(u)\frac{d^r}{du^r}\left[\varepsilon(u,x)(u-x)^{r+2}\right] du \\ &= f^{(r+1)}(x)\, n\widetilde{B}_{r,n,1} + \frac{f^{(r+2)}(x)}{2!} n\widetilde{B}_{r,n,2} + I_n, \end{aligned}$$

where

$$I_n = \frac{n(n-(r+1))}{\beta(n,r)}\sum_{k=0}^{\infty} p_{n+2,k}^{(r)}(x)\int_0^{\infty} p_{n,k+1}(u)\,\varepsilon(u,x)(u-x)^{r+2} du.$$

In order to prove the theorem it is sufficient the show that $I_n \to 0$ as $n \to \infty$. Using Lemma (4) we can get

$$\begin{aligned} |I_n| &\leq \frac{n\,(n-(r+1))}{\beta\,(n,r)} \sum_{k=0}^{\infty} \sum_{2i+j\leq r, i,j\geq 0} (n+2)^i \,|k-x(n+2)|^j \,\frac{|q_{i,j,r}\,(x)|}{(x\,(1+x))^r} p_{n+2,k}\,(x) \\ &\quad \times \int_0^{\infty} p_{n,k+1}\,(u)\,|\varepsilon\,(u,x)|\,|u-x|^{r+2}\,du \\ &\leq C\frac{n\,(n-(r+1))}{\beta\,(n,r)} \sum_{2i+j\leq r, i,j\geq 0} (n+2)^i \sum_{k=0}^{\infty} |k-x(n+2)|^j \,p_{n+2,k}\,(x) \\ &\quad \times \int_0^{\infty} p_{n,k+1}\,(u)\,|\varepsilon\,(u,x)|\,|u-x|^{r+2}\,du \\ &\leq C\frac{n\,(n-(r+1))}{\beta\,(n,r)} \sum_{2i+j\leq r, i,j\geq 0} (n+2)^i \left(\sum_{k=0}^{\infty} p_{n+2,k}\,(x)\,|k-x(n+2)|^{2j}\right)^{\frac{1}{2}} \\ &\quad \times \left[\sum_{k=0}^{\infty} p_{n+2,k}\,(x) \left(p_{n,k+1}\,(u)\,|\varepsilon\,(y,x)|\,|u-x|^{r+2}\,du\right)^2\right]^{\frac{1}{2}} \\ &\leq C\frac{n\,(n-(r+1))}{\beta\,(n,r)} O\left(n^{\frac{r}{2}}\right) \left[\sum_{k=0}^{\infty} p_{n+2,k}\,(x) \left(\int_0^{\infty} p_{n,k+1}\,(u)\,|\varepsilon\,(u,x)|\,|u-x|^{r+2}\,du\right)^2\right]^{\frac{1}{2}}, \end{aligned}$$

where $C = C\,(x) = \sup\limits_{2i+j\leq r, i,j\geq 0} \frac{|q_{i,j,r}(x)|}{(x(1+x))^r}$. For a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\varepsilon\,(u,x)| < \varepsilon$ whenever $0 < |u-x| < \delta$. For $|u-x| \geq \delta$, we have $|\varepsilon\,(u,x)| \leq K\,|u-x|^s$ for any $s \geq 0$. Therefore, we write

$$\begin{aligned} &\left(\int_0^{\infty} p_{n,k+1}\,(u)\,|\varepsilon\,(u,x)|\,|u-x|^{r+2}\,du\right)^2 \\ &\leq \int_0^{\infty} p_{n,k+1}\,(u)\,dy \int_0^{\infty} p_{n,k+1}\,(y)\,(\varepsilon\,(u,x))^2\,(u-x)^{2r+4}\,du \\ &= \frac{1}{(n-1)} \int_0^{\infty} p_{n,k+1}\,(y)\,(\varepsilon\,(u,x))^2\,(u-x)^{2r+4}\,du \\ &= \frac{1}{(n-1)} \left(\int_{|y-x|<\delta} p_{n,k+1}\,(u)\,(\varepsilon\,(u,x))^2\,(u-x)^{2r+4}\,du \right. \end{aligned}$$

$$+\int\limits_{|y-x|\geq\delta} p_{n,k+1}(u)\left(\varepsilon(u,x)\right)^2(y-x)^{2r+4}\,du\Bigg).$$

Thus, using Lemma 3, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} p_{n+2,k}(x)\left(\int\limits_0^{\infty} p_{n,k+1}(u)\left|\varepsilon(u,x)\right|\left|u-x\right|^{r+2}du\right)^2\\
\leq\ & \frac{(n-1)}{(n-1)^2}\int\limits_0^{\infty} p_{n,k+1}(u)\left(\varepsilon(u,x)\right)^2(u-x)^{2r+4}\,du\\
&+\frac{K^2(n-1)}{(n-1)^2}\int\limits_{|u-x|\geq\delta} p_{n,k+1}(u)(u-x)^{2s+2r+4}\\
=\ & (\varepsilon(y,x))^2\,O\left([n]^{-(r+4)}\right)+K^2O\left([n]^{-(r+s+4)}\right)\\
=\ & (\varepsilon(y,x))^2\,O\left([n]^{-(r+4)}\right)+O\left([n]^{-(r+s+4)}\right).
\end{aligned}$$

Hence by using Lemma 3 we get

$$\begin{aligned}
|I_n| &\leq C\frac{n(n-(r+1))}{\beta(n,r)}[n]^{-\left(\frac{r}{2}\right)}\left(\varepsilon(u,x)\right)^2O\left([n]^{-(r+4)}\right)^{\frac{1}{2}}+o(1)\\
&\leq \varepsilon+o(1)\ \text{ choosing } s>0.
\end{aligned}$$

Since ε is arbitrary, this implies that $I_n\to 0$ as $n\to\infty$.

Theorem 2 *Let $f\in\mathcal{H}$ be bounded on every finite subinterval of $[0,\infty)$ and $f(u)=\mathcal{O}(u^{\alpha})$ as $u\to\infty$ for some $0<\alpha\leq s$. If $f^{(r+1)}$ exist and is continuous on $(a-\delta,b+\delta)\subset[0,\infty)$, $\delta>0$, then for sufficiently large n,*

$$\begin{aligned}
\left\|\frac{d^r}{dx^r}(\widetilde{B}_nf)(x)-f^{(r)}(x)\right\| &\leq C_1n^{-1}\left(\left\|f^{(r)}\right\|+\left\|f^{(r+1)}\right\|\right)\\
&\quad+C_2n^{-\frac{1}{2}}\omega\left(f^{(r+1)},n^{-\frac{1}{2}}\right)+O\left(n^{\frac{r-s}{2}}\right)
\end{aligned}$$

holds for any $s>1$, where C_1 and C_2 are constants independent of f and n, and $\|.\|$ is sup-norm on $[a,b]$.

Proof. By finite Taylor's expansion of f we have

$$\begin{aligned}
f(u) &= \sum_{i=0}^{r+1}\frac{f^{(i)}(x)}{i!}(u-x)^i+\frac{\left\{f^{(r+1)}(\zeta)-f^{(r+1)}(x)\right\}}{(r+1)!}(u-x)^{r+1}\chi(u)\\
&\quad+h(u,x)(1-\chi(u)),
\end{aligned}$$

where ζ lies between u and x and $\chi(u)$ is the characteristic function of $(a-\delta, b+\delta)$.

For $u \in (a-\delta, b+\delta)$ and $x \in [a,b]$ we have

$$f(u) = \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!}(u-x)^i + \frac{\left\{f^{(r+1)}(\zeta) - f^{(r+1)}(x)\right\}}{(r+1)!}(u-x)^{r+1}.$$

For $u \in [0,\infty) \setminus (a-\delta, b+\delta)$ and $x \in [a,b]$ we define

$$h(u,x) = f(u) - \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!}(u-x)^i.$$

Now, we can write

$$\begin{aligned}
(\widetilde{B}_n^{(r)} f)(x) - f^{(r)}(x) &= (n-1)\,\beta(n,r) \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!} \sum_{k=0}^{\infty} p_{n+r+2,k}(x) \\
&\quad \times \int_0^\infty p_{n-r,k+r+1}(u) \frac{d^r}{du^r}(u-x)^i \, du - f^{(r)}(x) \\
&\quad + (n-1)\,\beta(n,r) \sum_{k=0}^{\infty} p_{n+r+2,k}(x) \int_0^\infty p_{n-r,k+r+1}(u) \\
&\quad \times \Bigg[\frac{\left\{f^{(r+1)}(\zeta) - f^{(r+1)}(x)\right\}}{(r+1)!}(u-x)^{r+1}\chi(u) \\
&\quad + h(u,x)\,(1-\chi(u)) \Bigg]^{(r)} du \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= (n-1)\,\beta(n,r) \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!} \sum_{k=0}^{\infty} p_{n+r+2,k}(x) \\
&\quad \times \int_0^\infty p_{n-r,k+r+1}(u) \frac{d^r}{du^r}(u-x)^i \, du - f^{(r)}(x) \\
I_2 &= (n-1) \sum_{k=0}^{\infty} \frac{d^r}{dx^r} p_{n+2,k}(x) \\
&\quad \times \int_0^\infty p_{n,k+1}(u) \frac{\left\{f^{(r+1)}(\zeta) - f^{(r+1)}(x)\right\}}{(r+1)!}(u-x)^{r+1}\chi(u)\, du \\
I_3 &= (n-1) \sum_{k=0}^{\infty} \frac{d^r}{dx^r} p_{n+2,k}(x) \int_0^\infty p_{n,k+1}(u)\, h(u,x)\,(1-\chi(u))\, du.
\end{aligned}$$

Using Lemma 3, we have

$$I_1 = f^{(r)}(x) \left\{ \frac{(n-1)\,\beta(n,r)}{n-(r+1)} \widetilde{B}_{r,n,0}(x) - 1 \right\} + \frac{(n-1)\,\beta(n,r)}{n-(r+1)} \widetilde{B}_{r,n,1}(x)\, f^{(r+1)}(x).$$

Now, using Lemma 4, we write

$$
\begin{aligned}
|I_2| &\leq (n-1) \sum_{2i+j\leq r, i,j\geq 0} (n+2)^i \sum_{k=0}^{\infty} p_{n+2,k}(x) \left|k-(n+2)x\right|^j \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r} \\
&\quad \times \int_0^{\infty} p_{n,k+1}(u) \frac{\left|f^{(r+1)}(\zeta) - f^{(r+1)}(x)\right|}{(r+1)!} \left|(u-x)\right|^{r+1} du \\
&\leq C(n-1) \sum_{2i+j\leq r, i,j\geq 0} (n+2)^i \sum_{k=0}^{\infty} p_{n+2,k}(x) \left|k-(n+2)x\right|^j \\
&\quad \times \int_0^{\infty} p_{n,k+1}(u) \left(1+\frac{|u-x|}{\delta}\right) \omega\left(f^{(r+1)},\delta\right) |u-x|^{r+1} du,
\end{aligned}
$$

for all $\delta > 0$, where C is a constant. Thus we obtain

$$
\begin{aligned}
|I_2| &= C(n-1)\omega\left(f^{(r+1)},\delta\right) \sum_{2i+j\leq r, i,j\geq 0} (n+2)^i \sum_{k=0}^{\infty} p_{n+2,k}(x) \left|k-(n+2)x\right|^j \\
&\quad \times \int_0^{\infty} p_{n,k+1}(u) \left(|u-x|^{r+1} + \frac{|u-x|}{\delta}^{r+2}\right) du.
\end{aligned}
$$

Applying Schwarz's inequality for integration and then for summation, in a similar way as in the proof of Theorem 1, we deduce

$$
\begin{aligned}
|I_2| &\leq C\omega\left(f^{(r+1)},\delta\right) \sum_{2i+j\leq r, i,j\geq 0} (n+2)^i \left(\sum_{k=0}^{\infty} p_{n+2,k}(x) \left(k-(n+2)x\right)^{2j}\right)^{1/2} \\
&\quad \times \left((n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x) \int_0^{\infty} p_{n,k+1}(u) |u-x|^{2r+2} du\right)^{\frac{1}{2}} \\
&\quad + \frac{1}{\delta} C\omega\left(f^{(r+1)},\delta\right) \sum_{2i+j\leq r, i,j\geq 0} (n+2)^i \left(\sum_{k=0}^{\infty} p_{n+2,k}(x) \left(k-(n+2)x\right)^{2j}\right)^{1/2} \\
&\quad \times \left((n-1)\sum_{k=0}^{\infty} p_{n+2,k}(x) \int_0^{\infty} p_{n,k+1}(u) |u-x|^{2r+4} du\right)^{\frac{1}{2}} \\
&\leq C\omega\left(f^{(r+1)},\delta\right) \left\{n^{-1/2} + \frac{1}{\delta} n^{-1}\right\} \\
&= Cn^{-1/2}\omega\left(f^{(r+1)},\delta\right).
\end{aligned}
$$

Thus, choosing $\delta = n^{-1/2}$ we have

$$
|I_2| \leq Cn^{-1/2}\omega\left(f^{(r+1)}, n^{-1/2}\right).
$$

Since $h(u,x) = O(u-x)^s$, for any $s \in \mathbb{N}$ with $s \geq \alpha$ we have

$$\begin{aligned} |I_3| &\leq K(n-1)\sum_{k=0}^{\infty}\sum_{2i+j\leq r,i,j\geq 0}(n+2)^i(k-(n+2)x)^j p_{n+2,k}(x) \\ &\times \int\limits_{|u-x|\geq\delta} p_{n,k+1}(u)|u-x|^s du \end{aligned}$$

where K is a constant. In a similar way as in I_2, we have

$$|I_3| \leq Kn^{-1/2}.$$

Choosing $s > r+1$, we obtain the limit $I_3 \to 0$ as $n \to \infty$.

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Dirichlet boundary value problem for a n$^{\text{th}}$ order complex partial differential equation

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Abstract

In this work, we investigate the solvability condition of the problem

$$\partial_{\bar{z}}^n w + c\partial_z \partial_{\bar{z}}^{n-1} w = f(z),\ f \in L_p(D, \mathbb{C}),\ p > 2,\ n = 1, 2, ...,$$
$$\partial_{\bar{z}}^{k-1} w|_{\partial D} = \gamma_k,\ \gamma_k \in C(\partial D; \mathbb{C}),\ 0 \leq k \leq n-1$$

in the unit disc of complex plane, for $|c| < 1$. Moreover, under this condition, we get the unique solution of the problem is given in explicit form.

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1 Introduction

In [1] and [2], Schwarz, Dirichlet and Neumann boundary value problems for the Beltrami equation $w_{\bar{z}} + cw_z = f$ with constant coefficient in the unit disc were investigated. Moreover, in [6] Vekua searched the Beltrami equation in the theory of quasi-conformal mappings. In additionly, in [1] it was given that the solvability of Schwarz and Dirichlet problems for the operators

$$w_{z\bar{z}} + cw_{\overline{zz}} = f,\ |c| < 1.$$

In this paper, we describe the solvability conditions and solutions of the Dirichlet problem for the following complex partial differential equation in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$$\partial_{\bar{z}}^n w + c\partial_z \partial_{\bar{z}}^{n-1} w = f(z), f \in L_p(\mathbb{D}, \mathbb{C}), p > 2, n = 1, 2, ..., \tag{1}$$

$$\partial_{\overline{z}}^k w|_{\partial D} = \gamma_k \ , 0 \leq k \leq n-1, \ \gamma_k \in C(\partial\mathbb{D}, \mathbb{C}), \tag{2}$$

where the complex partial differential operators ∂_z and $\partial_{\overline{z}}$ are defined by

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y), \quad z = x + iy, \ x, y \in \mathbb{R}.$$

2 Integral Representations

The fundemental tools for solving boundary value problems for complex partial differential equations are Gauss theorem and the Cauchy-Pompeiu representation formula. Let D be a regular domain of the complex plane $\mathbb{C}$, i.e. bounded domain with smooth boundary ∂D.

Theorem 1 *(Gauss theorem, complex form)[3] Let D be a regular domain of $\mathbb{C}$, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D}, \mathbb{C})$, then*

$$\int_D w_{\overline{z}}(z)dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz$$

and

$$\int_D w_z(z)dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\overline{z}.$$

A complex-valued function w which is independent of $\overline{z}$, i.e., satisfying the differential equation in open domain D in $\mathbb{C}$

$$w_{\overline{z}} = 0$$

is called an analytic function in D. For analytic functions the Cauchy theorem is valid [5].

Theorem 2 *(Cauchy theorem) Let γ be a simple closed smooth curve and D be the inner domain, bounded by γ. If w is analytic function in D, continuous in $\overline{D}$, then*

$$\int_\gamma w(z)dz = 0$$

holds.

The following representation of an analytic function can be obtained from the Cauchy theorem

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z}, z \in D.$$

One can deduced from the Gauss theorem the following representation formulas:

Theorem 3 ***(Cauchy- Pompeiu representations)*** *[3] Let D be a regular domain of $\mathbb{C}$, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D},\mathbb{C})$, $\zeta = \xi + i\eta$. Then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} \tag{3}$$

and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_{D} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} \tag{4}$$

hold for all $z \in D$.

3 The Dirichlet Boundary Value Problems

Theorem 4 *[1] The Dirichlet problem in the unit disc*

$$g_{\overline{z}} + c g_z = f, \quad g|_{\partial\mathbb{D}} = \gamma$$

for $f \in L^p(\overline{\mathbb{D}}), p > 2$, and $\gamma \in C(\partial\mathbb{D};\mathbb{C})$ is solvable iff

$$\begin{aligned} &\frac{1}{2\pi i}\int_{|\zeta|=1} \gamma(\zeta)\frac{2 + c\overline{z}\overline{\zeta}}{1 + c\overline{z}\overline{\zeta}}\frac{\overline{z}}{1 - \overline{z}\zeta}d\zeta \\ &\quad = \sum_{k=0}^{\infty}(-1)^k c^k \frac{1}{\pi}\int_{|\zeta|<1} f(\zeta)(\overline{\zeta - z})^k \frac{\overline{z}^{k+1}}{(1 - \overline{z}\zeta)^{k+1}}d\xi d\eta, \end{aligned} \tag{5}$$

and the solution is

$$g(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\zeta - z - c(\overline{\zeta - z})} - \frac{1}{\pi}\int_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{\zeta - z - c(\overline{\zeta - z})}. \tag{6}$$

Remark 1 *(5) solvability condition can be written as*

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \gamma(\zeta)\frac{2 + c\overline{z}\overline{\zeta}}{1 + c\overline{z}\overline{\zeta}}\frac{\overline{z}}{1 - \overline{z}\zeta}d\zeta = \frac{\overline{z}}{\pi}\int_{|\zeta|<1} \frac{f(\zeta)}{1 - \overline{z}\zeta + c\overline{z}(\overline{\zeta - z})}d\xi d\eta.$$

Theorem 5 *[4] The Dirichlet problem for the inhomogeneous polyanalytic equation in the unit disc*

$$\partial_{\overline{z}}^{n-1} w = g(z) \text{ in } \mathbb{D},\ \partial_{\overline{z}}^{k} w|_{\partial D} = \gamma_k \text{ on } \partial\mathbb{D},\ 0 \le k \le n-2,$$

is uniquely solvable for $g \in L_1(\mathbb{D};\mathbb{C}), \gamma_k \in C(\partial\mathbb{D},\mathbb{C}), 0 \leq k \leq n-2$ *if and only if for* $0 \leq k \leq n-2$

$$\sum_{\lambda=k}^{n-2} \frac{\bar{z}}{2\pi i} \int\limits_{|\zeta|=1} (-1)^{\lambda-k} \frac{\gamma_\lambda(\zeta)}{1-\overline{z}\zeta} \frac{(\overline{\zeta-z})^{\lambda-k}}{(\lambda-k)!} d\zeta + \frac{(-1)^{n-k-1}}{\pi} \bar{z} \int\limits_{|\zeta|<1} \frac{g(\zeta)}{1-\overline{z}\zeta} \frac{(\overline{\zeta-z})^{n-k-2}}{(n-k-2)!} d\xi d\eta = 0 \tag{7}$$

and the solution then is

$$w(z) = \sum_{k=0}^{n-2} \frac{(-1)^k}{2\pi i} \frac{1}{k!} \int\limits_{|\zeta|=1} \gamma_k(\zeta) \frac{(\overline{\zeta-z})^k}{\zeta-z} d\zeta + \frac{(-1)^{n-1}}{\pi} \frac{1}{(n-2)!} \int\limits_{|\zeta|<1} g(\zeta) \frac{(\overline{\zeta-z})^{n-2}}{\zeta-z} d\xi d\eta. \tag{8}$$

4 Main Results

In (1), introducing the new function

$$\partial_{\bar{z}}^{n-1} w = g, g \in L_1(\mathbb{D};\mathbb{C}), \tag{9}$$

the boundary problem is reduced to the following Dirichlet problem

$$g_{\bar{z}} + c g_z = f; f \in L_p(\mathbb{D},\mathbb{C}), p > 2 \tag{10}$$

$$g|_{\partial\mathbb{D}} = \gamma_{n-1} \tag{11}$$

which is Dirichlet boundary value problem for the Beltrami equation.

