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Some results on Orlicz spaces of entire sequences [1]

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Abstract

The spaces Γ_M and $\Gamma_M(p)$ were defined and studied by Rao and Subramanian in 2004, where Γ denote the space of entire sequences, M is a modulus function, and $p = (p_k)$ is a sequence of positive real numbers. In this paper, we introduce and study the spaces $\Gamma_M(\Delta_u^n)$ and $\Gamma_M(p, \Delta_u^n)$,where for any sequence $x = (x_k)$, the difference sequence Δx is defined as $\Delta x = (\Delta x_k)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$, n is a nonnegative integer, and $u = (u_k)$ is an arbitrary sequence such that $u_k \neq 0$ ($k = 1, 2, 3, \cdots$). We also study some topological properties of these spaces, some inclusion relations between them and some related results.

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1 Definitions and notations

Let w denote the set of all complex sequences $x = (x_k)$, and l_∞, c, and c_0 be the linear spaces of bounded, convergent, and null sequences with complex terms, respectively, normed by $\| x \| = \sup_k | x_k |$, where $k \in \mathbb{N}$, the set of positive integers.

A paranorm on a linear topological space X is a function $g : X \to \mathbb{R}$ which satisfies the following axioms :

for any $x, y, x_0 \in X$ and $\lambda, \lambda_0 \in \mathbb{C}$, the set of complex numbers,

(i) $g(\theta) = 0$, where $\theta = (0, 0, 0, \cdots)$, the zero sequence,

(ii) $g(x) = g(-x)$,

(iii) $g(x + y) \leq g(x) + g(y)$ (subadditivity),

and

(iv) the scalar multiplication is continuous, that is,

$$\lambda \to \lambda_0, x \to x_0 \text{ imply } \lambda x \to \lambda_0 x_0 \ ;$$

in other words,

$$| \lambda - \lambda_0 | \to 0, g(x - x_0) \to 0 \text{ imply } g(\lambda x - \lambda_0 x_0) \to 0.$$

A paranormed space is a linear space X with a paranorm g and is written (X, g), (see [5]).

Any function g which satisfies all the conditions (i)-(iv) together with the condition

(v) $g(x) = 0$ if and only if $x = \theta$,

is called a total paranorm on X, and the pair (X, g) is called a total paranormed space, (see [5]).

For any sequence $x = (x_k)$, the difference sequence Δx is defined by $\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty$.

Kizmaz [2] defined the sequence spaces

$$l_\infty(\Delta) = \{x \in w : \Delta x \in l_\infty\},$$

$$c\ (\Delta) = \{x \in w : \Delta x \in c\},$$

and

$$c_0(\Delta) = \{x \in w : \Delta x \in c_0\}.$$

Difference sequence spaces were studied by several authors (see [1, 7, 9]).

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$, (see [3]).

If convexity of M is replaced by $M(x + y) \leq M(x) + M(y)$, then it is called a modulus function, defined and studied by Nakano [8], Ruckle [11], Maddox [6] and others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of l, if there exist a constant $K > 0$ such that

$$M(2l) \leq KM(l)(l \geq 0);$$

equivalently,

$$M(hl) \leq KhM(l)$$

for every value of l and for $h > 1$.

An Orlicz function M can always be represented in the following integral form $M(x) = \int_0^x q(t)dt$, where q, known as the kernel of M, is right-differentiable for $t \geq 0, q(0) = 0, q(t) > 0$ for $t > 0, q$ is nondecreasing, and $q(t) \to \infty$ as $t \to \infty$.

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to define what is called an Orlicz sequence space :

$$l_M = \{x \in w : \sum_{k=1}^{\infty} M(\frac{\mid x_k \mid}{\rho}) < \infty, \text{ for some } \rho\ > 0\}$$

which is a Banach space with the norm :

$$\parallel x \parallel_M = \inf\{\rho\ > 0 : \sum_{k=1}^{\infty} M(\frac{\mid x_k \mid}{\rho}) \leq 1\}.$$

A sequence $x = (x_k)$ is said to be analytic if $\sup_k \mid x_k \mid^{\frac{1}{k}} < \infty$.