If $g(z)$ is plugged into in (1), we come across

$$\partial_{\bar{z}}^{n-1} w = g(z), g \in L_1(\mathbb{D};\mathbb{C}) \tag{12}$$

$$\partial_{\bar{z}}^{k} w|_{\partial D} = \gamma_k, \ 0 \leq k \leq n-2, \gamma_k \in C(\partial\mathbb{D},\mathbb{C}) \tag{13}$$

Dirichlet boundary value problem.

So, theorems in previous section will apply. By rewriting $g(\zeta)$ in (7), and changing order integration, we can get

$$\begin{aligned}(14)\qquad &\sum_{\lambda=k}^{n-2}\frac{\bar z}{2\pi i}\int\limits_{|\zeta|=1}(-1)^{\lambda-k}\frac{\gamma_\lambda(\zeta)}{1-\overline{z}\zeta}\frac{(\overline{\zeta-z})^{\lambda-k}}{(\lambda-k)!}d\zeta\\
&+\frac{(-1)^{n-k-1}}{2\pi i}\bar z\int\limits_{|t|=1}\gamma_{n-1}(t)\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-k-2}}{(n-k-2)!(1-\overline{z}\zeta)}\frac{d\xi d\eta}{t-\zeta}dt\\
&+\frac{(-1)^{n-k-1}}{2\pi i}\bar z\int\limits_{|t|=1}\gamma_{n-1}(t)\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-k-2}}{(n-k-2)!}\frac{c(\overline{t-\zeta})}{t-\zeta-c(\overline{t-\zeta})}\frac{d\xi d\eta}{(1-\overline{z}\zeta)}dt\\
&-\frac{(-1)^{n-k-1}}{\pi}\bar z\int\limits_{|t|<1}f(t)\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-k-2}}{(n-k-2)!}\frac{1}{t-\zeta-c(\overline{t-\zeta})}\frac{d\xi d\eta}{1-\overline{z}\zeta}dt_1dt_2\\
&=0.\end{aligned}$$

Evaluting with Cauchy-Pompeiu representations for second term of (14), one can get

$$\begin{aligned}(15)\qquad &\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-k-2}}{(n-k-2)!(1-\overline{z}\zeta)}\frac{d\xi d\eta}{t-\zeta}\\
&=\frac{(\overline{t-z})^{n-k-1}}{(n-k-1)!(1-\overline{z}t)}-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{\zeta-z})^{n-k-1}}{(n-k-1)!(1-\overline{z}\zeta)}\frac{d\zeta}{\zeta-t}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{\zeta-z})^{n-k-1}}{(1-\overline{z}\zeta)}\frac{d\zeta}{\zeta-t} &= -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{\zeta-z})^{n-k-1}}{(\overline{\zeta-z})}\frac{d\overline{\zeta}}{1-t\overline{\zeta}}\\
&= -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{\zeta-z})^{n-k-2}}{(1-t\overline{\zeta})}d\overline{\zeta}=0,\end{aligned}$$

we get

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-k-2}}{(n-k-2)!(1-\overline{z}\zeta)}\frac{d\xi d\eta}{t-\zeta}=\frac{(\overline{t-z})^{n-k-1}}{(n-k-1)!(1-\overline{z}t)}.$$

And similarly, the other term of (14) can be simplified that

$$\begin{aligned}&\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-k-2}}{(n-k-2)!}\frac{c(\overline{t-\zeta})}{t-\zeta-c(\overline{t-\zeta})}\frac{d\xi d\eta}{(1-\overline{z}\zeta)}\\
&=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}c^r}{(n-k-2)!}\frac{(\overline{z})^{r-1}}{(1-\overline{z}t)^r}\sum_{\nu=0}^{r}(-1)^\nu\binom{r}{\nu}\frac{(\overline{t-z})^{n-k+r-1}}{n-k+\nu-1}\end{aligned}$$

and

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-k-2}}{(n-k-2)!}\frac{1}{t-\zeta-c(\overline{t-\zeta})}\frac{d\xi d\eta}{(1-\overline{z}\zeta)}$$
$$=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}c^r}{(n-k-2)!}\frac{(\bar{z})^r}{(1-\bar{z}t)^{r+1}}\sum_{\nu=0}^{r}(-1)^{\nu+1}\binom{r}{\nu}\frac{(\overline{t-z})^{n-k+r-1}}{n-k+\nu-1}.$$

So, the solvability condition (7) can be found as

$$\sum_{\lambda=k}^{n-2}\frac{\bar{z}}{2\pi i}\int\limits_{|\zeta|=1}(-1)^{\lambda-k}\frac{\gamma_k(\zeta)}{1-\overline{z}\zeta}\frac{(\overline{\zeta-z})^{\lambda-k}}{(\lambda-k)!}d\zeta$$
$$+\frac{(-1)^{n-k-1}}{2\pi i}\bar{z}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{(\overline{\zeta-z})^{n-k-1}}{(n-k-1)!(1-\overline{z}\zeta)}d\zeta$$
$$+\frac{(-1)^{n-k-1}}{2\pi i}\bar{z}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\sum_{r=1}^{\infty}\frac{(-1)^{r-1}c^r}{(n-k-2)!}\frac{(\bar{z})^{r-1}}{(1-\bar{z}\zeta)^r}\sum_{\nu=0}^{r}(-1)^{\nu}\binom{r}{\nu}\frac{(\overline{\zeta-z})^{n-k+r-1}}{n-k+\nu-1}d\zeta$$
$$+\frac{(-1)^{n-k-1}}{\pi}\bar{z}\int\limits_{|\zeta|<1}f(\zeta)\sum_{r=1}^{\infty}\frac{(-1)^{r-1}c^r}{(n-k-2)!}\frac{(\bar{z})^r}{(1-\bar{z}\zeta)^{r+1}}\sum_{\nu=0}^{r}(-1)^{\nu+1}\binom{r}{\nu}\frac{(\overline{\zeta-z})^{n-k+r-1}}{n-k+\nu-1}d\xi d\eta$$
$$=0.$$

For the solution, by inserting $g(\zeta)$ into (8) and changing order integration, it gives that

$$\begin{aligned}
w(z) &= \sum_{k=0}^{n-2}\frac{(-1)^k}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma_k(\zeta)}{k!}\frac{(\overline{\zeta-z})^k}{(\zeta-z)}d\zeta \\
&+\frac{(-1)^{n-1}}{2\pi i}\int\limits_{|t|=1}\gamma_{n-1}(t)\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-2}}{(n-2)!(\zeta-z)}\frac{d\xi d\eta}{t-\zeta}dt \\
&+\frac{(-1)^{n-1}}{2\pi i}\int\limits_{|t|=1}\gamma_{n-1}(t)\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-2}}{(n-2)!(\zeta-z)}\frac{c(\overline{t-\zeta})}{t-\zeta-c(\overline{t-\zeta})}d\xi d\eta dt \\
&-\frac{(-1)^{n-1}}{\pi}\int\limits_{|t|<1}f(t)\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-2}}{(n-2)!(\zeta-z)}\frac{d\xi d\eta}{t-\zeta-c(\overline{t-\zeta})}dt_1dt_2.
\end{aligned}$$

Observing that

$$
\begin{aligned}
&-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-2}}{(n-2)!(\zeta-z)}\frac{d\xi d\eta}{\zeta-t}\\
&\quad=-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-2}}{(n-2)!(t-z)}\left(\frac{1}{\zeta-t}-\frac{1}{\zeta-z}\right)d\xi d\eta\\
&\quad=\frac{(\overline{t-z})^{n-1}}{(n-1)!(t-z)}-\frac{1}{2\pi i(n-1)!(t-z)}\frac{1}{\pi}\int\limits_{|\zeta|=1}\left(\frac{(\overline{\zeta-z})^{n-1}}{\zeta-t}-\frac{(\overline{\zeta-z})^{n-1}}{\zeta-z}\right)d\zeta\\
&\quad=\frac{(\overline{t-z})^{n-1}}{(n-1)!(t-z)}+\frac{1}{2\pi i(n-1)!(t-z)}\frac{1}{\pi}\int\limits_{|\zeta|=1}(\overline{\zeta-z})^{n-1}\left(\frac{1}{1-t\overline{\zeta}}-\frac{1}{1-z\overline{\zeta}}\right)\frac{d\zeta}{\overline{\zeta}}\\
&\quad=\frac{(\overline{t-z})^{n-1}}{(n-1)!(t-z)},
\end{aligned}
$$

by some calculations for the other integrals, one can find that

$$
\begin{aligned}
&\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-2}}{(n-2)!(\zeta-z)}\frac{c(\overline{t-\zeta})}{t-\zeta-c(\overline{t-\zeta})}d\xi d\eta\\
&\quad=\frac{1}{(n-2)!}\sum_{r=1}^{\infty}c^r\frac{(\overline{t-z})^{r-1}}{(t-z)^r}\sum_{\nu=0}^{r}\frac{(-1)^\nu}{n+\nu-1}\binom{r}{\nu}
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta-z})^{n-2}}{(n-2)!(\zeta-z)}\frac{d\xi d\eta}{t-\zeta-c(\overline{t-\zeta})}\\
&\quad=\frac{1}{(n-2)!}\sum_{r=0}^{\infty}c^r\frac{(\overline{t-z})^{r-1}}{(t-z)^{r+1}}\sum_{\nu=0}^{r}\frac{(-1)^\nu}{n+\nu-1}\binom{r}{\nu}.
\end{aligned}
$$

Finally, the solution is obtained that

$$\begin{aligned} w(z) &= \sum_{k=0}^{n-2}\frac{(-1)^k}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma_k(\zeta)}{k!}\frac{(\overline{\zeta-z})^k}{(\zeta-z)}d\zeta \\ &+\frac{(-1)^{n-1}}{2\pi i}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{(\overline{\zeta-z})^{n-1}}{(n-1)!(\zeta-z)}d\zeta \\ &+\frac{(-1)^{n-1}}{2\pi i}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{1}{(n-2)!}\sum_{r=1}^{\infty}c^r\frac{(\overline{\zeta-z})^{r-1}}{(\zeta-z)^r}\sum_{\nu=0}^{r}\frac{(-1)^\nu}{n+\nu-1}\binom{r}{\nu}d\zeta \\ &-\frac{(-1)^{n-1}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{1}{(n-2)!}\sum_{r=1}^{\infty}c^r\frac{(\overline{\zeta-z})^{r-1}}{(\zeta-z)^{r+1}}\sum_{\nu=0}^{r}\frac{(-1)^\nu}{n+\nu-1}\binom{r}{\nu}d\xi d\eta. \end{aligned}$$

We proved the following theorem:

Theorem 6 *In the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, the boundary value problem for $|c|<1$*

$$\partial_{\bar z}^n w+c\partial_z\partial_{\bar z}^{n-1}w=f(z),\ f\in L_p(\mathbb{D},\mathbb{C}),\ p>2,\ n=1,2,..., \tag{16}$$

$$\partial_{\bar z}^{k-1}w|_{\partial\mathbb{D}}=\gamma_k,\ \gamma_k\in C(\partial\mathbb{D};\mathbb{C}),\ 0\le k\le n-1 \tag{17}$$

is solvable, if and only if the functions f,γ satisfy that for all $z\in\mathbb{D}$

$$\begin{aligned} &\sum_{\lambda=k}^{n-2}\frac{\bar z}{2\pi i}\int\limits_{|\zeta|=1}(-1)^{\lambda-k}\frac{\gamma_k(\zeta)}{1-\bar z\zeta}\frac{(\overline{\zeta-z})^{\lambda-k}}{(\lambda-k)!}d\zeta \\ &+\frac{(-1)^{n-k-1}}{2\pi i}\bar z\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{(\overline{\zeta-z})^{n-k-1}}{(n-k-1)!(1-\bar z\zeta)}d\zeta \\ &+\frac{(-1)^{n-k-1}}{2\pi i}\bar z\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\sum_{r=1}^{\infty}\frac{(-1)^{r-1}c^r}{(n-k-2)!}\frac{(\bar z)^{r-1}}{(1-\bar z\zeta)^r}\sum_{\nu=0}^{r}(-1)^\nu\binom{r}{\nu}\frac{(\overline{\zeta-z})^{n-k+r-1}}{n-k+\nu-1}d\zeta \\ &+\frac{(-1)^{n-k-1}}{\pi}\bar z\int\limits_{|\zeta|<1}f(\zeta)\sum_{r=1}^{\infty}\frac{(-1)^{r-1}c^r}{(n-k-2)!}\frac{(\bar z)^r}{(1-\bar z\zeta)^{r+1}}\sum_{\nu=0}^{r}(-1)^{\nu+1}\binom{r}{\nu}\frac{(\overline{\zeta-z})^{n-k+r-1}}{n-k+\nu-1}d\xi d\eta \\ &=0 \end{aligned}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{2+c\overline{z\zeta}}{1+c\overline{z\zeta}}\frac{\bar z}{1-\bar z\zeta}d\zeta=\frac{\bar z}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{1-\bar z\zeta+c\bar z(\overline{\zeta-z})}d\xi d\eta.$$

In this case, the unique solution is

$$\begin{aligned} w(z) &= \sum_{k=0}^{n-2} \frac{(-1)^k}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_k(\zeta)}{k!} \frac{(\overline{\zeta - z})^k}{(\zeta - z)} d\zeta \\ &+ \frac{(-1)^{n-1}}{2\pi i} \int_{|\zeta|=1} \gamma_{n-1}(\zeta) \frac{(\overline{\zeta - z})^{n-1}}{(n-1)!(\zeta - z)} d\zeta \\ &+ \frac{(-1)^{n-1}}{2\pi i} \int_{|\zeta|=1} \gamma_{n-1}(\zeta) \frac{1}{(n-2)!} \sum_{r=1}^{\infty} c^r \frac{(\overline{\zeta - z})^{r-1}}{(\zeta - z)^r} \sum_{\nu=0}^{r} \frac{(-1)^\nu}{n+\nu-1} \binom{r}{\nu} d\zeta \\ &- \frac{(-1)^{n-1}}{\pi} \int_{|\zeta|<1} f(\zeta) \frac{1}{(n-2)!} \sum_{r=1}^{\infty} c^r \frac{(\overline{\zeta - z})^{r-1}}{(\zeta - z)^{r+1}} \sum_{\nu=0}^{r} \frac{(-1)^\nu}{n+\nu-1} \binom{r}{\nu} d\xi d\eta. \end{aligned}$$

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On certain subclasses of analytic functions involving a linear operator associated with the Fox-Wright Psi function

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Abstract

The objective of the present paper is to obtain some applications of first order differential subordinations and superordination results involving a linear operator associated with the Fox-Wright psi function and other linear operators for certain normalized analytic functions in the open unit discs.

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1 Introduction

Let H(U) be the class of analytic functions in the unit disk U = { z ∈ C : |z |< 1} and let H [a,k] be the subclass of H (U) consisting of functions presented in the following manner

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} + ... \qquad (a \in C). \tag{1}$$

Also, let A be the subclass of H(U) involving the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

If f and g are member of H(U), then the function f is known as subordinant to g, if there exists a Schwarz function ω, which (by definition) is analytic in U with $|\omega (z)| < 1$ for all z ∈ U, such that f(z) = g (ω (z)). In such case we write $f(z) \prec g(z)$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g. [3], [10] ; see also [11]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \; and \; f(U) \subset g(U).$$

Suppose p and h are member of H(U) and $\phi(r,s,t;z) : C^3 \times U \to C$. If p and ϕ (p(z), zp'(z), z^2 p"(z) ; z) are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z),\ zp'(z),\ z^2 p''(z)\ ;\ z), \tag{3}$$

then p is a solution of the differential superordination (3). Note that if f is subordinate to g, then g is superordinant to f. An analytic function q is termed a subordinant if q(z) $\prec$ p(z) for all p satisfying (3). An univalent subordinate $\tilde{q}$ that satisfies q $\prec \tilde{q}$ for all subordinants of (3) is known as the best subordinant. Miller and Macanu [12] derived conditions on the functions h, p and ϕ for which the following implication holds:

$$\mathrm{h(z)} \prec \phi\,(\mathrm{p(z)},\ \mathrm{zp'(z)},\ \mathrm{z^2p''(z)};\ \mathrm{z}) \Rightarrow \mathrm{q(z)} \prec \mathrm{p(z)}\,. \tag{4}$$

Ali et al. [1] have found sufficient conditions for certain normalized analytic functions f(z) to satisfy

$$q_1(z) \prec z\,\frac{f'(z)}{f(z)} \prec q_2(z), \tag{5}$$

where q_1 and q_2 are given univalent functions in U. Sanmugam et al. [16] derived sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{z\,f'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2\,f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

Furthermore, they also obtained results for functions defined with the help of Carlson-Shaffer operator. The Fox-Wright psi function is defined and represented as follows [18, p.50]

$$ {}_q\psi_s\left[\begin{smallmatrix}(\alpha_i,A_i)_{1,q}\,;\\ (\beta_i,B_i)_{1,s}\ \ ;\end{smallmatrix}\ z\right] = {}_q\psi_s\left[\begin{smallmatrix}(\alpha_1,A_1)\,,...,\,(\alpha_q,\,A_q)\,;\\ (\beta_1,B_1)\ \,,...,\,(\beta_s,B_s)\ \ ;\end{smallmatrix}\ z\right] \tag{6}$$

$$= \sum_{n=0}^{\infty}\left(\prod_{i=1}^{q}\Gamma(\alpha_i + A_i n)\right)\left(\prod_{i=1}^{s}(\beta_i + B_i n\right)^{-1}\frac{z^n}{n\,!}\,,$$

where $\alpha_i \in C\ (i = 1,...,q)$, $\beta_i \in C\ (i = 1,...,s)$and the coefficients $A_i \in R_+(i = 1,...,q)$ and $B_i \in R_+\ (i = 1,...,s)$ such that

$$1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{q} A_i \geq 0\,(q,s\ \in N_0 = N\ U\ \{0\}).$$

The normalized Fox-Wright psi function ${}_q\psi_s^*$(z) in series form is represented as

$$ {}_q\psi_s^*\left[\begin{smallmatrix}(\alpha_i,A_i)_{1,q}\,;\\ (\beta_i,B_i)_{1,s}\ \ ;\end{smallmatrix}\ z\right] = \frac{\prod_{i=1}^{s}\Gamma(\beta_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i)}\ {}_q\psi_s\left[\begin{smallmatrix}(\alpha_1,A_1)\,,...,\,(\alpha_q,\,A_q)\,;\\ (\beta_1,B_1)\ \,,...,\,(\beta_s,B_s)\ \ ;\end{smallmatrix}\ z\right]. \tag{7}$$

The ${}_q\psi_s$(z) is a special case of Fox's H-function $H^{m,n}_{k,\ell}(z)$(see [18, p.50]) and ${}_q\psi^*_s(z)$ is a generalization of the familiar generalized hypergeometric function ${}_qF_s$(z),

$$(8) \qquad {}_qF_s\left[{(\alpha_i)_{1,q}\,; \atop (\beta_i)_{1,s}\ \ ;}\, z\right] = {}_qF_s\left[{(\alpha_1)\,,...,(\alpha_q)\,; \atop (\beta_1)\ \ ,...,(\beta_s)\ \ ;}\, z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n} \frac{z^n}{n\,!},$$

where $(\alpha)_n$ is the Pochhammer symbol defined in terms of the gamma function Γ by

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

Corresponding to a function $L\ (\alpha_1, ..., \alpha_q; A_1, ..., A_q;\ \beta_1, ..., \beta_s;\ B_1, ..., B_s; z)$ defined by

$$(9) \qquad \mathrm{L}\ (\alpha_1, ..., \alpha_\mathrm{q};\ \mathrm{A}_1, ..., \mathrm{A}_\mathrm{q};\ \beta_1, ..., \beta_\mathrm{s};\ \mathrm{B}_1, ..., \mathrm{B}_\mathrm{s}; \mathrm{z}) = \mathrm{z}\ {}_\mathrm{q}\psi^*_\mathrm{s}(\mathrm{z}).$$

We consider a linear operator

$$L_{q,s}(\alpha_1, ..., \alpha_q;\ A_1, ..., A_q;\ \beta_1, ..., \beta_s;\ B_1, ..., B_s)\ :\ A \to A$$

defined by the convolution

$$(10) \qquad L_{q,s}(\alpha_1, ..., \alpha_q;\ A_1, ..., A_q;\ \beta_1, ..., \beta_s;\ B_1, ..., B_s)\, f(z)$$

$$= L\ (\alpha_1, ..., \alpha_q;\ A_1, ..., A_q;\ \beta_1, ..., \beta_s;\ B_1, ..., B_s;\ z) * f(z).$$

For brevity, we write

$$(11) \qquad L_{q,s}(\alpha_1) = L_{q,s}(\alpha_1, ..., \alpha_q;\ A_1, ..., A_q;\ \beta_1, ..., \beta_s;\ B_1, ..., B_s),$$

then one can easily verify from the definition (10) that

$$(12) \qquad z\ (A_1 L_{q,s}(\alpha_1)\ f(z))' = \alpha_1 L_{q,s}(\alpha_1+1)\ f(z)$$

$$-\,(\alpha_1 - A_1)\ L_{q,s}(\alpha_1)\ f(z).$$

Special cases of the operator $L_{q,s}(\alpha_1)\, f(z)$ includes Dziok-Srivastava linear operator (cf. [8]), the Carlson-Shaffer linear operator [5], the Cho-Kwon-Srivastava operator [6], Choi-Saigo-Srivastava operator [7], Libera operator [9] and the $\delta-$Ruschewey derivative operator [13].