The vector space of all analytic functions will be denoted by Λ.

A sequence $x = (x_k)$ is called an entire sequence if $lim_{k \to \infty} \mid x_k \mid^{\frac{1}{k}} = 0$.

The vector space of all entire sequences will be denoted by Γ which is defined by

$$\Gamma = \{x \in w : \ \mid x_k \mid^{\frac{1}{k}} \to 0 \text{ as } k \to \infty\}.$$

Let $p = (p_k)$ be a sequence of positive integers, then the spaces Γ_M and $\Gamma_M(p)$ were defined and studied by Rao and Subramanian [10], which are given by

$$\Gamma_M = \{x \in w : \ M(\frac{\mid x_k \mid^{\frac{1}{k}}}{\rho}) \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \ \},$$

and

$$\Gamma_M(p) = \{x \in w : \ [M(\frac{\mid x_k \mid^{\frac{1}{k}}}{\rho})]^{p_k} \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \ \}.$$

In this paper, we generalize Γ_M and $\Gamma_M(p)$ to $\Gamma_M(\Delta_u^n)$ and $\Gamma_M(p, \Delta_u^n)$ as follows :

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ $(k = 1, 2, \cdots)$ and let n be a nonnegative integer, then

$\Delta_u^n x_k = (\Delta_u^{n-1} x_k - \Delta_u^{n-1} x_{k+1})$ so that $\Delta_u^n x_k = \sum\limits_{r=0}^{n} (-1)^r \binom{n}{r} u_{k+r} x_{k+r}$.

$\Delta_u^0 x_k = (u_k x_k)$, $\Delta_u x_k = (u_k x_k - u_{k+1} x_{k+1})$.

Now, we define the sequence spaces

$$\Gamma_M(\Delta_u^n) = \{x \in w : \ M(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho}) \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \ \},$$

and

$$\Gamma_M(p, \Delta_u^n) = \{x \in w : \ [M(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho})]^{p_k} \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0 \ \}.$$

If $n = 0, \Delta x_k = x_k$ for each $k \in \mathbb{N}$, and $u = e = (1, 1, 1, ...)$, then $\Gamma_M(\Delta_u^n)$ and $\Gamma_M(p, \Delta_u^n)$ reduce to Γ_M and $\Gamma_M(p)$ respectively.

2 Main results

We prove the following theorems :

Theorem 1 $\Gamma(\Delta_u^n) \subset \Gamma_M(\Delta_u^n)$, *with the hypothesis that* $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \leq | \Delta_u^n x_k |^{\frac{1}{k}}$,*where*

$$\Gamma(\Delta_u^n) = \{x \in w : \ | \Delta_u^n x_k |^{\frac{1}{k}} \rightarrow 0 \ as \ k \rightarrow \infty\}.$$

Proof. Let $x \in \Gamma(\Delta_u^n)$. Then $| \Delta_u^n x_k |^{\frac{1}{k}} \rightarrow 0$ as $k \rightarrow \infty$.

Since $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \leq | \Delta_u^n x_k |^{\frac{1}{k}}$, then we get that $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \rightarrow 0$ as $k \rightarrow \infty$. Therefore $x \in \Gamma_M(\Delta_u^n)$.

Theorem 2 *The space* $\Gamma_M(p, \Delta_u^n)$ *is solid.*

Proof. Let $| \Delta_u^n x_k | \leq | \Delta_u^n y_k |$ and $y = (y_k) \in \Gamma_M(p, \Delta_u^n)$. Then $(M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}))^{p_k} \leq (M(\frac{|\Delta_u^n y_k|^{\frac{1}{k}}}{\rho}))^{p_k}$ as M is nondecreasing.

Now, since $y = (y_k) \in \Gamma_M(p, \Delta_u^n)$, we have $(M(\frac{|\Delta_u^n y_k|^{\frac{1}{k}}}{\rho}))^{p_k} \rightarrow 0$ as $k \rightarrow \infty$.

Since $(M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}))^{p_k} \leq (M(\frac{|\Delta_u^n y_k|^{\frac{1}{k}}}{\rho}))^{p_k}$, we get that $(M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}))^{p_k} \rightarrow 0$ as $k \rightarrow \infty$, so that $x = (x_k) \in \Gamma_M(p, \Delta_u^n)$.

Theorem 3 *Let M be an Orlicz function which satisfies the Δ_2-condition. Then* $\Gamma(\Delta_u^n) \subset \Gamma_M(\Delta_u^n)$.

Proof. Let $x \in \Gamma(\Delta_u^n)$. Then $| \Delta_u^n x_k |^{\frac{1}{k}} \leq \varepsilon$ for sufficiently large k and every $\varepsilon > 0$. Therefore by taking $\rho \geq \frac{1}{2}$, we see that $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \leq M(\frac{\varepsilon}{\rho}) \leq M(2\varepsilon)$, as M is nondecreasing, so that $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \leq KM(\varepsilon) \leq \varepsilon$, for some $K > 0$, using the Δ_2-condition and defining $M(\varepsilon) < \frac{\varepsilon}{K}$.

Hence we obtain that $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \rightarrow 0$ as $k \rightarrow \infty$ that is $x \in \Gamma_M(\Delta_u^n)$.

Theorem 4 *If M is a modulus function. Then $\Gamma_M(p, \Delta_u^n)$ is a linear set over the set of complex numbers* $\mathbb{C}$.

Proof. Let $x, y \in \Gamma_M(p, \Delta_u^n)$ and $\alpha, \beta \in \mathbb{C}$. Then to prove the theorem we need to find some ρ_3 such that $[M(\frac{|\alpha\Delta_u^n x_k+\beta\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_3})]^{p_k} \to 0$ as $k \to \infty$. Since $x, y \in \Gamma_M(p, \Delta_u^n)$, there exists some positive numbers ρ_1, ρ_2 such that $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho_1})]^{p_k} \to 0$ as $k \to \infty$,

$[M(\frac{|\Delta_u^n y_k|^{\frac{1}{k}}}{\rho 2})]^{p_k} \to 0$ as $k \to \infty$. Since M is nondecreasing modulus function, we get that

$$\begin{aligned}
[M(\frac{|\alpha\Delta_u^n x_k+\beta\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_3})]^{p_k} &\le [M(\frac{|\alpha|^{\frac{1}{k}}|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho_3}+\frac{|\beta|^{\frac{1}{k}}|\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_3})]^{p_k}\\
&\le [M(\frac{|\alpha||\Delta_u^n x_k|^{\frac{1}{k}}}{\rho_3})+\frac{|\beta||\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_3})]^{p_k}
\end{aligned}$$

Take ρ_3 such that $\frac{1}{\rho_3} = \min\{\frac{1}{|\alpha|}\frac{1}{\rho_1}, \frac{1}{|\beta|}\frac{1}{\rho 2}\}$. Then

$$\begin{aligned}
[M(\frac{|\alpha\Delta_u^n x_k+\beta\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_3})]^{p_k} &\le [M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho_1}+\frac{|\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_2})]^{p_k}\\
&\le [M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho_1})]^{p_k}+[M(\frac{|\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_2})]^{p_k}\\
&\to 0 \text{ as } k \to \infty.
\end{aligned}$$

Hence $[M(\frac{|\alpha\Delta_u^n x_k+\beta\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_3})]^{p_k} \to 0$ as $k \to \infty$, so that $(\alpha x + \beta y) \in \Gamma_M(p, \Delta_u^n)$.