In this paper, we obtain sufficient conditions for the normalized analytic function f defined by using the linear operator $L_{q,s}(\alpha_1)f(z)$to satisfy

$$q_1(z) \prec \left(\frac{z}{L_{q,s}(\alpha_1)\, f(z)}\right)^{\mu} \prec q_2(z)$$

and q_1 and q_2 are given univalent functions in U.

2 Definitions and Preliminaries

In order to prove our main results, we require the following known results.
Definition 1 [12]. Denote Q the set of all functions f that are analytic and injective on $\bar{U}\backslash E(f)$, where $E(f)$ is given by

$$E\,(f)\;=\;\{\,\xi\,\in\,\partial U\;:\;\lim_{z\,\to\,\xi}\,f(z)\;=\;\infty\},$$

and are such that $f^{'}(\xi)\,\neq\,0\;for\;\xi\,\in\,\partial U\backslash E(f)$.
Lemma 1 [11]. Assume that q be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing q(U) with $\phi(\omega) \neq 0$ when $\omega \in$ q(U). Set

$$\psi(z)\;=\;z\,q^{'}(z)\,\phi\,(q(z))\;and\;h(z)\;=\;\theta\,(q(z))\;+\;\psi(z)\,. \tag{13}$$

Suppose that
(i) ψ(z)is starlike univalent in U.
(ii) $Re\left\{\frac{z\,h^{'}(z)}{\psi(z)}\right\}\;>\;0\,for\;z\;\in\;U.$
If p is analytic with p(0) = q(0), p(U) $\subseteq$ D and

$$\theta\,(p(z))\;+\;z\,p^{'}(z)\,\phi\,(p(z))\;\prec\;\theta\,(q(z))\;+\;z\,q^{'}(z)\,\phi\,(q(z)), \tag{14}$$

then p(z) $\prec$ q(z) and q is the best dominant.
Lemma 2 [2]. If q be convex univalent in U and v and ϕ be analytic in a domain D containing q(U). Suppose that:
(i) $Re\,\{v^{'}(q(z))/\phi(q(z))\}\;>\;0\;for\;z\;\in\;U,$
(ii) $Q(z)\;=\;z\,q^{'}(z)\;\phi(z)$is starlike univalent in U.
If p(z) $\in$ H [q(0),1] $\bigcap$ Q with p(U) $\subseteq$ D, and $v(p(z)) + zp'(z)$ is univalent in U and

$$v(q(z))\;+\;z\,q^{'}(z)\,\phi\,(q(z))\;\prec\;v\,(q(z))+\;zp^{'}(z)\,\phi\,(p(z)), \tag{15}$$

then q(z) $\prec$ p(z) and q is the best subordinant.

3 Application to linear operator associated with the Fox-Wright psi function

Unless otherwise stated, we shall assume the reminder of this paper that γ, ξ, $\delta \in$ C and β, $\mu \in$ C* = C$\backslash\{0\}$.
Theorem 1. If q be analytic univalent in U with q(z) $\neq$ 0. Suppose that $\frac{z\,q^{'}(z)}{q(z)}$ is starlike univalent in U. Let γ, ξ, $\delta \in$ C; β, $\mu \in$ C* satisfy:

$$Re\left\{1+\frac{\xi}{\beta}\,q(z)\;+\;\frac{2\delta}{\beta}\,(q(z))^2-\;\frac{z\,q^{'}(z)}{q(z)}\;+\;\frac{z\,q^{''}(z)}{q^{'}(z)}\right\}\;>\;0, \tag{16}$$

and

$$\psi\left(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f\right) \;=\; \gamma+\xi\left(\frac{z}{L_{q,s}(\alpha_1)\;f(z)}\right)^{\mu} \tag{17}$$

$$+\,\delta\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{2\mu}+\frac{\beta\mu\alpha_1}{A_1}\left[1-\frac{L_{q,s}(\alpha_1+1)\;f(z)}{L_{q,s}(\alpha_1)\;f(z)}\right].$$

If q satisfies the following subordination:

$$\psi(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)\;\prec\;\gamma\;+\;\xi\,q(z)\;+\;\delta\left(q(z)\right)^2+\;\beta z\,\frac{q^{'}(z)}{q(z)}, \tag{18}$$

then

$$\left(\frac{z}{L_{q,s}(\alpha_1)\;f(z)}\right)^{\mu}\;\prec\;q(z)\quad(\mu\;\in\;C^{*}) \tag{19}$$

and q is the best dominant.
Proof. Let us define a function p as follows

$$p(z)\;=\;\left(\frac{z}{L_{q,s}(\alpha_1)\;f(z)}\right)^{\mu}\qquad(z\;\in\;U\;;\;\mu\;\in\;C^{*}). \tag{20}$$

Then the function p is analytic in U and p(0) = 1. So, on differentiating (20) logarithmically with respect to z and using the identity (12) in the resulting equation, we get the following result

$$\gamma+\xi\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu}+\delta\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{2\mu} \tag{21}$$

$$+\frac{\beta\mu\alpha_1}{A_1}\left[1-\frac{L_{q,s}(\alpha_1+1)\;f(z)}{L_{q,s}(\alpha_1)\;f(z)}\right]=\gamma+\xi\,p(z)\;+\;\delta\left(p(z)\right)^2+\;\beta\,z\,\frac{p^{'}(z)}{p(z)}.$$

Now using the results (21) and (18), we have

$$\gamma+\xi\,p(z)\;+\;\delta(p(z))^2+\beta\,\frac{zp^{'}(z)}{p(z)}\;\prec\;\gamma+\;\xi\,q(z)\;+\;\delta(q(z))^2\;+\;\beta z\,\frac{q^{'}(z)}{q(z)} \tag{22}$$

Setting $\theta(\omega)\;=\;\gamma+\xi\omega+\delta\omega^2 and\;\phi(\omega)\;=\;\frac{\beta}{\omega}$, it can be easily observed that θ is analytic in C, ϕ is analytic in C* and $\phi(\omega)\neq 0$ ($\omega\in$ C*). Hence, the result now follows by using Lemma 1.

Putting q(z) = (1+Az)/(1+Bz) ($-1\leq$ B $<$ A $\leq$1) in Theorem 1, we get the following result presented in the form of corollary.
Corollary 1. Let $-1\leq$ B $<$ A $\leq$ 1 and

$$Re\left\{1+\frac{\xi}{\beta}\left(\frac{1+Az}{1+Bz}\right)+\frac{2\delta}{\beta}\left(\frac{1+Az}{1+Bz}\right)^2-\frac{(A+B+3AB)}{(1+Az)\;(1+Bz)}\right\}\;>\;0$$

holds. If f(z) $\in$ A, and

$$\gamma+\xi\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu}+\delta\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{2\mu}+\frac{\beta\mu\alpha_1}{A_1}\left[1-\frac{L_{q,s}(\alpha_1+1)\,f(z)}{L_{q,s}(\alpha_1)\,f(z)}\right]$$

$$\prec\ \gamma+\xi\left(\frac{1+Az}{1+Bz}\right)+\delta\left(\frac{1+Az}{1+Bz}\right)^{2}+\beta\,\frac{(A-B)z}{(1+Az)\,(1+Bz)},$$

then

$$\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu}\prec\ \frac{1+Az}{1+Bz}\quad(\mu\ \in\ C^{*})$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $q(z)\ =\ \left(\frac{1+z}{1-z}\right)^{\nu}$ $(0\ <\ \nu\ \leq\ 1)$ in Theorem 1, we obtain the following interesting result given in the form of corollary

Corollary 2. Assume that (16) holds. If f $\in$ A, and

$$\gamma+\xi\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu}+\delta\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{2\mu}+\frac{\beta\mu\alpha_1}{A_1}\left[1-\frac{L_{q,s}(\alpha_1+1)\,f(z)}{L_{q,s}(\alpha_1)\,f(z)}\right]$$

$$\prec\ \gamma+\xi\ \left(\frac{1+z}{1-z}\right)^{\nu}+\delta\ \left(\frac{1+z}{1-z}\right)^{2\nu}+\ \frac{\beta 2\nu z}{(1-z)^{2}},$$

then

$$\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu}\prec\ \left(\frac{1+z}{1-z}\right)^{\nu}\quad(\mu\ \in\ C^{*},\ 0\ <\ \nu\ \leq\ 1\ )$$

and $\left(\frac{1+z}{1-z}\right)^{\nu}$ is the best dominant.

Taking $A_i\ =\ 1$(i = 1,. . . ,q) and $B_i\ =\ 1$ (i = 1,. . . ,s) in Theorem 1, we have the following Corollary which is the same result recently obtained by Mostafa [13].

Corollary 3. Let q be analytic univalent in U with q(z) $\neq$ 0. Suppose that $\frac{z\,q'(z)}{q(z)}$ is starlike univalent in U. Let γ, ξ, $\delta \in$ C; β, $\mu \in$ C* satisfy:

$$Re\left\{1+\ \frac{\xi}{\beta}\,q(z)\ +\ \frac{2\delta}{\beta}\,(q(z))^{2}-\ \frac{z\,q'(z)}{q(z)}\ +\ \frac{z\,q''(z)}{q'(z)}\right\}\ >\ 0,$$

and

(23) $$\zeta(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)\ =\ \gamma+\xi\left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu}$$

$$+\delta\left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{2\mu}+\ \beta\mu\alpha_1\left[1-\ \frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)\ f(z)}\right].$$

If q satisfies the following subordination

$$\zeta(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)\ \prec\ \gamma+\xi\,q(z)\ +\ \delta\,(q(z))^{2}+\ \beta\,z\,\frac{q'(z)}{q(z)},$$

then

$$\left(\frac{z}{H_{q,s}(\alpha_1)\ f(z)}\right)^{\mu} \prec\ q\ (z) \qquad (\mu\ \in C^*)$$

and q is best dominant.
Taking $A_i = 1$ (i = 1,. . . ,q), $B_i = 1$ (i = 1,. . . ,s), α_1 = a > 0, β_1 = c > 0, $\alpha_j = 1$ (j = 2,. . . ,q) and $\beta_j = 1$ (j = 2,. . . ,s) in Theorem 1, we have the following Corollary which improves the result obtained by Shanmugam et al. [15, Theorem 3.1].
Corollary 4. Let q be analytic univalent in U with q(z) ≠ 0 and condition (16) holds. Suppose also that $\frac{z\,q^{'}(z)}{q(z)}$ is starlike univalent in U and

$$V\ (\gamma,\xi,\delta,\beta,\mu)\ =\ \gamma+\xi\left(\frac{z}{L(a,c)f(z)}\right)^{\mu}+\ \delta\left(\frac{z}{L(a,c)\ f(z)}\right)^{2\mu} + \beta\mu a\left[1-\frac{L(a+1,c)}{L(a,c)\ f(z)}\right]. \tag{24}$$

If q satisfies the following subordination

$$V\ (\gamma,\xi,\delta,\beta,\mu) \prec\ \ \gamma+\xi\, q(z)\ +\ \delta(q(z))^2+\ \beta z\,\frac{q^{'}(z)}{q(z)},$$

then

$$\left(\frac{z}{L(a,c)\,f(z)}\right)^{\mu} \prec\ q(z) \qquad (\mu\ \in\ C^*)$$

and q is the best dominant.
Theorem 2. If q be convex univalent in U, $q(z)\ \neq\ 0$ *and* $\frac{z\,q^{'}(z)}{qz}$ be starlike univalent in U. Assume that

$$Re\left\{\frac{2\delta}{\beta}\,(q(z))^2+\frac{\xi}{\beta}\,q(z)\right\}\ >\ 0\quad (z\ \in\ U). \tag{25}$$

If $f\ \in\ A,\ 0\ \neq\ \left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu}\ \in\ H\,[q(0),1]\bigcap Q, \psi(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)$is univalent in U, and $\gamma+\xi\,q(z)\ +\ \delta\,(q(z))^2+\ \beta\,z\,\frac{q^{'}(z)}{q(z)}\ \prec\ \psi(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)$, where $\psi(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)$ is given by (17), then

$$q(z)\ \prec\ \left(\frac{z}{L_{q,s}(\alpha_1)\ f(z)}\right)^{\mu} \qquad (\mu\ \in C^*) \tag{26}$$

and q is the best subordinant.
Proof. Taking

$$v(\omega)\ =\ \gamma+\xi\omega+\delta\omega^2\ and\ \phi(\omega)\ =\ \frac{\beta}{\omega},$$

it is easily observed that v is analytic in C, ϕ is analytic in C* and $\phi(\omega)\neq 0$ ($\omega\in$ C*). Since q is convex (univalent) function it follows that

$$Re\left\{\frac{v^{'}(q(z))}{\phi\,(q(z))}\right\}\ =\ Re\left\{\frac{2\delta}{\beta}\,(q(z))^2+\frac{\xi}{\beta}\,q(z)\right\}q^{'}(z)\ >\ 0\quad (z\ \in U).$$

Thus the assertion (26) of Theorem 2 follows by an application of Lemma 2.
Taking $A_i = 1\,(i = 1,...,q)$ *and* $B_i = 1\,(i = 1,...,s)$ in Theorem 2, we have the following corollary which is the same result recently obtained by Mostafa [13].

Corollary 5. Let q be convex univalent in U, $q(z) \neq 0$ *and* $\frac{z\,q'(z)}{qz}$ be starlike univalent in U. Assume that (25) holds. If f ∈ A, $0 \neq \left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \in H[q(0),1] \bigcap Q$, $\zeta\,(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)$,is univalent in U, and
$\gamma + \xi\,q(z) \;+\; \delta\,(q(z))^2 + \beta\,z\,\frac{q'(z)}{q(z)} \prec \zeta(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)$,
where $\zeta(\alpha_1,\gamma,\xi,\delta,\beta,\mu,f)$ is given by (23), then

$$q(z) \prec \left(\frac{z}{H_{q,s}(\alpha_1)\;f(z)}\right)^{\mu} \qquad (\mu \in C^*)$$

and q is the best subordinant.
Taking $A_i = 1$ (i = 1,. . . ,q), $B_i = 1$ (i = 1,. . . ,s), α_1 = a > 0, β_1 = c > 0, $\alpha_j = 1$ (j = 2,. . . ,q) and $\beta_j = 1$ (j = 2,. . . ,s) in Theorem 2, we have the following corollary which improves the result of Shanmugam et al. [15, Theorem 3.6].

Corollary 6. Let q be convex univalent in U, $q(z) \neq 0$ *and* $\frac{z\,q'(z)}{qz}$ be starlike univalent in U. Assume that (25) holds. If f ∈ A, $0 \neq \left(\frac{z}{L(a,c)f(z)}\right)^{\mu} \in H\;[q(0),1] \bigcap Q$, $V(\gamma,\xi,\delta,\beta,\mu)$ is univalent in U and
$\gamma + \xi\,q(z) \;+\; \delta\,(q(z))^2 + \beta\,z\,\frac{q'(z)}{q(z)} \prec V(\gamma,\xi,\delta,\beta,\mu)$, *where* $V(\gamma,\xi,\delta,\beta,\mu)$ is given by (24) then

$$q(z) \prec \left(\frac{z}{L(a,c)\;\;f(z)}\right)^{\mu} \qquad (\mu \in C^*)$$

and q is the best subordinant.

4 Sandwich result

On combining the Theorems 1 and 2, we get the following theorem.

Theorem 3. Let q_1 be convex univalent in U and q_2 be univalent in U, q_1(z) ≠ 0 and q_2(z) ≠ 0 in U. Suppose that q_2 and q_1 satisfy (16) and (25), respectively.
If $f \in A$, $0 \neq \left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu} \in H\;[q(0),1] \bigcap Q$ and
$\gamma + \xi\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu} + \delta\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{2\mu} + \frac{\beta\mu\alpha_1}{A_1}\left[1 - \frac{L_{q,s}(\alpha_1+1)f(z)}{L_{q,s}(\alpha_1)f(z)}\right]$ is univalent in U. Then

$$\gamma \,+\, \xi\,q_1(z) \;+\; \delta\,(q_1(z))^2 + \beta\,z\,\frac{q_1'(z)}{q_1(z)}$$

$$\prec \gamma + \xi\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{\mu} + \delta\left(\frac{z}{L_{q,s}(\alpha_1)f(z)}\right)^{2\mu} + \frac{\beta\mu\alpha_1}{A_1}\left[1 - \frac{L_{q,s}(\alpha_1+1)f(z)}{L_{q,s}(\alpha_1)f(z)}\right]$$

$$\prec\; \gamma \,+\, \xi\,q_2(z) \;+\; \delta\,(q_2(z))^2 + \beta\,z\,\frac{q_2'(z)}{q_2(z)}$$

implies that

$$q_1(z) \prec \left(\frac{z}{L_{q,s}(\alpha_1)\ f(z)}\right)^{\mu} \prec q_2(z) \qquad (\mu \in C^*)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

5 Conclusions

In the present article, we have obtained the subordination and superordination results involving a linear operator associated with the Fox-Wright psi function for a family of analytic univalent functions in the open unit disk. Further, these results have been applied to obtain sandwich results. It is interesting to note that the subordination and superordination results contain a linear operator associated with the Fox-Wright psi function, which is the most generalized operator. Finally, we mentioned that the results obtained in the present work provide an extension of the results available in the literature and improve the existing results.

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About on equality

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Abstract

In this paper is given an equality for Bernstein polynomial.

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1 Introduction

Let $\Pi_n(\mathbb{R})$ the set of all polynomials on degree at the most n with real coefficients and the polynomial Taylor

$$(T_n f)(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0), f \in \Pi_n(\mathbb{R})\,([3]).$$

2 Principal results

Since $f \in \Pi_n(\mathbb{R})$, it follows

$$f(x) = (T_n f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \cdot 1^{n-k}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \cdot \sum_{j=0}^{n-k} \binom{n-k}{j} x^j \cdot (1-x)^{n-k-j}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \cdot \sum_{j=k}^{n} \binom{n-k}{j-k} x^{j-k} \cdot (1-x)^{n-j}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \sum_{j=k}^{n} \binom{n-k}{j-k} x^{j} \cdot (1-x)^{n-j}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \sum_{j=k}^{n} \frac{\binom{n-k}{j-k}}{\binom{n}{j}} \cdot \binom{n}{j} x^j \cdot (1-x)^{n-j}.$$

Denote $b_{j,k} = \dfrac{\binom{n-k}{j-k}}{\binom{n}{j}}$, $c_j = \binom{n}{j} x^j \cdot (1-x)^{n-j}$, $a_k = \dfrac{f^{(k)}(0)}{k!}$.

Propertie 1 *For $a_k, c_j, b_{k,j} \in \mathbb{R}$, $k, j \in \{0,1,...,n\}$, we have*

$$\sum_{k=0}^{n} a_k \sum_{j=k}^{n} b_{j,k} c_j = \sum_{k=0}^{n} c_k \sum_{j=0}^{k} b_{k,j} a_j. \tag{1}$$

Proof. We can write

$$\sum_{k=0}^{n} a_k \sum_{j=k}^{n} b_{j,k} c_j a_0 \sum_{j=0}^{n} b_{j,0} c_j$$

$$+a_1 \sum_{j=1}^{n} b_{j,1} c_j + a_2 \sum_{j=2}^{n} b_{j,2} c_j + ... + a_n \sum_{j=n}^{n} b_{j,n} c_j$$

$$= a_0(b_{0,0}c_0 + b_{1,0}c_1 + b_{2,0}c_2 + ... + b_{n,0}c_n) + a_1(b_{1,1}c_1 + b_{2,1}c_2 + ... + b_{n,1}c_n)$$

$$+a_2(b_{2,2}c_2 + b_{3,2}c_3 + ... + b_{n,2}c_n) + ... + a_n b_{n,n} c_n$$

$$= c_0 \cdot a_0 b_{0,0} + c_1(a_0 b_{1,0} + a_1 b_{1,1}) + c_2(a_0 b_{2,0} + a_1 b_{2,1} + a_2 b_{2,2}) + ...$$

$$+c_n(a_0 b_{n,0} + a_1 b_{n,1} + a_2 b_{n,2} + ... + a_n b_{n,n}) = \sum_{k=0}^{n} c_k \sum_{j=0}^{k} a_j b_{j,k}.$$

We obtain

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} \cdot \frac{\binom{n-j}{k-j}}{\binom{n}{k}}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} \cdot \frac{\frac{k!}{(k-j)!}}{\frac{n!}{(n-j)!}}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} \cdot \frac{k(k-1)...(k-j+1)}{n(n-1)...(n-j+1)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} \cdot \frac{(-k)(-k+1)...(-k+j-1)}{(-n)(-n+1)...(-n+j-1)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} \cdot \frac{(-k)_j}{(-n)_j},$$

where

$$(z)_m = z(z+1)...(z+m-1), \; (z)_0 = 1.$$

Denote $s_{n,k}(f)=\sum_{j=0}^{k}\frac{f^{(j)}(0)}{j!}\cdot\frac{(-k)_j}{(-n)_j}$. We consider the linear operator: $Q_n:\Pi_n\to\Pi_n$, define by

$$(Q_nf)(x)=\sum_{j=0}^{n}\frac{(-nx)_j}{(-n)_j}\cdot\frac{f^{(j)}(0)}{j!}.$$

We observe

$$s_{n,k}(f)=(Q_nf)\left(\frac{k}{n}\right).$$

Therefore

$$f(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\cdot(Q_nf)\left(\frac{k}{n}\right),$$

$$f(x)=(B_n(Q_nf))(x),$$

where

$$(B_nf)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),$$

is Bernstein polynomial ([1], [2]).

For $f(x)=x^l, l\in\{0,1,...,n\}$ we obtain

$$(Q_nf)(x)=\sum_{j=0}^{n}\frac{(-nx)_j}{(-n)_j}\cdot\frac{l(l-1)...(l-j+1)}{j!}t^{l-j}\mid_{t=0}=\frac{(-nx)_l}{(-n)_l},$$

$$x^l=(B_n(Q_nf))(f),\ l\in\{0,1,2,...,n\}.$$

Particular cases:

1) $l=0\Rightarrow 1=(B_ng)(x),\ g(x)=1.$

2) $l=1\Rightarrow x=(B_ng)(x),\ g(x)=x.$

3) $l=2\Rightarrow x^2=(B_ng)(x),\ g(x)=\frac{x(nx-1)}{n-1}.$

4) $l=3\Rightarrow x^3=(B_ng)(x),\ g(x)=\frac{x(nx-1)(nx-2)}{(n-1)(n-2)}.$

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Subclasses of p-valent meromorphic functions defined by linear operator

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Abstract

In this paper, we introduce and investigate some properties of the class $\mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$ and its subclass $\overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$ of meromorphic p-valent functions with positive coefficients, which are defined by linear operator. In particular, some inclusion relations, coefficients estimates, distortion theos, radii of meromorphically p-valent, neighborhoods, partial sums and Hadamard product are proven here for these functions classes.

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1 Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad \left(p \in \mathbb{N} = \{1, 2, ...\}\right), \tag{1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U\backslash\{0\}$.

A function $f \in \Sigma_p$ is said to be meromorphically p-valent reverse starlike of order α if it satisfies

$$-Re\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha \quad (0 \leq \alpha < \frac{1}{p}).$$

Recently, more and more researchers are interested in the reciprocal case of the starlike functions (see [10], [20], [26], [15], [16], [25]).

For two functions $f_j(z) \in \Sigma_p (j = 1, 2)$ are given by

$$f_j(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p}.$$

Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ in Σ_p is given by

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p,1} a_{n-p,2} z^{n-p} = (f_2 * f_1)(z).$$

For two functions $f(z)$ and $F(z)$, analytic in U, we say that $f(z)$ is subordinate to $F(z)$, written symbolically as follows:

$$f \prec F \text{ in } U \text{ or } f(z) \prec F(z) \qquad (z \in U),$$

if there exists a Schwarz function $\omega(z) \in \Omega$, which (by definition) is analytic in U with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \qquad (z \in U)$$

such that

$$f(z) = F(\omega(z)) \qquad (z \in U).$$

Indeed it is known that

$$f(z) \prec F(z)(z \in U) \implies f(0) = F(0) \text{ and } f(U) \subset F(U).$$

In particular, if the function $F(z)$ is univalent in U, we have the following equivalence

$$f(z) \prec F(z)(z \in U) \iff f(0) = F(0) \text{ and } f(U) \subset F(U).$$

El-Ashwah and Bulboaca [6] defined the linear operator:

$$\mathcal{L}^s_{p,d}(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{d}{n+d}\right)^s z^{n-p}$$

$$(s \in \mathbb{C}; d \in \mathbb{C}^* = \mathbb{C}\backslash\{0, -1, -2, ...\}; z \in U^*)$$

by setting

$$\mathcal{J}^s_{p,d}(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{n+d}{d}\right)^s z^{n-p}$$

and

$$\left(\mathcal{J}^{s,\lambda}_{p,d} * \mathcal{J}^s_{p,d}\right)(z) = \frac{1}{z^p (1-z)^\lambda} \qquad (\lambda > 0),$$

we, obtain the linear operator

$$\mathcal{J}^{s,\lambda}_{p,d}(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{d}{n+d}\right)^s \frac{(\lambda)_n}{(1)_n} z^{n-p},$$

which is defined by

$$\mathcal{J}_{p,d}^{s,\lambda} f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{d}{n+d}\right)^s \frac{(\lambda)_n}{(1)_n} a_{n-p} z^{n-p}. \tag{2}$$

$$(\lambda > 0, s \in \mathbb{C}; d \in \mathbb{C}^*; z \in U^*),$$

where $f \in \Sigma_p$ is in the form (1) and $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N}). \end{cases}$$

It is readily verified from (2) that

$$z(\mathcal{J}_{p,d}^{s,\lambda} f(z))' = \lambda \mathcal{J}_{p,d}^{s,\lambda+1} f(z) - (\lambda+p)\mathcal{J}_{p,d}^{s,\lambda} f(z) \quad (\lambda > 0) \tag{3}$$

and

$$z(\mathcal{J}_{p,d}^{s+1,\lambda} f(z))' = d\mathcal{J}_{p,d}^{s,\lambda} f(z) - (d+p)\mathcal{J}_{p,d}^{s+1,\lambda} f(z). \tag{4}$$

Remark 1 *(i)* $\mathcal{J}_{1,d}^{s,1} f(z) = P_\beta^\alpha f(z)$ $(\alpha, \beta > 0)$ *(see Lashin [12]);*

(ii) $\mathcal{J}_{p,1}^{s,1} f(z) = P^\alpha f(z)$ $(\alpha > 0)$ (see Aqlan et al. [5]);
(iii) $\mathcal{J}_{1,v}^{1,1} f(z) = F_v f(z)$ $(v > 0)$ (see [[17], p.11 and 389]);
(iv) $\mathcal{J}_{1,d}^{s,\lambda} f(z) = \mathcal{J}_d^{s,\lambda} f(z)$ (see [6]);
(v) $\mathcal{J}_{1,d}^{-1,1} f(z) = \mathcal{F}_d^s(a,c;z) f(z)$ (see El-Ashwah [7]);
(vi) $\mathcal{J}_{1,d}^{s,1} f(z) = L_d^s f(z) f(z)$ (see El-Ashwah [8]).

By using the linear operator defined by (2), we define a new subclass $\mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$ of Σ_p as follows: For $p\beta > 1, -1 \le B < A \le 1, v$

$$\frac{p}{1-p\beta}\left(\frac{\mathcal{J}_{p,d}^{s,\lambda} f(z)}{z\left(\mathcal{J}_{p,d}^{s,\lambda} f(z)\right)'} + \beta\right) \prec -\frac{1+Az}{1+Bz} \tag{5}$$

which is equivalent to

$$\frac{z\left(\mathcal{J}_{p,d}^{s,\lambda} f(z)\right)'}{p\mathcal{J}_{p,d}^{s,\lambda} f(z)} \prec -\frac{1+Bz}{1+[A-p\beta(A-B)]z} \tag{6}$$

or equivalently, the following inequality holds true

$$\left| - \frac{p\mathcal{J}_{p,d}^{s,\lambda} f(z) + z\left(\mathcal{J}_{p,d}^{s,\lambda} f(z)\right)'}{B[p\mathcal{J}_{p,d}^{s,\lambda} f(z) + z\left(\mathcal{J}_{p,d}^{s,\lambda} f(z)\right)'] + (1-p\beta)(A-B)z\left(\mathcal{J}_{p,d}^{s,\lambda} f(z)\right)'} \right| < 1 \tag{7}$$

Remark 2 *(i) Putting* $s=0,\lambda=1, A=1-2\alpha(0\leq\alpha<\frac{1}{p}), B=-1, \mathcal{J}_{p,d}^{s,\lambda}f(z)=f(z)$, *if* $f\in\mathcal{M}_d^{0,1}(p;\beta,1-2\alpha,-1)$, *we have*

$$\frac{p}{1-p\beta}\left(\frac{f(z)}{zf'(z)}+\beta\right)\prec-\frac{1+(1-2\alpha)z}{1-z},$$

which is equivalent to

$$\frac{p}{1-p\beta}Re\left(\frac{f(z)}{zf'(z)}+\beta\right)>\alpha;$$

(ii) Putting $p=1$, we have

$$\frac{1}{1-\beta}Re\left(\frac{f(z)}{zf'(z)}+\beta\right)>\alpha.$$

Also, we say that a function $f\in\mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$ is in the analogous class $\mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$ whenever $f(z)$ is given by

$$f(z)=z^{-p}+\sum_{n=p}^{\infty}|a_n|\,z^n \quad (p\in\mathbb{N}) \tag{8}$$

Inclusion Properties of The Class $\mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$

We need the following lema which is popularly known as Jack's lema to prove our theorem.

Lemma 1 *[9]. Let* $\omega(z)$ *be a non-constant function analytic in* U *with* $\omega(0)=0$. *If* $|\omega(z)|$ *attains its maximum value on the circle* $|z|=r<1$ *at* $z_0\in U$, *then*

$$z_0\omega'(z_0)=\gamma\omega(z_0),$$

where $\gamma\geq1$ is a real number.

Theorem 1 *If*

$$\lambda\geq\frac{p(A-B)(p\beta-1)}{1+[A-p\beta(A-B)]}\quad\left(-1\leq B<A\leq1;\frac{1}{p}<\beta<\frac{1+A}{p(A-B)};p\in\mathbb{N}\right),$$

then

$$\mathcal{M}_d^{s,\lambda+1}(p;\beta,A,B)\subset\mathcal{M}_d^{s,\lambda}(p;\beta,A,B).$$

Proof. Let $f\in\mathcal{M}_d^{s,\lambda+1}(p;\beta,A,B)$ and suppose that

$$\frac{p}{1-p\beta}\left(\frac{\mathcal{J}_{p,d}^{s,\lambda}f(z)}{z\left(\mathcal{J}_{p,d}^{s,\lambda}f(z)\right)'}+\beta\right)=-\frac{1+A\omega(z)}{1+B\omega(z)}, \tag{9}$$

where the function $\omega(z)$ is either analytic or meromorphic in U with $\omega(0)=0$.

Then, by using (3) and (9), we have

$$\frac{\lambda \mathcal{J}_{p,d}^{s,\lambda+1} f(z)}{\mathcal{J}_{p,d}^{s,\lambda} f(z)} = \frac{\lambda + \{(\lambda+p)[A - p\beta(A-B)] - Bp\}\omega(z)}{1 + [A - p\beta(A-B)]\omega(z)}. \tag{10}$$

Differentiating (10) logarithmically with respect to z and making use of (6) once again, we obtain

$$\frac{z\left(\mathcal{J}_{p,d}^{s,\lambda+1} f(z)\right)'}{\mathcal{J}_{p,d}^{s,\lambda+1} f(z)} = \frac{p(A-B)(1-p\beta)z\omega'(z)}{(\lambda+\{(\lambda+p)[A-p\beta(A-B)]-Bp\}\omega(z))(1+[A-p\beta(A-B)]\omega(z))}$$

$$-\frac{p(1+B\omega(z))}{1+[A-p\beta(A-B)]\omega(z)} \tag{11}$$

Now, assuming that

$$\max_{|z|\le|z_0|} |\omega(z)| = |\omega(z_0)| = 1 \, (z_0 \in U) \tag{12}$$

and applying Jack's lema, we have

$$z_0\omega'(z_0) = \gamma\omega(z_0) \quad (\gamma \ge 1). \tag{13}$$

If we set $\omega(z_0) = e^{i\theta}$ $(0 \le \theta < 2\pi)$ in (11), we have

$$\left| -\frac{p\mathcal{J}_{p,d}^{s,\lambda+1} f(z_0) + z_0\left(\mathcal{J}_{p,d}^{s,\lambda+1} f(z_0)\right)'}{B[p\mathcal{J}_{p,d}^{s,\lambda+1} f(z_0) + z_0\left(\mathcal{J}_{p,d}^{s,\lambda+1} f(z_0)\right)'] + (1-p\beta)(A-B)z_0\left(\mathcal{J}_{p,d}^{s,\lambda+1} f(z_0)\right)'}\right|^2 - 1$$

$$= \left|\frac{(\gamma+\lambda) + \{(\lambda+p)[A-p\beta(A-B)] - Bp\}e^{i\theta}}{\lambda + \{[A-p\beta(A-B)](\lambda+p-\gamma) - Bp\}e^{i\theta}}\right|^2 - 1$$

$$= \frac{\gamma^2 + 2\lambda\gamma + \gamma\{(2\lambda+2p-\gamma)M - 2Bp\} + 2\gamma((2\lambda+p)M - Bp)\cos\theta}{|\lambda + \{M(\lambda+p-\gamma) - Bp\}e^{i\theta}|^2},$$

where $M = A - p\beta(A-B)$.

Set

$$g(t) = \gamma^2 + 2\lambda\gamma + \gamma\{(2\lambda+2p-\gamma)M - 2Bp\} + 2\gamma((2\lambda+p)M - Bp)t \tag{14}$$

Then

$$g(1) = \gamma(1+M)\{2\lambda(1+M) + \gamma(1-M) - 2p(B-M)\} > 0$$

and

$$g(-1) = \gamma(1-M)\{2\lambda(1-M) + \gamma(1+M) + 2p(B-M)\} > 0,$$

which, together, imply that

$$g(\cos\theta) \geq 0 \quad (0 \leq \theta < 2\pi). \tag{15}$$

In view of (14) and (15), it would obviously contradict our hypothesis that

$$f \in \mathcal{M}_d^{s,\lambda+1}(p;\beta,A,B).$$

Thus we must have

$$|\omega(z)| < 1 \quad (z \in U),$$

and we conclude from (9) that

$$f \in \mathcal{M}_d^{s,\lambda}(p;\beta,A,B),$$

which evidently completes the proof of Theorem 1.

Theorem 2 *If*

$$Re\,(d) \geq \frac{p(A-B)(p\beta-1)}{1+[A-p\beta(A-B)]} \quad (-1 \leq B < A \leq 1; \frac{1}{p} < \beta < \frac{1+A}{p(A-B)}; p \in \mathbb{N}),$$

then

$$\mathcal{M}_d^{s,\lambda}(p;\beta,A,B) \subset \mathcal{M}_d^{s+1,\lambda}(p;\beta,A,B).$$

Proof. Making use of (4), the proof of Theorem 2 is similar to that of Theorem 1, so it is omitted.

Theorem 3 *Let* $\delta \in \mathbb{C}$ *such that*

$$Re(\delta) \geq \frac{p(A-B)(p\beta-1)}{1+[A-p\beta(A-B)]} \quad (-1 \leq B < A \leq 1; \frac{1}{p} < \beta < \frac{1+A}{p(A-B)}; p \in \mathbb{N}).$$

If $f(z) \in \mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$, then the function $F_\delta(z)$ is defined by

$$F_\delta(z) = \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} f(t)dt \tag{16}$$

also belongs to the class $\mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$.