Theorem 5 *(i) Let* $0 < \inf p_k \le p_k \le 1$. *Then* $\Gamma_M(p, \Delta_u^n) \subset \Gamma_M(\Delta_u^n)$

(ii) Let $1 \le p_k \le \sup_k p < \infty$. *Then* $\Gamma_M(\Delta_u^n) \subset \Gamma_M(p, \Delta_u^n)$.

Proof. (i) Let $x \in \Gamma_M(p, \Delta_u^n)$. Then $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k} \to 0$ as $k \to \infty$.

Since $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k} \to 0$ as $k \to \infty$, we have $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k} < 1$ for sufficiently large k. Since $0 < \inf p_k \le p_k \le 1$ for all k, we have $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \le [M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k}$, which yields that $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \to 0$ as $k \to \infty$. Hence $x \in \Gamma_M(\Delta_u^n)$ which proves that $\Gamma_M(p, \Delta_u^n) \subset \Gamma_M(\Delta_u^n)$.

(ii) Let $p_k \ge 1$ for each k , $\sup_k p < \infty$ and let $x \in \Gamma_M(\Delta_u^n)$.

Then $M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho}) \to 0$ as $k \to \infty$. Since $1 \le p_k \le \sup_k p < \infty$, we get that $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k} \le M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})$, which yields that $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k} \to 0$ as $k \to \infty$. Hence $x \in \Gamma_M(p, \Delta_u^n)$.

Theorem 6 *Let* $0 < p_k \le q_k$ *for each* k. *Then* $\Gamma_M(p, \Delta_u^n) \subset \Gamma_M(q, \Delta_u^n)$.

Proof. Let $x \in \Gamma_M(p, \Delta_u^n)$. Then $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k} \to 0$ as $k \to \infty$, so we have $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k} < 1$ for sufficiently large k. Since $0 < p_k \le q_k$ for each k, we have $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{q_k} \le [M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k}$, which yields that $[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{q_k} \to 0$ as $k \to \infty$.

Hence $x \in \Gamma_M(q, \Delta_u^n)$ which proves that $\Gamma_M(p, \Delta_u^n) \subset \Gamma_M(q, \Delta_u^n)$.

Theorem 7 *Let* M_1 *and* M_2 *be two Orlicz functions, then* $\Gamma_{M_1}(p, \Delta_u^n) \cap \Gamma_{M_2}(p, \Delta_u^n) \subseteq \Gamma_{M_1+M_2}(p, \Delta_u^n)$.

Proof. Let $x \in \Gamma_{M_1}(p, \Delta_u^n) \cap \Gamma_{M_2}(p, \Delta_u^n)$. Then there exists ρ_1 and ρ_2 such that $[M_1(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho_1})]^{p_k} \to 0$ as $k \to \infty$ and $[M_2(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho_2})]^{p_k} \to 0$ as $k \to \infty$.

Take ρ such that $\frac{1}{\rho} = \min\{\frac{1}{\rho_1}, \frac{1}{\rho 2}\}$. Then we have

$$\begin{aligned} [(M_1 + M_2)(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho})]^{p_k} &\le [M_1(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho_1})]^{p_k} + [M(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho_2})]^{p_k} \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence $[(M_1+M_2)(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho})]^{p_k} \to 0$ as $k \to \infty$, that is $x \in \Gamma_{M_1+M_2}(p, \Delta_u^n)$.

Theorem 8 *The space* $\Gamma_M(p, \Delta_u^n)$ *is paranormed space (not totally paranormed) with*

$$g_\Delta(x) = \inf\{\rho^{p n/H} : \sup_{k \ge 1}[M(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho})]^{p_k} \le 1; \rho > 0\},$$

where $H = \max(1, \sup p_k)$.