Proof. Supoose that $f \in \mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$ and put

$$\frac{p}{1-p\beta}\left(\frac{\mathcal{J}_{p,d}^{s,\lambda}F_\delta(z)}{z\left(\mathcal{J}_{p,d}^{s,\lambda}F_\delta(z)\right)'} + \beta\right) = -\frac{1+A\omega(z)}{1+B\omega(z)}. \tag{17}$$

From (16), we have

$$z\left(\mathcal{J}_{p,d}^{s,\lambda}F_\delta(z)\right)' = \delta\mathcal{J}_{p,d}^{s,\lambda}f(z) - (\delta+p)(\mathcal{J}_{p,d}^{s,\lambda}F_\delta(z))$$

which implies that

$$\frac{z\left(\mathcal{J}_{p,d}^{s,\lambda}F_\delta(z)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}F_\delta\,(z)}=\frac{\delta\mathcal{J}_{p,d}^{s,\lambda}f(z)}{\mathcal{J}_{p,d}^{s,\lambda}F_\delta(z)}-(\delta+p)\ . \tag{18}$$

where the function $\omega(z)$ is either analytic or meromorphic in U with $\omega(0)=0$. Then, by using (17) and (18), we have

$$\begin{aligned}&\frac{z\left(\mathcal{J}_{p,d}^{s,\lambda}f(z)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}f\,(z)}\\ =\;&\frac{p(A-B)(1-p\beta)z\omega'(z)}{(\delta+\{(\delta+p)[A-p\beta(A-B)]-Bp\}\omega(z))(1+[A-p\beta(A-B)]\omega(z))}\\ &-\frac{p(1+B\omega(z))}{1+[A-p\beta(A-B)]\omega(z)}\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\frac{z\left(\mathcal{J}_{p,d}^{s,\lambda}f(z)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}f\,(z)}+p\\ =\;&\frac{p(A-B)(1-p\beta)z\omega'(z)}{(\delta+\{(\delta+p)[A-p\beta(A-B)]-Bp\}\omega(z))(1+[A-p\beta(A-B)]\omega(z))}\\ &+\frac{p(A-B)(1-p\beta)\omega(z)}{1+[A-p\beta(A-B)]\omega(z)}.\end{aligned} \tag{20}$$

Thus, the proof follows similar that proof of Theorem 1 and assume that (12) and (13) hold true. Putting $\omega(z_0)=e^{i\theta}(0\le\theta<2\pi)$ and setting $z=z_0$ in (19) and (20), we have

$$\begin{aligned}&\left|\frac{p\mathcal{J}_{p,d}^{s,\lambda}f\,(z_0)+z_0\left(\mathcal{J}_{p,d}^{s,\lambda}f\,(z_0)\right)'}{B[p\mathcal{J}_{p,d}^{s,\lambda}f\,(z_0)+z_0\left(\mathcal{J}_{p,d}^{s,\lambda}f\,(z_0)\right)']+(1-p\beta)(A-B)z_0\left(\mathcal{J}_{p,d}^{s,\lambda}f\,(z_0)\right)'}\right|^2-1\\ =&\left|\frac{p+\frac{z_0\left(\mathcal{J}_{p,d}^{s,\lambda}f(z_0)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}f(z_0)}}{B[p+\frac{z_0\left(\mathcal{J}_{p,d}^{s,\lambda}f(z_0)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}f(z_0)}]+(1-p\beta)(A-B)\frac{z_0\left(\mathcal{J}_{p,d}^{s,\lambda}f(z_0)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}f(z_0)}}\right|^2-1\\ =&\left|\frac{(\gamma+\delta)+\{(\delta+p)[A-p\beta(A-B)]-Bp\}e^{i\theta}}{\delta+\{[A-p\beta(A-B)](\delta+p-\gamma)-Bp\}e^{i\theta}}\right|^2-1\\ =&\frac{\Omega(\theta)}{|\delta+\{[A-p\beta(A-B)](\delta+p-\gamma)-Bp\}e^{i\theta}|^2},\end{aligned} \tag{21}$$

where

$$\begin{aligned}\Omega(\theta) &= \left|(\gamma+\delta)+\{(\delta+p)M-Bp\}e^{i\theta}\right|^2-\left|\delta+\{M(\delta+p-\gamma)-Bp\}e^{i\theta}\right|^2\\ &= 2\gamma Re(\delta)+\gamma^2+\gamma(2Re(\delta)+2p-\gamma)M^2-2pB\gamma M\\ &\quad +2\gamma\cos\theta(M(2Re(\delta)+p)-Bp)\end{aligned}$$

(22) $$(M=A-p\beta(A-B);-1\leq B<A\leq 1;\gamma\geq 0;0\leq\theta<2\pi).$$

Then by condition

$$Re(\delta)\geq\frac{p(A-B)(p\beta-1)}{1+[A-p\beta(A-B)]},$$

we have

$$\Omega(0)=2\gamma Re(\delta)+\gamma^2+\gamma(2Re(\delta)+2p-\gamma)M^2-2pB\gamma M+2\gamma(M(2Re(\delta)+p)-Bp)\geq 0$$

and

$$\Omega(\pi)=2\gamma Re(\delta)+\gamma^2+\gamma(2Re(\delta)+2p-\gamma)M^2-2pB\gamma M-2\gamma(M(2Re(\delta)+p)-Bp)\geq 0$$

which imply that

(23) $$\Omega(\theta)\geq 0\qquad(0\leq\theta<2\pi)$$

In view of (23) and (21), it would obviously contradict our hypothesis that

$$f\in\mathcal{M}_d^{s,\lambda}(p;\beta,A,B).$$

Thus we must have

$$|\omega(z)|<1\qquad(z\in U),$$

and we conclude from (17) that

$$F_\delta(z)\in\mathcal{M}_d^{s,\lambda}(p;\beta,A,B),$$

where the function is given by (16).

The proof of Theorem 3 is completed.

Theorem 4 *Set* $-1\leq B<A\leq 1;p\beta>1;\lambda,d\in\mathbb{R}$, *the function* $f\in\mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$ *if and only if* $F_\lambda(z)$ *given by*

(24) $$F_\lambda(z)=\frac{\lambda}{z^{\lambda+p}}\int_0^z t^{\lambda+p-1}f(t)dt$$

belongs to the class $\mathcal{M}_d^{s,\lambda+1}(p;\beta,A,B)$.

Proof. By using (24), we have

$$\lambda f(z) = (\lambda+p)F_\lambda(z) + z(F_\lambda(z))'$$

which, according to (3), we have

$$\lambda \mathcal{J}_{p,d}^{s,\lambda} f(z) = (\lambda+p)\mathcal{J}_{p,d}^{s,\lambda} F_\lambda(z) + z(\mathcal{J}_{p,d}^{s,\lambda} F_\lambda(z))' = \lambda \mathcal{J}_{p,d}^{s,\lambda+1} F_\lambda(z).$$

Therefore, we have

$$\mathcal{J}_{p,d}^{s,\lambda} f(z) = \mathcal{J}_{p,d}^{s,\lambda+1} F_\lambda(z)$$

which gives the result.

Theorem 5 *Set $-1 \leq B < A \leq 1; p\beta > 1$ the function $f \in \mathcal{M}_d^{s+1,\lambda}(p;\beta,A,B)$ if and only if $F_d(z)$ given by*

$$F_d(z) = \frac{d}{z^{d+p}} \int_0^z t^{d+p-1} f(t)dt \tag{25}$$

belongs to the class $\mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$.

Proof. The proof of Theorem 5 is similar to that of Theorem 4, so it is omitted.

Basic Properties of The Class $\overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$

Throughout this section, we assume that

$$-1 \leq B < A \leq 1; p\beta > 1; \lambda > 0, d \in \mathbb{C}^*, s \in \mathbb{C}.$$

We first determine a necessary and sufficient condition for a function $f(z) \in \Sigma_p$ of the form (8) to be in the class $\overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$.

Theorem 6 *Let the function $f(z)$ be given by (8). Then $f(z) \in \overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$ if and only if*

$$\sum_{n=p}^{\infty}[(n+p)(1-B) + n(A-B)(p\beta-1)] \left| \left(\frac{d}{n+d}\right)^s \right| \frac{(\lambda)_n}{(1)_n} |a_n|$$

$$\leq p(A-B)(p\beta-1). \tag{26}$$

Proof Suppose that $f(z)$ be given by (8) in the class $\overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$. Then from (7) and (8), we have

$$\left| -\frac{p\mathcal{J}_{p,d}^{s,\lambda} f(z) + z\left(\mathcal{J}_{p,d}^{s,\lambda} f(z)\right)'}{B[p\mathcal{J}_{p,d}^{s,\lambda} f(z) + z\left(\mathcal{J}_{p,d}^{s,\lambda} f(z)\right)'] + (1-p\beta)(A-B)z\left(\mathcal{J}_{p,d}^{s,\lambda} f(z)\right)'} \right|$$

(27)

$$= \left| \frac{\sum\limits_{n=p}^{\infty}(n+p)\left(\frac{d}{n+d}\right)^{s}\frac{(\lambda)_n}{(1)_n}|a_n|z^{n+p}}{p(p\beta-1)(A-B)+\sum\limits_{n=p}^{\infty}(Bp+nB)\left(\frac{d}{n+d}\right)^{s}\frac{(\lambda)_n}{(1)_n}|a_n|z^{n+p}-\sum\limits_{n=p}^{\infty}n(p\beta-1)(A-B)\left(\frac{d}{n+d}\right)^{s}\frac{(\lambda)_n}{(1)_n}|a_n|z^{n+p}} \right| < 1$$

$(z \in U)$. Since $|Re(z)| < |z|$ for all z, choosing z to be real and letting $z \longrightarrow 1^-$ through real values, (27) yields

$$\sum_{n=p}^{\infty}[(n+p)(1-B)+n(A-B)(p\beta-1)]\left|\left(\frac{d}{n+d}\right)^{s}\right|\frac{(\lambda)_n}{(1)_n}|a_n|$$

$$\le p(A-B)(p\beta-1). \tag{28}$$

Conversely, we assume that the inequality (26) holds true.

Then, if we let $z \in \partial U$ we find from (8) and (26) that

$$\left| \frac{p\mathcal{J}_{p,d}^{s,\lambda}f(z)+z\left(\mathcal{J}_{p,d}^{s,\lambda}f(z)\right)'}{B[p\mathcal{J}_{p,d}^{s,\lambda}f(z)+z\left(\mathcal{J}_{p,d}^{s,\lambda}f(z)\right)']+(1-p\beta)(A-B)z\left(\mathcal{J}_{p,d}^{s,\lambda}f(z)\right)'} \right|$$

$$< \frac{\sum\limits_{n=p}^{\infty}(n+p)\left|\left(\frac{d}{n+d}\right)^{s}\right|\frac{(\lambda)_n}{(1)_n}|a_n|}{p(p\beta-1)(A-B)+\sum\limits_{n=p}^{\infty}[B(n+p)-n(A-B)(p\beta-1)]\left|\left(\frac{d}{n+d}\right)^{s}\right|\frac{(\lambda)_n}{(1)_n}|a_n|}$$

$$\le 1 \qquad (z \in \partial U = \{z \in \mathbb{C} : |z| = 1\}). \tag{29}$$

Hence, by the Maximum Modulus Theorem, we conclude $f(z) \in \overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$.

This completes the proof of Theorem 6.

Corollary 1 *Let the function $f(z)$ be given by (8). If $f(z) \in \overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$, then*

$$|a_n| \le \frac{p(A-B)(p\beta-1)}{[(n+p)(1-B)+n(A-B)(p\beta-1)]}\cdot\left|\left(\frac{n+d}{d}\right)^{s}\right|\frac{(1)_n}{(\lambda)_n}$$

$$(n = p, p+1, p+2, ...; p \in \mathbb{N}).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{p(A-B)(p\beta-1)}{[(n+p)(1-B)+n(A-B)(p\beta-1)]}\left|\left(\frac{n+d}{d}\right)^{s}\right|\frac{(1)_n}{(\lambda)_n}z^n \tag{30}$$

$$(n = p, p+1, p+2, ...; p \in \mathbb{N}).$$

Now, we prove the following growth and distortion theorems for the class $\overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$.

Theorem 7 *Let a function $f(z)$ defined by (8) is in the class $\mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$. If the sequence $\{C_n\}_{n=p}^{\infty}$ is nondecreasing, then*

$$r^{-p}-\frac{p(A-B)(p\beta-1)}{C_p}r^p\le|f(z)|\le r^{-p}+\frac{p(A-B)(p\beta-1)}{C_p}r^p,(0<|z|=r<1)\,, \tag{31}$$

where

$$C_n=[(n+p)\,(1-B)+n(A-B)(p\beta-1)]\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n} \tag{32}$$

$$(n=p,p+1,p+2,...;p\in\mathbb{N})\,.$$

If the sequence $\{C_n/n\}_{n=p}^{\infty}$ is nondecreasing, then

$$pr^{-p-1}-\frac{p^2(A-B)(p\beta-1)}{C_p}r^{p-1}\le\left|f'(z)\right|\le pr^{-p-1}+\frac{p^2(A-B)(p\beta-1)}{C_p}r^{p-1},(0<|z|=r<1)\,. \tag{33}$$

The results are sharp for the function given by (30.

Proof. Let the function of the form (8) in the class $\mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$. If the sequence $\{C_n\}_{n=p}^{\infty}$ is nondecreasing, then by Theorem 6, we have

$$\sum_{n=p}^{\infty}|a_n|\le\frac{p(A-B)(p\beta-1)}{C_p} \tag{34}$$

and the sequence $\{C_n/n\}_{n=p}^{\infty}$ is nondecreasing, Theorem 3 also implies

$$\sum_{n=p}^{\infty}n\,|a_n|\le\frac{p^2(A-B)(p\beta-1)}{C_p} \tag{35}$$

Thus, assertion (31) and (33) follow immediately.

Finally, it is easy to see that the bounds in (31) and (33) are attained for the function given by (30), with $n=p$.

Next, we determine the radii of meromorphically p-valent starlikeness and meromorphically p-valent convexity of the class $\mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$.

Theorem 8 *Let a function $f(z)$ defined by (8) is in the class $\mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$. Then we have*

(i) f is meromorphically p-valent starlike of order $\delta(0\le\delta<1)$ in $|z|<r_1$, that is

$$Re\left\{-\frac{zf'(z)}{pf(z)}\right\}>\delta\ \ (|z|<r_1;0\le\delta<1;p\in\mathbb{N}),$$

where

$$r_1=\inf_{n\ge p}\left\{\frac{(1-\delta)[(n+p)(1-B)+n(A-B)(p\beta-1)]}{(A-B)(p\beta-1)(n+\delta p)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}\right\}^{\frac{1}{n+p}}; \tag{36}$$

(ii) f is meromorphically p-valent convex of order $\delta (0 \leq \delta < 1)$ in $|z| < r_2$, that is

$$Re\left\{ -\frac{(zf'(z))'}{pf'(z)} \right\} > \delta \quad (|z| < r_2; 0 \leq \delta < 1; p \in \mathbb{N}),$$

where

$$r_2 = \inf_{n \geq p} \left\{ \frac{p(1-\delta)[(n+p)(1-B)+n(A-B)(p\beta-1)]}{n(A-B)(p\beta-1)(n+\delta p)} \cdot \left| \left(\frac{d}{n+d} \right)^s \right| \frac{(\lambda)_n}{(1)_n} \right\}^{\frac{1}{n+p}}. \tag{37}$$

Each of these results is sharp for the function given by (30).

Proof. (i) From the definition (8), we easily get

$$\left| \frac{1 + \frac{zf'(z)}{pf(z)}}{\frac{zf'(z)}{pf(z)} + 2\delta - 1} \right| \leq \frac{\sum\limits_{n=p}^{\infty} (n+p)\,|a_n|\,|z|^{n+p}}{2p\,(1-\delta) - \sum\limits_{n=p}^{\infty} [n - p(1-2\delta)]\,|a_n|\,|z|^{n+p}} \tag{38}$$

Thus, we have the desired inequality

$$\left| \frac{1 + \frac{zf'(z)}{pf(z)}}{\frac{zf'(z)}{pf(z)} + 2\delta - 1} \right| \leq 1 \qquad (0 \leq \delta < 1; p \in \mathbb{N}), \tag{39}$$

if

$$\sum_{n=p}^{\infty} \frac{(n+\delta p)}{p\,(1-\delta)}\,|a_n|\,|z|^{n+p} \leq 1, \tag{40}$$

that is, if

$$\frac{(n+\delta p)}{p(1-\delta)}\,|z|^{n+p} \leq \frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \left| \left(\frac{d}{n+d} \right)^s \right| \frac{(\lambda)_n}{(1)_n} \tag{41}$$

$$(n = p, p+1, p+2, ...; p \in \mathbb{N}).$$

The last inequality (41) leads us immediately to the disc $|z| < r_1$, where r_1 is given by (36).

(ii) In order to prove the second assertion of Theorem 8, we find from the definition (8) that

$$\left| \frac{1 + \frac{(zf'(z))'}{pf'(z)}}{\frac{(zf'(z))'}{pf'(z)} + 2\delta - 1} \right| \leq \frac{\sum\limits_{n=p}^{\infty} n\,(n+p)\,|a_n|\,|z|^{n+p}}{2p^2\,(1-\delta) - \sum\limits_{n=p}^{\infty} n\,[n - p(1-2\delta)]\,|a_n|\,|z|^{n+p}} \tag{42}$$

Thus, we have the desired inequality

$$\left| \frac{1 + \frac{(zf'(z))'}{pf'(z)}}{\frac{(zf'(z))'}{pf'(z)} + 2\delta - 1} \right| \leq 1 \qquad (0 \leq \delta < 1; p \in \mathbb{N}), \tag{43}$$

if

$$\sum_{n=p}^{\infty} \frac{n\,(n+\delta p)}{p^2\,(1-\delta)}\,|a_n|\,|z|^{n+p} \leq 1, \tag{44}$$

that is, if

$$\frac{n(n+\delta p)}{p^2(1-\delta)}\,|z|^{n+p} \leq \frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)}.\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n} \tag{45}$$

$$(n = p, p+1, p+2, ...; p \in \mathbb{N}).$$

The last inequality (45) leads us immediately to the disc $|z| < r_2$, where r_2 is given by (37). The proof of Theorem 8 is completed.

Theorem 9 *Let $\nu > 0$. If $f \in \overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$, the function $F(z)$ given by*

$$F(z) = \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t)dt$$

belongs to the class $\overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$.

Proof. Suppose that $f \in \overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$, from (26), we have

$$\sum_{n=p}^{\infty} \frac{[(n+p)\,(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_n}{(1)_n}\,|a_n| \leq 1.$$

Since

$$F(z) = \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu-p-1} f(t)dt = z^{-p} + \sum_{n=p}^{\infty} \frac{\nu}{\nu+n+p}.\,|a_n|\,z^k,$$

we obtain

$$\begin{aligned}
&\sum_{n=p}^{\infty} \frac{[(n+p)\,(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_n}{(1)_n}.\frac{\nu}{\nu+n+p}.\,|a_n| \\
\leq\ &\sum_{n=p}^{\infty} \frac{[(n+p)\,(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_n}{(1)_n}\,|a_n| \\
\leq\ &1,
\end{aligned}$$

which implies that $F(z) \in \overset{+}{\mathcal{M}}{}_d^{s,\lambda}(p;\beta,A,B)$.

This evidently completes the proof of Theorem 9.

Neighborhoods and Partial Sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Rusceheweyh [21], and (more recently) by Altintas [2], J. L. Liu and H. M. Srivastava [14], and M. S. Liu and N. S. Song [13], (see also

[1, 3, 4, 11, 18, 19, 22, 23, 24]), we define the $\delta-$neighborhood of a function $f(z) \in \Sigma_p$ of the form (1) by

$$\mathcal{N}_\delta(f) = \Bigg\{ g \in \Sigma_p : g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \text{ and } \sum_{n=1}^{\infty} \tfrac{n(1+|A-p\beta(A-B)|)+p(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \left| \left(\tfrac{d}{n+d}\right)^s \right| \tfrac{(\lambda)_n}{(1)_n} ||a_n| - |b_n|| \le \delta \Bigg\} \tag{46}$$

$$(d \in \mathbb{C}^*, s \in \mathbb{C}, \lambda > 0, p \in \mathbb{N}; -1 \le B < A \le 1; p\beta > 1)$$

and if

$$e(z) = z^{-p} \quad (p \in \mathbb{N}),$$

then

$$\mathcal{N}_\delta(e) = \Bigg\{ g \in \Sigma_p : g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \text{ and } \sum_{n=1}^{\infty} \tfrac{n(1+|A-p\beta(A-B)|)+p(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \left| \left(\tfrac{d}{n+d}\right)^s \right| \tfrac{(\lambda)_n}{(1)_n} |b_n| \le \delta \Bigg\}.$$

Theorem 10 *Let $f \in \mathcal{M}_d^{s,\lambda}(p; \beta, A, B), (-1 \le B < A \le 1; p\beta > 1)$ be given by (1). If f satisfies the condition*

$$\frac{f(z) + \epsilon z^{-p}}{1+\epsilon} \in \mathcal{M}_d^{s,\lambda}(p; \beta, A, B) \qquad (\epsilon \in \mathbb{C}; |\epsilon| < \delta; \delta > 0) \tag{47}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{M}_d^{s,\lambda}(p; \beta, A, B). \tag{48}$$

Proof. It is obvious from (7) that $g \in \mathcal{M}_d^{s,\lambda}(p; \beta, A, B)$ if and only if

$$-\frac{p\mathcal{J}_{p,d}^{s,\lambda} g(z) + z\left(\mathcal{J}_{p,d}^{s,\lambda} g(z)\right)'}{B[p\mathcal{J}_{p,d}^{s,\lambda} g(z) + z\left(\mathcal{J}_{p,d}^{s,\lambda} g(z)\right)'] + (1-p\beta)(A-B)z\left(\mathcal{J}_{p,d}^{s,\lambda} g(z)\right)'} \neq \sigma \tag{49}$$

$$(\sigma \in \mathbb{C}; |\sigma| = 1; z \in U^*),$$

which is equivalent to

$$\frac{(g * h)(z)}{z^{-p}} \neq 0 \qquad (z \in U^*), \tag{50}$$

where, for convenience,

$$h(z) = z^{-p} + \sum_{n=1}^{\infty} c_n z^{n-p}$$

$$(51)\quad = \sum_{n=1}^{\infty} \frac{n[1+\sigma(A-p\beta\,(A-B))]+p\sigma(A-B)(p\beta-1)}{p\sigma(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_n}{(1)_n} z^{n-p}.$$

We find from (51) that

$$|c_n| = \left|\frac{n[1+\sigma(A-p\beta\,(A-B))]+p\sigma(A-B)(p\beta-1)}{p\sigma(A-B)(p\beta-1)}\right| \left|\left(\frac{d}{n+d}\right)^s\right| \left|\frac{(\lambda)_n}{(1)_n}\right|$$

$$(52)\qquad \leq \frac{n(1+|A-p\beta\,(A-B)|)+p(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_n}{(1)_n}$$

$$(n=p,p+1,p+2,...;p\in\mathbb{N}).$$

Under the hypothesis of Theorem 10, (50) yields

$$(53)\qquad \left|\frac{(f*h)(z)}{z^{-p}}\right| \geq \delta \qquad (\delta>0;z\in U^*)\,.$$

Setting

$$(54)\qquad g(z)=z^{-p}+\sum_{n=1}^{\infty} b_n z^{n-p} \in \mathcal{N}_\delta(f),$$

we have

$$\left|\frac{(f(z)-g(z))*h(z)}{z^{-p}}\right|$$

$$=\sum_{n=1}^{\infty} |a_n-b_n|\, c_n z^n$$

(55)

$$\leq |z| \sum_{n=1}^{\infty} \frac{n(1+|A-p\beta\,(A-B)|)+p(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_n}{(1)_n} |a_n-b_n| < \delta$$

$$(56)\qquad (\delta>0;z\in U^*)\,.$$

Thus we have (50) and hence (49) for any $\sigma\in\mathbb{C}$ such that $|\sigma|=1$, which implies that $\mathcal{N}_\delta(f)\subset\mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$. This evidently proves the assertion (48) of Theorem 10.

We now define the $\delta-$neighborhood of a function $f(z)\in\Sigma_p$ of the form (8) by

$$\mathcal{N}_\delta^+(f) = \Bigg\{ g\in\Sigma_p : g(z)=z^{-p}+\sum_{n=p}^{\infty} |b_n|\, z^{n-p} \text{ and }$$

$$\sum_{n=p}^{\infty} \tfrac{n(1+|A-p\beta(A-B)|)+p(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \left|\left(\tfrac{d}{n+d}\right)^s\right| \tfrac{(\lambda)_n}{(1)_n} \,||a_n|-|b_n||$$

$$(57)\qquad \leq \delta \quad (d\in\mathbb{C}^*, s\in\mathbb{C}, \lambda>0, p\in\mathbb{N}; -1\leq B<A\leq 1; p\beta>1) \Bigg\},$$

Theorem 11 *If* $f \in \mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$ *and* $-1 \le B \le 0$, *then*

$$\mathcal{N}_\delta^+(f) \subset \mathcal{M}_d^{+s,\lambda}(p;\beta,A,B) \quad (\delta = \frac{2p}{\lambda+2p}). \tag{58}$$

The result is sharp.

Proof. Making use of the same method as in the proof of Theorem 10, we can show that

$$\begin{aligned} h(z) &= z^{-p} + \sum_{n=p}^{\infty} c_n z^n \\ &= \sum_{n=p}^{\infty} \frac{[(n+p)(1-\sigma B) + \sigma n(A-B)(p\beta-1)]}{p\sigma(B-A)(p\beta-1)} \left| \left(\frac{d}{n+d}\right)^s \right| \frac{(\lambda)_{n+p}}{(1)_{n+p}} z^n \end{aligned} \tag{59}$$

we find from (59) that

$$\begin{aligned} |c_n| &= \left| \frac{[(n+p)(1+|B|) + n(A-B)(p\beta-1)]}{p(B-A)(p\beta-1)} \left| \left(\frac{d}{n+d}\right)^s \right| \frac{(\lambda)_{n+p}}{(1)_{n+p}} \right| \\ &\le \frac{[(n+p)(1+|B|) + n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left| \left(\frac{d}{n+d}\right)^s \right| \frac{(\lambda)_{n+p}}{(1)_{n+p}} \\ &\qquad (n = p, p+1, ...; p \in \mathbb{N}). \end{aligned}$$

Thus, under hypothesis $-1 \le B \le 0$, if $f \in \mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$ is given by (8), we have

$$\left| \frac{(f*h)(z)}{z^{-p}} \right| = \left| 1 + \sum_{n=p}^{\infty} c_n |a_n| z^{n+p} \right|$$

$$\begin{aligned} &\ge 1 - \sum_{n=p}^{\infty} |c_n| |a_n| \\ &\ge 1 - \sum_{n=p}^{\infty} \tfrac{[(n+p)(1+|B|)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left| \left(\tfrac{d}{n+d}\right)^s \right| \frac{(\lambda+1)_{n+p}}{(1)_{n+p}} . \tfrac{\lambda}{\lambda+n+p} |a_n| \\ &\ge 1 - \tfrac{\lambda}{\lambda+2p} \sum_{n=p}^{\infty} \tfrac{[(n+p)(1+|B|)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left| \left(\tfrac{d}{n+d}\right)^s \right| \tfrac{(\lambda+1)_{n+p}}{(1)_{n+p}} |a_n| \\ &= 1 - \frac{\lambda}{\lambda+2p} \sum_{n=p}^{\infty} \frac{[(n+p)(1-B) + n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left| \left(\frac{d}{n+d}\right)^s \right| \frac{(\lambda+1)_{n+p}}{(1)_{n+p}} |a_n| . \end{aligned}$$

By Theorem 6, we obtain

$$\left| \frac{(f*h)(z)}{z^{-p}} \right| \ge 1 - \frac{\lambda}{\lambda+2p} = \frac{2p}{\lambda+2p} = \delta.$$

The remainder of the proof of Theorem 11 is similar to that of Theorem 10, and we skip the details involved.

Theorem 12 *Let* $(-1 \leq B < A \leq 1; p\beta > 1)$ *and* δ *be a real number with*

$$\delta > \frac{p(A-B)(p\beta-1)}{1+[A-p\beta(A-B)]}.$$

If the function $f(z)$ given by (8) is in the class $\mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$, then $F_\delta(z)$ defined by (16) belongs to $\mathcal{N}_1^+(f)$. The result is sharp.

Proof. Suppose that $f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_n| z^n \in \mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$, then it follows from (16) and Theorem 9 that

$$F_\delta(z) = z^{-p} + \sum_{n=p}^{\infty} |b_n| z^n = z^{-p} + \sum_{n=p}^{\infty} \frac{\delta}{\delta+p+n} |a_n| z^n \in \mathcal{M}_d^{+s,\lambda}(p;\beta,A,B). \tag{60}$$

From the hypothesis of Theorem 12, we have

$$\begin{aligned}
& \sum_{n=p}^{\infty} \tfrac{[(n+p)(1+|B|)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \left|\left(\tfrac{d}{n+d}\right)^s\right| \tfrac{(\lambda)_{n+p}}{(1)_{n+p}} \left||a_n|-|b_n|\right| \\
= & \sum_{n=p}^{\infty} \tfrac{[(n+p)(1+|B|)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \left|\left(\tfrac{d}{n+d}\right)^s\right| \tfrac{(\lambda)_{n+p}}{(1)_{n+p}} \cdot \tfrac{n+p}{\delta+n+p} |a_n| \\
\leq & \sum_{n=p}^{\infty} \tfrac{[(n+p)(1+|B|)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \left|\left(\tfrac{d}{n+d}\right)^s\right| \frac{(\lambda)_{n+p}}{(1)_{n+p}} |a_n| \\
\leq & \ 1 \quad (f \in \mathcal{M}_d^{+s,\lambda}(p;\beta,A,B),
\end{aligned}$$

which shows that $F_\delta(z) \in \mathcal{N}_1^+(f)$.

In order to verify the sharpness of the assertion Theorem 12, we consider the function $f(z)$ given by (30). From (30) and (60), we have

$$\begin{aligned}
F_\delta(z) &= \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} f(t)dt \\
&= \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} \left(t^{-p} + \frac{p(A-B)(p\beta-1)}{[(n+p)(1-B)+n(A-B)(p\beta-1)]} \cdot \left|\left(\frac{n+d}{d}\right)^s\right| \tfrac{(1)_n}{(\lambda)_n} t^n \right) dt \\
&= z^{-p} + \tfrac{p(A-B)(p\beta-1)}{[(n+p)(1-B)+n(A-B)(p\beta-1)]} \cdot \left|\left(\frac{n+d}{d}\right)^s\right| \frac{(1)_n}{(\lambda)_n} \cdot \frac{\delta}{\delta+n+p} z^n
\end{aligned}$$

$$(n = p, p+1, p+2, ...; p \in \mathbb{N}).$$

Thus, by making use of (57), we have

$$\begin{aligned}
& \sum_{n=p}^{\infty} \frac{n(1+|A-p\beta(A-B)|)+p(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_n}{(1)_n} \left||a_n|-|b_n|\right| \\
= & \ \frac{p+n}{\delta+n+p} \rightarrow 1 (n \longrightarrow \infty),
\end{aligned}$$

This completes the proof of Theorem 12.

Theorem 13 *Let* $(-1 \le B < A \le 1; p\beta > 1)$ *and* δ *be a real number with*

$$\delta > \frac{p(A-B)(p\beta-1)}{1+[A-p\beta(A-B)]}.$$

If $f \in \overset{+}{\mathcal{M}}_d^{s,\lambda+1}(p;\beta,A,B)$, then

$$\mathcal{N}_{\delta'}^{+}(F_\delta) \subset \overset{+}{\mathcal{M}}_d^{s,\lambda}(p;\beta,A,B) \qquad \left(\delta' = \frac{2p\,(\lambda+\delta+2p)}{(\lambda+2p)\,(\delta+2p)}\right) \tag{61}$$

and $F_\delta(z)$ defined by (16).

Proof. Making use of the same method as in the proof of Theorem 11, we can show that

$$\begin{aligned} h(z) &= z^{-p} + \sum_{n=p}^{\infty} c_n z^n \\ &= \sum_{n=p}^{\infty} \frac{[(n+p)\,(1-\sigma B)+\sigma n(A-B)(p\beta-1)]}{p\sigma(B-A)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_{n+p}}{(1)_{n+p}} z^n \end{aligned} \tag{62}$$

we find from (62) that

$$\begin{aligned} |c_n| &= \left| \frac{[(n+p)\,(1+|B|)+n(A-B)(p\beta-1)]}{p(B-A)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_{n+p}}{(1)_{n+p}} \right| \\ &\le \frac{[(n+p)\,(1+|B|)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda)_{n+p}}{(1)_{n+p}} \end{aligned}$$

$$(n = p, p+1, ...; p \in \mathbb{N}).$$

Thus, under hypothesis $-1 \le B \le 0$, if $F_\delta \in \overset{+}{\mathcal{M}}_d^{s,\lambda}(p;\beta,A,B)$ is given by (60), we have

$$\left|\frac{(F_\delta * h)(z)}{z^{-p}}\right| = \left|1 + \sum_{n=p}^{\infty} c_n\, |a_n|\, z^{n+p}\right|$$

$$\ge 1 - \sum_{n=p}^{\infty} |c_n|\, |a_n|$$

$$\ge 1 - \sum_{n=p}^{\infty} \frac{[(n+p)\,(1+|B|)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda+1)_{n+p}}{(1)_{n+p}} \frac{\delta}{\delta+p+n} \cdot \frac{\lambda}{\lambda+n+p} |a_n|$$

$$\ge 1 - \frac{\delta\lambda}{(\lambda+2p)\,(\delta+2p)} \sum_{n=p}^{\infty} \frac{[(n+p)\,(1+|B|)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda+1)_{n+p}}{(1)_{n+p}} |a_n|$$

$$= 1 - \frac{\delta\lambda}{(\lambda+2p)\,(\delta+2p)} \sum_{n=p}^{\infty} \frac{[(n+p)\,(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \left|\left(\frac{d}{n+d}\right)^s\right| \frac{(\lambda+1)_{n+p}}{(1)_{n+p}} |a_n|.$$

By Theorem 6, we obtain

$$\left|\frac{(F_\delta * h)(z)}{z^{-p}}\right| \geq 1 - \tfrac{\delta\lambda}{(\lambda+2p)(\delta+2p)} = \frac{2p\,(\lambda+\delta+2p)}{(\lambda+2p)\,(\delta+2p)} = \delta'.$$

Theorem 14 *Let $(-1 \leq B < A \leq 1; p\beta > 1)$ and $f \in \Sigma_p$ is given by (1) and define the partial sums $s_1(z)$ and $s_m(z)$ by*

$$s_1(z) = z^{-p} \text{ and } s_m(z) = z^{-p} + \sum_{n=1}^{m-1} a_n z^{n-p} \quad (m \in \mathbb{N}\setminus\{1\}). \tag{63}$$

Suppose that

$$\sum_{n=1}^{\infty} d_n\,|a_n| \leq 1 \quad (d_n = \tfrac{n(1+|A-p\beta(A-B)|)+p(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \left|\left(\tfrac{d}{n+d}\right)^s\right| \tfrac{(\lambda)_n}{(1)_n} \tag{64}$$

(i) If $s > 0, \lambda > 0$ and $d > 0$, then $f(z) \in \mathcal{M}_d^{s,\lambda}(p;\beta,A,B)$;
(ii) If $\lambda > 0$ and $d > 0$, then

$$Re\left\{\frac{f(z)}{s_m(z)}\right\} > 1 - \frac{1}{d_m} \qquad (m \in \mathbb{N}; z \in U) \tag{65}$$

and

$$Re\left\{\frac{s_m(z)}{f(z)}\right\} > \frac{d_m}{1+d_m} \qquad (m \in \mathbb{N}; z \in U) \tag{66}$$

Each of these bounds in (65) and (66) is the best possible for each $m \in \mathbb{N}$.
Proof. (i) It is not difficult to see that

$$z^{-p} \in \mathcal{M}_d^{s,\lambda}(p;\beta,A,B) \qquad (p \in \mathbb{N}).$$

Thus from Theorem 10 and hypothesis (64), we have

$$f(z) \in \mathcal{N}_1(e) \subset \mathcal{M}_d^{s,\lambda}(p;\beta,A,B) \quad (s > 0, \lambda > 0, d > 0; p \in \mathbb{N}) \tag{67}$$

as asserted by Theorem 14.

(ii) For the coefficient d_n given by (64), it easy to verify that

$$d_{n+1} > d_n \quad (s > 0, \lambda > 0, d > 0; n = p, p+1, p+2, ...; p \in \mathbb{N})\,. \tag{68}$$

So, we have

$$\sum_{n=1}^{m-1} |a_n| + d_m \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} d_n\,|a_n| \leq 1 \tag{69}$$

by using the hypothesis (64) again.

Setting

$$g_1(z) = d_m \left[\frac{f(z)}{s_m(z)} - (1 - \frac{1}{d_m}) \right] = 1 + \frac{d_m \sum\limits_{n=m}^{\infty} a_n z^n}{\sum\limits_{n=1}^{m-1} a_n z^n} \tag{70}$$

and applying (63), we have

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_m \sum\limits_{n=m}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{m-1} |a_n| - d_m \sum\limits_{n=m}^{\infty} |a_n|} \leq \frac{d_m \sum\limits_{n=m}^{\infty} |a_n|}{1 - \sum\limits_{n=1}^{m-1} |a_n|} \leq 1 \quad (z \in U) \tag{71}$$

which readily yields the assertion (65) of Theorem 14.

If we take

$$f(z) = z^{-p} + \frac{1}{d_m} z^{m-p}, \tag{72}$$

then, for

$$z = re^{i\frac{\pi}{m}},$$

we have

$$\frac{f(z)}{s_m(z)} = 1 + \frac{z^m}{d_m} \longrightarrow 1 - \frac{1}{d_m} \text{ as } z \longrightarrow 1^-,$$

which shows that the bound in (65) is the best possible for each $n \in \mathbb{N}$.

Similarly, if we put

$$g_2(z) = (1 + d_m) \left[\frac{s_m(z)}{f(z)} - \frac{d_m}{1 + d_m} \right] = 1 - \frac{(1 + d_m) \sum\limits_{n=m}^{\infty} a_n z^n}{1 + \sum\limits_{n=1}^{m-1} a_n z^n} \tag{73}$$

and make use (69) we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_m) \sum\limits_{n=m}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{m-1} |a_n| + (1 - d_m) \sum\limits_{n=m}^{\infty} |a_n|} \leq 1 \quad (z \in U) \tag{74}$$

which leads us immediately to the assertion (66) of Theorem 14.

The bound in (66) is sharp for each $m \in \mathbb{N}$, with the extremal function $f \in \Sigma_p$ given by (72). The proof of Theorem 14 is completed.

Properties involving modified Hadamard Product

Let the functions $f_j(z)$ $(j = 1, 2)$ be defined by

$$f_j(z) = z^{-p} + \sum_{n=p}^{\infty} |a_{n,j}| \, z^n \tag{75}$$

The modified Hadamard Product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=p}^{\infty} |a_{n,1}|\,|a_{n,2}|\, z^n \tag{76}$$

Theorem 15 *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (75) be in the class $\mathcal{M}_d^{+s,\lambda}(p; \beta, A, B)$. Then $(f_1 * f_2)(z) \in \mathcal{M}_d^{+s,\lambda}(p; \zeta, A, B)$, where*

$$\zeta = \frac{1}{p} - \frac{2(A-B)(p\beta-1)^2(1-B)}{p(A-B)^2(p\beta-1)^2 - p[2(1-B)+(A-B)(p\beta-1)]^2\left|\left(\frac{d}{p+d}\right)^s\right|\frac{(\lambda)_p}{(1)_p}} \tag{77}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \sum_{n=p}^{\infty} \frac{(A-B)(p\beta-1)}{2(1-B)+(A-B)(p\beta-1)}\left|\left(\frac{p+d}{d}\right)^s\right|\frac{(1)_p}{(\lambda)_p} z^n \tag{78}$$
$$(j = 1, 2; p \in \mathbb{N}).$$

Proof. Employing the techniques used earlier by Schild and Silverman [22], we need to find the largest $\zeta = (p; \zeta, A, B)$ such that,

$$\sum_{n=p}^{\infty} \frac{[(n+p)(1-B)+n(A-B)(p\zeta-1)]}{p(A-B)(p\zeta-1)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}|a_{n,1}|\,|a_{n,2}| \leq 1. \tag{79}$$

Since $f_j(z) \in \mathcal{M}_d^{+s,\lambda}(p; \beta, A, B)$ $(j = 1, 2)$, we have

$$\sum_{n=p}^{\infty} \frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}|a_{n,j}| \leq 1 \quad (j = 1, 2). \tag{80}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=p}^{\infty} \frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq 1. \tag{81}$$

Thus it is sufficient to show that

$$\begin{aligned} &\frac{[(n+p)(1-B)+n(A-B)(p\zeta-1)]}{(p\zeta-1)}|a_{n,1}|\,|a_{n,2}| \\ \leq\ &\frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{(p\beta-1)}\sqrt{|a_{n,1}|\,|a_{n,2}|}, \end{aligned} \tag{82}$$

or, equivalently that

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{[(n+p)(1-B)+n(A-B)(p\beta-1)](p\zeta-1)}{[(n+p)(1-B)+n(A-B)(p\zeta-1)](p\beta-1)}\ (k \geq p). \tag{83}$$

Hence, by inequality (81), it is sufficient to prove that

$$\sum_{n=p}^{\infty}\frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)\left|\left(\frac{d}{n+d}\right)^{s}\right|\frac{(\lambda)_n}{(1)_n}} \leq \frac{[(n+p)(1-B)+n(A-B)(p\beta-1)](p\zeta-1)}{[(n+p)(1-B)+n(A-B)(p\zeta-1)](p\beta-1)}. \tag{84}$$

It follows from (84) that

$$\zeta \leq \frac{1}{p}-\frac{(A-B)(p\beta-1)^2(1-B)(n+p)}{np(A-B)^2(p\beta-1)^2-[(n+p)(1-B)+n(A-B)(p\beta-1)]^2\left|\left(\frac{d}{n+d}\right)^{s}\right|\frac{(\lambda)_n}{(1)_n}}. \tag{85}$$

Now, defining the function $\varphi(n)$ by

$$\varphi(n)=\frac{1}{p}-\frac{(A-B)(p\beta-1)^2(1-B)(n+p)}{np(A-B)^2(p\beta-1)^2-[(n+p)(1-B)+n(A-B)(p\beta-1)]^2\left|\left(\frac{d}{p+d}\right)^{s}\right|\frac{(\lambda)_n}{(1)_n}}\quad (n\geq p). \tag{86}$$

We see that $\varphi(n)$ is an increasing function of n $(n \geq p)$. Therefore we conclude that

$$\zeta \leq \varphi(p)=\frac{1}{p}-\frac{2(A-B)(p\beta-1)^2(1-B)}{p(A-B)^2(p\beta-1)^2-p[2(1-B)+(A-B)(p\beta-1)]^2\left|\left(\frac{d}{p+d}\right)^{s}\right|\frac{(\lambda)_p}{(1)_p}}, \tag{87}$$

which completes the proof of Theorem 15.

Theorem 16 *Let the function $f_1(z)$ defined by (75) be in the class $\mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$ and the function $f_2(z)$ defined by (75) be in the class $\mathcal{M}_d^{+s,\lambda}(p;\mu,A,B)$. Then $(f_1 * f_2)(z) \in \mathcal{M}_d^{+s,\lambda}(p;\gamma,A,B)$, where*

$$\gamma=\frac{1}{p}-\frac{2(A-B)(p\beta-1)(1-B)(p\mu-1)}{p(A-B)^2(p\beta-1)(p\mu-1)-pMN\left|\left(\frac{d}{p+d}\right)^{s}\right|\frac{(\lambda)_p}{(1)_p}}, \tag{88}$$

where $M=[2(1-B)+(A-B)(p\beta-1)]$ and $N=[2(1-B)+(A-B)(p\mu-1)]$. The result is sharp for the function $f_j(z)\quad(j=1,2)$ given by

$$f_1(z)=z^{-p}+\sum_{n=p}^{\infty}\frac{(A-B)(p\beta-1)}{2(1-B)+(A-B)(p\beta-1)}\left|\left(\frac{p+d}{d}\right)^{s}\right|\frac{(1)_p}{(\lambda)_p}z^n\quad(p\in\mathbb{N}). \tag{89}$$

and

$$f_2(z)=z^{-p}+\sum_{n=p}^{\infty}\frac{(A-B)(p\mu-1)}{2(1-B)+(A-B)(p\mu-1)}\left|\left(\frac{p+d}{d}\right)^{s}\right|\frac{(1)_p}{(\lambda)_p}z^n\quad(p\in\mathbb{N}). \tag{90}$$

Theorem 17 *Let the functions $f_j(z)$ $(j=1,2)$ defined by (75) be in the class , then the function $h(z)$ defined by*

$$h(z)=z^{-p}+\sum_{n=p}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n \tag{91}$$

belongs to the class $\mathcal{M}_d^{+s,\lambda}(p;\eta,A,B)$, where

$$\eta=\frac{1}{p}-\frac{4(A-B)(p\beta-1)^2(1-B)}{2p(A-B)^2(p\beta-1)^2-p[2(1-B)+(A-B)(p\beta-1)]^2\left|\left(\frac{d}{p+d}\right)\right|^s\frac{(\lambda)p}{(1)p}}$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ are given by (78).

Proof. For $f_j(z)\in\mathcal{M}_d^{+s,\lambda}(p;\beta,A,B)$ $\quad(j=1,2)$, we have

$$\sum_{n=p}^{\infty}\frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}|a_{n,j}|\leq 1.$$

Therefore,

$$\sum_{n=p}^{\infty}\left[\frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}\right]^2|a_{n,j}|^2$$

$$\left[\sum_{n=p}^{\infty}\frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}|a_{n,j}|\right]^2$$

$$(92)\quad\leq\quad 1\quad (j=1,2).$$

So,

$$\sum_{n=p}^{\infty}\frac{1}{2}\left[\frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}\right]^2\left[|a_{n,j}|^2+|a_{n,j}|^2\right]\leq 1.$$

In order to obtain our result, we have to find the largest η such that

$$\sum_{n=p}^{\infty}\frac{[(n+p)(1-B)+n(A-B)(p\eta-1)]}{p(A-B)(p\eta-1)}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}\left[|a_{n,j}|^2+|a_{n,j}|^2\right]\leq 1.$$

It is sure if

$$\frac{[(n+p)(1-B)+n(A-B)(p\eta-1)]}{p(A-B)(p\eta-1)}\leq\frac{1}{2}\frac{[(n+p)(1-B)+n(A-B)(p\beta-1)]^2}{p(A-B)(p\beta-1)^2}\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n},$$

so that

$$\eta\leq\frac{1}{p}-\frac{(n+p)(A-B)(p\beta-1)^2(1-B)}{np(A-B)^2(p\beta-1)^2-\frac{1}{2}[(n+p)(1-B)+n(A-B)(p\beta-1)]^2\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}}.$$

Now, we define the function $\psi(n)$ by

$$\psi(n)=\frac{1}{p}-\frac{4(A-B)(p\beta-1)^2(1-B)}{np(A-B)^2(p\beta-1)^2-\frac{1}{2}[(n+p)(1-B)+n(A-B)(p\beta-1)]^2\left|\left(\frac{d}{n+d}\right)^s\right|\frac{(\lambda)_n}{(1)_n}}\quad(n\geq p).$$

We see that $\psi(n)$ is an increasing function of n $(n\geq p)$. Therefore we conclude that

$$\eta\leq\psi(n)=\frac{1}{p}-\frac{4(A-B)(p\beta-1)^2(1-B)}{2p(A-B)^2(p\beta-1)^2-p[2(1-B)+(A-B)(p\beta-1)]^2\left|\left(\frac{d}{p+d}\right)\right|^s\frac{(\lambda)p}{(1)p}},$$

which completes the proof of Theorem 17.

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Riesz triple almost lacunary χ^3 sequence spaces defined by a Orlicz function

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Abstract

In this paper we introduce a new concept for Riesz almost lacunary χ^3 sequence spaces strong $P-$ convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of Riesz almost lacunary χ^3 sequence spaces and also some inclusion theorems are discussed.

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1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [7], Subramanian et al. [8-14], and many others. Later on investigated by some initial work on triple sequence spaces is found in Sahiner et al. [15] , Esi et al. [2-6], Subramanian et al. [16-25], V.N. Mishra [31] and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence $(S_{mnk})$is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} (m, n, k = 1, 2, 3, ...) .$$

A sequence $x = (x_{mnk})$is said to be triple analytic if

$$sup_{m,n,k} \left| x_{mnk} \right|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$\left| x_{mnk} \right|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric

$$d(x, y) = sup_{m,n,k} \left\{ \left| x_{mnk} - y_{mnk} \right|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, ... \right\}, \quad (1)$$

forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ $in\, \Gamma^3$. Let $\phi = \{finite\ sequences\}$.

Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \delta_{ijq}$ for all $m, n, k \in \mathbb{N}$,

$$\delta_{ijq} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & . & .... & . & ... \end{bmatrix}$$

with 1 in the $[i, j, q]^{th}$ section and zero otherwise.

A sequence $x = (x_{mnk})$ is called triple gai sequence if $((m + n + k)! \left| x_{mnk} \right|)^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The triple gai sequences will be denoted by χ^3.

2 Definitions and Preliminaries

A triple sequence $x = (x_{mnk})$ has limit 0 (denoted by $P - limx = 0$)
(i.e) $((m + n + k)! \left| x_{mnk} \right|)^{1/m+n+k} \to 0$ as $m, n, k \to \infty$. We shall write more briefly as $P - convergent\ \ to\ \ 0$.

Definition 1 *A modulus function was introduced by Nakano [26]. We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that*
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from here that f is continuous on $[0, \infty)$.

Definition 2 *A triple sequence $x = (x_{mnk})$ of real numbers is called almost $P-$ convergent to a limit 0 if*

$$P-lim_{p,q,u\to\infty} sup_{r,s,t\geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} \left((m+n+k)!\,|x_{mnk}|\right)^{1/m+n+k} \to 0.$$

that is, the average value of (x_{mnk}) taken over any rectangle $\{(m,n,k) : r \leq m \leq r+p-1, s \leq n \leq s+q-1, t \leq k \leq t+u-1\}$ tends to 0 as both p, q and u to ∞, and this $P-$ convergence is uniform in i, ℓ and j. Let denot the set of sequences with this property as $\left[\widehat{\chi^3}\right]$.

Definition 3 *Let $(q_{rst}), (\overline{q_{rst}}), (\overline{\overline{q_{rst}}})$ be sequences of positive numbers and*

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0\ldots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = q_{11} + q_{12} + \ldots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0\ldots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \cdots & \overline{\overline{q}}_{1s} & 0\ldots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \cdots & \overline{\overline{q}}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \cdots & \overline{\overline{q}}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0.$$ *Then the transformation is given by*

$T_{rst} = \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left((m+n+k)!\,|x_{mnk}|\right)^{1/m+n+k}$ *is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - lim x = 0$.*

Definition 4 *The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that*

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by
$I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell \, and \, k_{j-1} < k \leq k_j\}, q_k = \frac{m_k}{m_{k-1}}$, $\overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.
Using the notations of lacunary sequence and Riesz mean for triple sequences.
$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive real numbers such that $Q_{m_i} = \sum_{m \in (0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n \in (0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k \in (0,k_j]} p_{k_j}$ and $H_i = \sum_{m \in (0,m_i]} p_{m_i}, \overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k \in (0,k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty, \overline{\overline{H}} = \sum_{k \in (0,k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \{(Q_{m_i} Q_{n_j} Q_{k_k})\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty, \overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.
Throughout the paper, we assume that $Q_r = q_{11} + q_{12} + \ldots + q_{rs} \to \infty \, (r \to \infty), \overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \to \infty \, (s \to \infty), \overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \ldots + \overline{\overline{q}}_{rs} \to \infty \, (t \to \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ as $j \to \infty$.
Let $Q_{m_i, n_\ell, k_j} = Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i \overline{H}_\ell \overline{\overline{H}}_j$,
$I'_{i\ell j} = \left\{(m,n,k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell} \, and \, \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\}$,
$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ and $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$. and $V_{i\ell j} = V_i \overline{V}_\ell \overline{\overline{V}}_j$.
If we take $q_m = 1, \overline{q}_n = 1 \, and \, \overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.

Let f be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:
$\left[\chi^3_R, \theta_{i\ell j}, q, f, p\right] = P - lim_{i,\ell,j \to \infty}$
$\frac{1}{H_{i,\ell j}} \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0$, uniformly in i, ℓ and j.

$$\left[\Lambda^3_R, \theta_{i\ell j}, q, f, p\right] = \left\{ x = (x_{mnk}) : P - sup_{i,\ell,j} \frac{1}{H_{i,\ell j}} \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \right.$$

$$\left. \left[f \left|x_{m+i,n+\ell,k+j}\right|^{p_{mnk}}\right] < \infty \right\},$$

uniformly in i, ℓ and j.
Let f be an Orlicz function, $p = p_{mnk}$ be any factorable double sequence of strictly positive real numbers and and $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11} \cdots \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} \cdots \overline{\overline{q}}_{rs}$,
If we choose $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k, then we obtain the

following sequence spaces.
$\left[\chi^3_R, q, f, p\right] = P - lim_{r,s,t\to\infty}$
$\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0$, uniformly in i, ℓ and j.

$$\left[\Lambda^3_R, q, f, p\right] = \left\{P - sup_{r,s,t}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\right.$$

$$\left.\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] < \infty\right\},$$

uniformly in i, ℓ and j.

Definition 5 *Let f be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space: $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence*

$$\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right] = \left\{P - lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\right.$$

$$\left.\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0,\right\},$$

uniformly in i, ℓ and j.

We shall denote $\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right]$ as $\chi^3\left[AC_{\theta_{i,\ell,j}}, p\right]$ respectively when $p_{mnk} = 1$ for all m, n and k If x is in $\chi^3\left[AC_{\theta_{i,\ell,j}}, p\right]$, we shall say that x is almost lacunary χ^3 strongly $p-$convergent with respect to the Orlicz function f. Also note if $f(x) = x, p_{mnk} = 1$ for all m, n and k then $\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right] = \chi^3\left[AC_{\theta_{i,\ell,j}}\right]$ which are defined as follows:

$$\chi^3\left[AC_{\theta_{i,\ell,j}}\right] = \left\{P - lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\right.$$

$$\left.\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right] = 0,\right\},$$

uniformly in i, ℓ and j.
Again note if $p_{mnk} = 1$ for all m, n and k then $\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right] = \chi^3_f\left[AC_{\theta_{i,\ell,j}}\right]$, we define

$$\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right] = \left\{P - lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{k,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\right.$$

$$\left.\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0,\right\},$$

uniformly in i, ℓ and j.

Definition 6 *Let f be an Orlicz function $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space:*

$$\chi^3_f[p] = \left\{ P - lim_{r,s,t\to\infty} \frac{1}{rst} \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} \right.$$

$$\left. \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} = 0 \right\},$$

uniformly in i, ℓ and j.

If we take $f(x) = x, p_{mnk} = 1$ for all m, n and k then $\chi^3_f[p] = \chi^3$.

Definition 7 *Let $\theta_{i,\ell,j}$ be a triple lacunary sequence; the triple number sequence x is $\widehat{S_{\theta i,\ell,j}} - p-$ convergent to 0 then*

$$P - lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} max_{i,\ell,j} \left| \{ (m,n,k) \in I_{i,\ell,j} : \right.$$

$$\left. f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} - 0 \right| \right)^{1/m+n+k} \right\} \Big| = 0.$$

In this case we write $\widehat{S_{\theta i,\ell,j}} - lim \left(f(m+n+k)! \left| x_{m+i,n+\ell,k+j} - 0 \right| \right)^{1/m+n+k} = 0.$

3 Main Results

Theorem 1 *If f be any Orlicz function and a bounded factorable positive triple number sequence p_{mnk} then $\chi^3_f \left[AC_{\theta_{i,\ell,j}}, P \right]$ is linear space*
Proof: The proof is easy. Theorefore, we omit the proof.

Theorem 2 *For any Orlicz function f, we have $\chi^3 \left[AC_{\theta_{i,\ell,j}} \right] \subset \chi^3_f \left[AC_{\theta_{i,\ell,j}} \right]$*
Proof: Let $x \in \chi^3 \left[AC_{\theta_{i,\ell,j}} \right]$ so that for each i, ℓ and j
$\chi^3 \left[AC_{\theta_{i,\ell,j}} \right] =$
$\left\{ lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}} \left[\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right] = 0 \right\}.$
Since f is continuous at zero, for $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \le t \le \delta$. We obtain the following,
$\frac{1}{h_{i\ell j}} (h_{i\ell j}\epsilon) +$
$\frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}\, and \left| x_{m+i,n+\ell,k+j} - 0 \right| > \delta} f \left[\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]$
$\frac{1}{h_{i\ell j}} (h_{i\ell j}\epsilon) + \frac{1}{h_{i\ell j}} K\delta^{-1} f(2)\, h_{i\ell j}\, \chi^3 \left[AC_{\theta_{i,\ell,j}} \right].$
Hence i, ℓ and j goes to infinity, we are granted $x \in \chi^3_f \left[AC_{\theta_{i,\ell,j}} \right].$

Theorem 3 *Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a triple lacunary sequence with $liminf_i q_i > 1$, $liminf_\ell \overline{q_\ell} > 1$ and $liminf_j q_j > 1$ then for any Orlicz function f, $\chi^3_f(P) \subset \chi^3_f \left(AC_{\theta_{i,\ell,j}}, P \right)$*
Proof: *Suppose $liminf_i q_i > 1$, $liminf_\ell \overline{q_\ell} > 1$ and $liminf_j q_j > 1$ then there exists*

$\delta > 0$ *such that* $q_i > 1+\delta$, $\overline{q_\ell} > 1+\delta$ *and* $q_j > 1+\delta$ *This implies* $\frac{h_i}{m_i} \geq \frac{\delta}{1+\delta}$, $\frac{h_\ell}{n_\ell} \geq \frac{\delta}{1+\delta}$ *and* $\frac{h_j}{k_j} \geq \frac{\delta}{1+\delta}$ *Then for* $x \in \chi^3_f(P)$, *we can write for each* r, s *and* u.

$$B_{i,\ell,j} = \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} =$$

$$\frac{1}{h_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$$

$$\frac{1}{h_{i\ell j}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{i-1}} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$$

$$\frac{1}{h_{i\ell j}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$$

$$\frac{1}{h_{i\ell j}} \sum_{k=k_j+1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$

$$= \frac{m_i n_\ell k_j}{h_{i\ell j}} \left(\frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$$

$$\frac{m_{k-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \left(\frac{1}{m_{i-1} n_{\ell-1} kj - 1} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

$$-\frac{k_{j-1}}{h_{i\ell j}} \left(\frac{1}{k_{j-1}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

$$-\frac{n_{\ell-1}}{h_{i\ell j}} \left(\frac{1}{n_{\ell-1}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$$

$$\frac{m_{k-1}}{h_{i\ell j}} \left(\frac{1}{m_{k-1}} \sum_{k=1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right).$$

Since $x \in \chi^3_f(P)$ *the last three terms tend to zero uniformly in* m, n, k *in the sense, thus, for each* r, s *and* u

$$B_{i,\ell,j} = \frac{m_i n_\ell k_j}{h_{i\ell j}} \left(\frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$$

$$\frac{m_{i-1}n_{\ell-1}k_{j-1}}{h_{i\ell j}}\left(\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}}\right.$$
$$\left. f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)+O\left(1\right).$$

Since $h_{i\ell j}=m_in_\ell k_j-m_{i-1}n_{\ell-1}k_{j-1}$ *we are granted for each* i,ℓ *and* j *the following*

$$\tfrac{m_in_\ell kj}{h_{i\ell j}}\leq\tfrac{1+\delta}{\delta}\text{ and }\tfrac{m_{i-1}n_{\ell-1}k_{j-1}}{h_{i\ell j}}\leq\tfrac{1}{\delta}.$$

The terms $\left(\frac{1}{m_in_\ell k_j}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j}f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$ *and* $\left(\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}}f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$ *are both gai sequences for all* i,ℓ *and* j. *Thus* $B_{i\ell j}$ *is a gai sequence for each* i,ℓ *and* j. *Hence* $x\in\chi^3_f\left(AC_{\theta_{i,\ell,j}},P\right)$.

Theorem 4 *Let* $\theta_{i,\ell,j}=\{m,n,k\}$ *be a triple lacunary sequence with* $limsup_\eta q_\eta<\infty$ *and* $limsup_i\overline{q_i}<\infty$ *then for any Orlicz function* f, $\chi^3_f\left(AC_{\theta_{i,\ell,j}},P\right)\subset\chi^3_f\left(p\right)$.

Proof. *Since* $limsup_iq_i<\infty$ *and* $limsup_i\overline{q_i}<\infty$ *there exists* $H>0$ *such that* $q_i<H$, $\overline{q_\ell}<H$ *and* $q_j<H$ *for all* i,ℓ *and* j. *Let* $x\in\chi^3_f\left(AC_{\theta_{i,\ell,j}},P\right)$. *Also there exist* $i_0>0,\ell_0>0$ *and* $j_0>0$ *such that for every* $a\geq i_0$ $b\geq\ell_0$ *and* $c\geq j_0$ *and* i,ℓ *and* j.

$$A'_{abc}=\frac{1}{h_{abc}}\sum_{m\in I_{a,b,c}}\sum_{n\in I_{a,b,c}}\sum_{k\in I_{a,b,c}}f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\to 0\,as\,m,n,k\to\infty.$$

Let $G'=max\left\{A'_{a,b,c}:1\leq a\leq i_0,\ 1\leq b\leq\ell_0\text{ and }1\leq c\leq j_0\right\}$ *and* p,q *and* t *be such that* $m_{i-1}<p\leq m_i$, $n_{\ell-1}<q\leq n_\ell$ *and* $m_{j-1}<t\leq m_j$. *Thus we obtain the following:*

$$\frac{1}{pqt}\sum_{m=1}^{p}\sum_{n=1}^{q}\sum_{k=1}^{t}\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$\leq\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j}\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$\leq\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{a=1}^{i}\sum_{b=1}^{\ell}\sum_{c=1}^{j}$$
$$\left(\sum_{m\in I_{a,b,c}}\sum_{n\in I_{a,b,c}}\sum_{k\in I_{a,b,c}}\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$
$$=\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{a=1}^{i_0}\sum_{b=1}^{\ell_0}\sum_{c=1}^{j_0}h_{a,b,c}A'_{a,b,c}+$$
$$\frac{1}{m_{k-1}n_{\ell-1}k_{j-1}}\sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)}h_{a,b,c}A'_{a,b,c}$$
$$\leq\frac{G'}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{a=1}^{i_0}\sum_{b=1}^{\ell_0}\sum_{c=1}^{j_0}h_{a,b,c}$$
$$+\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq J)}h_{a,b,c}A'_{a,b,c}$$
$$\leq\frac{G'm_{i_0}n_{\ell_0}k_{j_0}i_0\ell_0j_0}{m_{i-1}n_{\ell-1}k_{j-1}}+\frac{1}{m_{i-1}n_{\ell-1}j_{j-1}}\sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)}h_{a,b,c}A'_{a,b,c}$$
$$\leq\frac{G'm_{i_0}n_{\ell_0}k_{j_0}i_0\ell_0j_0}{m_{i-1}n_{\ell-1}k_{j-1}}$$

$+ \left(sup_{a \geq i_0 \bigcup b \geq \ell_0 \bigcup c \geq j_0} A'_{a,b,c} \right) \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{(i_0 < a \leq i) \bigcup (\ell_0 < b \leq \ell) \bigcup (j_0 < c \leq j)} h_{a,b,c}$
$\leq \frac{G' m_{i_0} n_{\ell_0 k_{j_0}} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \frac{\epsilon}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{(i_0 < a \leq i) \bigcup (\ell_0 < b \leq \ell) \bigcup (j_0 < c \leq j)} h_{a,b,c}$
$\leq \frac{G' m_{i_0} n_{\ell_0 k_{j_0}} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \epsilon H^3.$

Since m_i, n_ℓ and k_j both approaches infinity as both p, q and t approaches infinity, it follows that

$$\frac{1}{pqt} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} \left[((m+n+k)! \, |x_{m+i, n+\ell, k+j}|)^{1/m+n+k} \right]^{p_{mnk}} = 0,$$

uniformly in i, ℓ and j.

Hence $x \in \chi^3_f (P)$.

Theorem 5 *Let $\theta_{i,\ell,j}$ be a triple lacunary sequence then*
(i) $(x_{mnk}) \stackrel{P}{\to} \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$
(ii)$\left(AC_{\theta_{i,\ell,j}} \right)$ is a proper subset of $\left(\widehat{S_{\theta_{i,\ell,j}}} \right)$
(iii) If $x \in \Lambda^3$ and $(x_{mnk}) \stackrel{P}{\to} \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$ then $(x_{mnk}) \stackrel{P}{\to} \chi^3 \left(AC_{\theta_{i,\ell,j}} \right)$
(iv) $\chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right) \bigcap \Lambda^3 = \chi^3 \left[AC_{\theta_{i,,\ell,j}} \right] \bigcap \Lambda^3$.

Proof. *(i) Since for all i, ℓ and j*
$\left| \left\{ (m,n,k) \in I_{i,\ell,j} : ((m+n+k)! \, |x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} \right\} = 0 \right| \leq$
$\sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \, and \, |x_{m+i,n+\ell,k+j}|=0} ((m+n+k)! \, |x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}$
$\leq \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} ((m+n)! \, |x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}$, *for all i, ℓ and j*
$P - lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} ((m+n+k)! \, |x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0$
This implies that for all i, ℓ and j

$$P - lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left| \{ (m,n,k) \in I_{i,\ell,j} : \right.$$

$$\left. ((m+n+k)! \, |x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0 \right\} \Big| = 0.$$

(ii)*let $x = (x_{mnk})$ be defined as follows:*

$$(x_{mnk}) = \begin{bmatrix} 1 & 2 & 3 & \ldots \frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots \\ 1 & 2 & 3 & \ldots \frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots \\ \vdots & & & & & \\ 1 & 2 & 3 & \ldots \frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots \\ \vdots & & & & & \end{bmatrix};$$

Here x is an trible sequence and for all i,ℓ and j
$P\!-\!lim_{i,\ell,j}\frac{1}{h_{k,\ell,j}}\left|\left\{(m,n,k)\in I_{i,\ell,j}:((m+n+k)!\left|x_{m+i,n+\ell,k+j}-0\right|)^{1/m+n+k}=0\right\}\right|=$

$P-lim_{i,\ell,j}\frac{1}{h_{i,\ell,j}}\left(\frac{(m+n+k)!\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right)^{1/m+n+k}=0.$

Therefore $(x_{mnk})\overset{P}{\to}\chi^3\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$. *Also*
$P-lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|)^{1/m+n+k}=$
$P-\frac{1}{2}\left(lim_{i,\ell,j}\frac{1}{h_{i,\ell,j}}\left(\frac{(m+n+k)!\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right)^{1/m+n+k}+1\right)=\frac{1}{2}.$

Therefore $(x_{mnk})\overset{P}{\not\to}\chi^3\left(AC_{\theta_{i,\ell,j}}\right)$.
(iii) *If* $x\in\Lambda^3$ *and* $(x_{mnk})\overset{P}{\to}\chi^3\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$ *then* $(x_{mnk})\overset{P}{\to}\chi^3\left(AC_{\theta_{i,\ell,j}}\right)$.
Suppose $x\in\Lambda^3$ *then for all* i,ℓ *and* j, $((m+n+k)!\left|x_{m+i,n+\ell,k+j}-0\right|)^{1/m+n+k}\leq M$ *for all* m,n,k. *Also for given* $\epsilon>0$ *and* i,ℓ *and* j *large for all* i,ℓ *and* j *we obtain the following:*

$$\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}((m+n+k)!\left|x_{m+i,n+\ell,k+j}-0\right|)^{1/m+n+k}=$$

$$\frac{1}{h_{i\ell j}}\sum_{m\in I_{k,\ell}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{k,\ell,j}\ and\ \left|x_{m+i,n+\ell,k+j}\right|\geq 0}$$

$$((m+n+k)!\left|x_{m+i,n+\ell,k+j}-0\right|)^{1/m+n+k}+$$

$$\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}\ and\ \left|x_{m+i,n+\ell,k+j}\right|\leq 0}$$

$$((m+n+k)!\left|x_{m+i,n+\ell,k+j}-0\right|)^{1/m+n+k}$$

$$\leq\frac{M}{h_{i\ell j}}\left|\left\{(m,n,k)\in I_{i,\ell,j}:((m+n+k)!\left|x_{m+i,n+\ell,k+j}-0\right|)^{1/m+n+k}\right\}=0\right|+\epsilon.$$

Therefore $x\in\Lambda^3$ *and* $(x_{mnk})\overset{P}{\to}\chi^3\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$ *then* $(x_{mnk})\overset{P}{\to}\chi^3\left(AC_{\theta_{i,\ell,j}}\right)$.
(iv)$\chi^3\left(\widehat{S_{\theta_{i,\ell,j}}}\right)\bigcap\Lambda^3=\chi^3\left[AC_{\theta_{i,\ell,j}}\right]\bigcap\Lambda^3$. *follows from (i),(ii) and (iii).*

Theorem 6 *If f be any Orlicz function then* $\chi^3_f\left[AC_{\theta_{i,\ell,j}}\right]\notin\chi^3\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$
Proof: *Let* $x\in\chi^3_f\left[AC_{\theta_{i,\ell,j}}\right]$, *for all* i,ℓ *and* j.
Therefore we have
$\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}f\left[((m+n+k)!\left|x_{m+i,n+\ell,k+j}-0\right|)^{1/m+n+k}\right]\geq$
$\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}\ and\ \left|x_{m+r,n+s,k+u}\right|=0}$
$f\left[((m+n+k)!\left|x_{m+i,n+\ell,k+j}-0\right|)^{1/m+n+k}\right]>$

$\frac{1}{h_{i\ell j}} f(0) \left| \left\{ (m,n,k) \in I_{i,\ell,j} : ((m+n+k)! \left| x_{m+i,n+\ell,k+j} - 0 \right|)^{1/m+n+k} \right\} = 0 \right|$.
Hence $x \notin \chi^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$.

4 Conclusions and Future Work

We introduced Riesz almost lacunary χ^3 sequence spaces strong $P-$ convergent to zero with respect to an Orlicz function and study statistical convergence of Riesz almost lacunary χ^3 sequence spaces, also some inclusion theorems. For the reference sections, consider the following introduction described the main results are motivating the research.

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Section of a summability method

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Abstract

In this paper, section of a summability method are determined, using random variables which follow Pascal′s distribution law and the central limit theorem.

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Key words and phrases: summability, Pascal, Toeplitz, Lyapunov, random variables, the regular transformation.

1 Introduction

Let $S = \|a_{n,k}\|_{n,k\in\mathbb{N}}$ be a real elements matrix. A sequence $(s_n)_{n\in\mathbb{N}}$ is said to be A-summable to the value $s \in \mathbb{R}$ if each of the series $\sigma_n = \sum_{k=0}^{n} a_{n,k}s_k$, $n = 0, 1, ...$ is convergent and if $\sigma_n \to s$ for $n \to \infty$. The method A is called regular if each convergent sequence is A-summable to its limit.

Theorem 1 *(Toeplitz) (see[2]) The summation method A is regular if and only if:*

$$\lim_{n\to\infty} a_{n,k} = 0,\ for\, every\ k\ natural, \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{n,k} = 1, \tag{2}$$

$$\sum_{k=0}^{n} |a_{n,k}| \le M, \tag{3}$$

for every n natural, M being a constant independent of n.

Note:

- $M(f)$ represents the expectation of a random variable f,
- $D^2(f)$ represents the variance (dispersion) of a random variable f,
- K represents the set of real sequences,
- $[x]$ represents the wolle part of x.

Theorem 2 *(Lyapunov) (see[3]) Let $(f_n)_{n\in\mathbb{N}}$ a sequence of independent random variables. Let us suppose that $M_k = M(f_k)$, $D_k^2 = D^2(f_k)$, $H_k = \sqrt[3]{M(|f_k - M_k|^3)}$ exists for every k natural. Note with $S_n = \sqrt{D_1^2 + ... + D_n^2}$, $K_n = \sqrt[3]{H_1^3 + ... + H_n^3}$, $\beta_n = f_1 + ... + f_n$, $\beta_n^* = \dfrac{\beta_n - M(\beta_n)}{D(\beta_n)}$ and with $F_{n,\beta_n^*}(x)$ the distribution function of variable β_n^*. Thus, if*

$$\lim_{n\to\infty} \frac{K_n}{S_n} = 0,\ we\,have \tag{4}$$

$$\lim_{n\to\infty} F_{n,\beta_n^*}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \Phi(x),\ for\,every\ x\ natural. \tag{5}$$

Function $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-t^2/2} dt$ represents the standard normal distribution function.

Theorem 2 is also true in the case when the independent random variables have the same distribution.

Theorem 3 *(see[1]) If a sequence of characteristic functions $f_1(t), f_2(t), ..., f_n(t), ...$ converge to the continuous functions $f(t)$, then the sequence of distribution functions $F_1(x), F_2(x), ..., F_n(x), ...$ converges weakly to some distribution function $F(x)\Phi$ [by virtue of the direct limit theorem $f(t) = \int e^{itx} dF(x)$].*

If the random variables η_1 and η_2 verify the condition $\eta_2 = a \cdot \eta_1 + b$ with a, b real, then

i) the characteristic functions verify

$$\varphi_{\eta_2}(t) = e^{itb} \varphi_{\eta_1}(at), \tag{6}$$

ii) the distribution functions of these random variable verify

$$F_{\eta_2}(x) = F_{\eta_1}\left(\frac{x-b}{a}\right),\ for\,a > 0. \tag{7}$$

Definition 1 *We say that the discrete random variable X follows Pascal's law if it has the distribution* $\begin{pmatrix} k \\ P(n,k) \end{pmatrix}_{k=0,1,2,...}$, *where* $P(n,k) = \binom{n+k-1}{k} p^n q^k$, $n \in \mathbb{N}^*$, $p \in (0,1)$, $q = 1-p$ *(see[2]).*

For $n = 1$, *the discrete random variable follows the geometric law, that is it has the distribution*

$$\begin{pmatrix} k \\ p \cdot q^k \end{pmatrix}_{k=0,1,2,...}, \; p \in (0,1), \; q = 1-p.$$

2 Principal results

We consider the formula

$$(8) \qquad \frac{1}{(1-z)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} z^k, \; |z| < 1, \; \alpha \in \mathbb{R}_+,$$

$$(\alpha)_k = \alpha(\alpha+1) \cdot ... \cdot (\alpha+k-1) \text{ with the convention } (\alpha)_0 = 1.$$

If we put $\alpha = n\lambda, n \in \mathbb{N}, \lambda \in \mathbb{R}_+$ we obtain

$$(9) \qquad \frac{1}{(1-z)^{n\lambda}} = \sum_{k=0}^{\infty} \frac{(n\lambda)_k}{k!} z^k, \; |z| < 1.$$

We define the transformation $T_1(n, \cdot, \lambda, z) : K \to K$,

$$(10) \qquad T_1(n, s; \lambda, z) = \sum_{k=0}^{\infty} c_1(n, k; \lambda, z) s_k,$$

where

$$c_1(n, k; \lambda, z) = \frac{(n\lambda)_k}{k!} z^k (1-z)^{n\lambda}, z \in (0,1), n \in \mathbb{N}, \lambda \in \mathbb{R}_+.$$

Proposition 1 *The transformation* $T_1(n, s; \lambda, z)$ *is a regular.*

Proof. We check the conditions (1), (2) and (3) from theorem one:

$$\lim_{n\to\infty} c_1(n, k; \lambda, z) = \lim_{n\to\infty} \frac{(n\lambda)_k}{k!} z^k (1-z)^{n\lambda}$$

$$= \frac{z^k}{k!} \lim_{n\to\infty} [n\lambda(n\lambda+1)...(n\lambda+k-1)(1-z)^{n\lambda}]$$

$$= \frac{z^k}{k!} \lim_{n\to\infty} \left[\lambda \left(\lambda + \frac{1}{n} \right) ... \left(\lambda + \frac{k-1}{n} \right) \cdot \frac{n^k}{(1-z)^{-n\lambda}} \right] = 0;$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} c_1(n, k; \lambda, z) = 1, \; (see[9]);$$

$$\sum_{k=0}^{\infty} |c_1(n,k;\lambda,z)| = \sum_{k=0}^{\infty} c_1(n,k;\lambda,z) = 1 = M.$$

Next, we will determine a finite section of the regular transformation $T_1(n,s;\lambda,z)$. Let the independent random variable $f_1, f_2, ..., f_n$ which follows the geometric law and $\beta_n = f_1 + f_2 + ... + f_n$. The random variable β_n follows Pascal′s law and the distribution function is
$F_{n,\beta} : \mathbb{R} \to \mathbb{R}$,

$$F(x) = P(X < x) = \sum_{k=0}^{\infty} P(n,k)\Theta(x-k) \tag{11}$$

where $P(n,k)$ is defined in Definition 1.1 and $\Theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

Since $M(\beta_n) = \dfrac{nq}{p}$ and $D^2(\beta_n) = \dfrac{nq}{p^2}$, from (11) and from the relations between the distribution functions, we obtain the fact the distribution function of the normed random variable $\beta_n^* = \dfrac{\beta_n - M(\beta_n)}{D(\beta_n)}$ has the form

$$F_{n,\beta^*} = \sum_{k=0}^{\infty} P(n,k)\Theta\left(x\frac{\sqrt{nq}}{p} + \frac{nq\sqrt{nq}}{p^2} - k\right) \tag{12}$$

$$= \sum_{k=0}^{\left[x\frac{\sqrt{nq}}{p} - \frac{nq\sqrt{nq}}{p^2}\right]} \binom{n+k-1}{k} p^n q^k = \Phi(x)\ ([6]).$$

We defined the transformation $T_2(n,\cdot;p,x) : K \to K$,

$$T_2(n,s;p,x) = \sum_{k=0}^{\left[x\frac{\sqrt{nq}}{p} + \frac{nq\sqrt{nq}}{p^2}\right]} c_2(n,k;p,x) \cdot s_k \tag{13}$$

where

$$c_2(n,k;p,x) = \frac{1}{\Phi(x)} \cdot \binom{n+k-1}{k} p^n q^k,\ x \geq 0,\ \ q = 1-p,\ \ p \in (0,1).$$

Theorem 4 *The transformation $T_2(n,s;p,x)$ is regular.*

Proof. We check the conditions from theorem Toeplitz:

$$\lim_{n\to\infty} c_2(n,k;p,x) = \frac{q^k}{\Phi(x)} \lim_{n\to\infty} \binom{n+k-1}{k} p^n \tag{14}$$

$$= \frac{q^k}{\Phi(x)} \lim_{n\to\infty} \frac{(n+k-1)(n+k-2)...n}{k!} p^n$$

$$= \frac{q^k}{\Phi(x)k!} \lim_{n\to\infty} \left(1+\frac{k-1}{n}\right)\left(1+\frac{k-2}{n}\right)...\left(1+\frac{1}{n}\right) n^k p^n = 0;$$

From (12) we have

$$\sum_{k=0}^{\left[x\frac{\sqrt{nq}}{p}+\frac{nq\sqrt{nq}}{p^2}\right]} c_2(n,k;p,x) = 1; \tag{15}$$

$$\sum_{k=0}^{\left[x\frac{\sqrt{nq}}{p}+\frac{nq\sqrt{nq}}{p^2}\right]} |c_2(n,k;p,x)| = \sum_{k=0}^{\left[x\frac{\sqrt{nq}}{p}+\frac{nq\sqrt{nq}}{p^2}\right]} c_2(n,k;p,x) = 1. \tag{16}$$

In (13), let $x = 0, q = z$ and $n = n\lambda, \lambda > 0$; it follows that

$$T_2(n\lambda, s; z, 0) = 2(1-z)^{n\lambda} \cdot \sum_{k=0}^{\left[\frac{n\lambda z\sqrt{n\lambda z}}{(1-z)^2}\right]} \binom{n\lambda+k-1}{k} z^k s_k,\ z \in (0,1). \tag{17}$$

Remark 1 *The transformation* $\frac{1}{2}T_2(n\lambda; z, 0)$ *denotes a finite section of the transformation* $T_1(n,s;\lambda,z)$*, (see (10)).*

Particular case: In (10) and (14), let $z = \frac{1}{2}$; the following representations

$$T_1(n,s;\lambda,\frac{1}{2}) = \frac{1}{2^{n\lambda}} \sum_{k=0}^{\infty} \frac{(n\lambda)_k}{k!} \cdot \frac{s_k}{2^k} = \frac{1}{2^{n\lambda}} \sum_{k=0}^{\infty} \binom{n\lambda+k-1}{k} \frac{s_k}{2^k},$$

$$T_2(n\lambda, s; \frac{1}{2}, 0) = \frac{2}{2^{n\lambda}} \sum_{k=0}^{[n\lambda\sqrt{2n\lambda}]} \binom{n\lambda+k-1}{k} \frac{s_k}{2^k}.$$

In (13) let $x = 0$,

$$T_2(n,s;p,0) = 2 \sum_{k=0}^{\frac{[nq\sqrt{nq}]}{p^2}} \binom{n+k-1}{k} p^n q^k s_k,\ q = 1-p, p \in (0,1).$$

For $p = q = \frac{1}{2}$, we have

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{[n\sqrt{2n}]} \binom{n+k-1}{k} \frac{1}{2^k} = \frac{1}{2}.$$

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