Proof. Clearly $g_\Delta(x) \geq 0$, $g_\Delta(x) = g_\Delta(-x)$ and $g_\Delta(\theta) = 0$, where θ is the zero sequence. Let $(x_k), (y_k) \in \Gamma_M(p, \Delta_u^n)$. Let $\rho_1, \rho_2 > 0$ be such that

$\sup_{k\geq 1}[M(\frac{|\Delta_u^n x_k|^{\frac{1}{k}}}{\rho_1})]^{p_k} \leq 1$ and $\sup_{k\geq 1}[M(\frac{|\Delta_u^n y_k|^{\frac{1}{k}}}{\rho_2})]^{p_k} \leq 1$. Then taking ρ such that $\rho = \rho_1 + \rho_2$, we see that

$$\sup_{k\geq 1}[M(\frac{\mid \Delta_u^n x_k + \Delta_u^n y_k \mid^{\frac{1}{k}}}{\rho})]^{p_k}$$

$$\leq (\frac{\rho_1}{\rho_1+\rho_2})\sup_{k\geq 1}[M(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho_1})]^{p_k} + (\frac{\rho_1}{\rho_1+\rho_2})\sup_{k\geq 1}[M(\frac{\mid \Delta_u^n y_k \mid^{\frac{1}{k}}}{\rho_2})]^{p_k} \leq 1.$$

$$\begin{aligned} g_\Delta(x+y) &\leq \inf\{\rho_1^{pn/H} : \sup_{k\geq 1}[M(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho_1})]^{p_k} \leq 1; \rho_1 > 0\} \\ + \inf\{\rho_2^{pn/H} &: \sup_{k\geq 1}[M(\frac{\mid \Delta_u^n y_k \mid^{\frac{1}{k}}}{\rho_2})]^{p_k} \leq 1; \rho_2 > 0\}. \end{aligned}$$

Thus $g_\Delta(x+y) \leq g_\Delta(x) + g_\Delta(y)$.

Now

$$\begin{aligned} g_\Delta(\lambda x) &= \inf\{\rho^{pn/H} : \sup_{k\geq 1}[M(\frac{\mid \lambda\Delta_u^n x_k \mid^{\frac{1}{k}}}{\rho})]^{p_k} \leq 1; \rho > 0\} \\ &= \inf\{(r \mid \lambda \mid)^{pn/H} : \sup_{k\geq 1}[M(\frac{\mid \Delta_u^n x_k \mid^{\frac{1}{k}}}{r})]^{p_k} \leq 1; r > 0\}, \end{aligned}$$

where $r = \frac{\rho}{|\lambda|}$. Hence $\Gamma_M(p, \Delta_u^n)$ is a paranormed space.

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A note concerning the Euler totient [1]

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Abstract

We complete and improve certain results from paper [4] related to some properties of the Euler totient and similar arithmetic functions.

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Key words and phrases: totatives of a number, Euler's totient, arithmetic functions.

1 Introduction

Let $n \geq 1$ be a positive integer, and denote by $\varphi(n)$ the number of "totatives" of n, i.e. those positive integers $r < n$ such that $r < n$. The totatives of a number (a notion due to J.J. Sylvester, from 1879) have a long history. For example, in 1888 H.W. Lloyd Tanner (see [3], [8]) studied the group G of totatives of a number, finding all its subgroups and the simple groups whose direct product is G.

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It is easy to see that, the sum of totatives of n is $\dfrac{n\varphi(n)}{2}$ $(n \geq 2)$, due to A.L. Crelle from 1845 (see [3], [8]). This sum is recalled in [4] as Proposition 2. Let $t(n) = \{k : \ 1 \leq k < n, \ (k, n) = 1\}$ be the set of totatives of n.

In 1850 A. Thacker introduced the function (see [3], [8])

$$\phi_j(n) = \sum_{t \in t(n)} t^j, \quad j \geq 0 \tag{1}$$

(i.e., the sum of jth powers of totatives of n), and noted that for $j = 0$ it reduces to Euler's totient. Clearly $\phi_1(n) = \dfrac{n\varphi(n)}{2}, n \geq 2$.

In 1852 W. Brennecke proved that

$$\phi_2(n) = \frac{1}{3}\varphi(n)\left[n^2 + (-1)^{\omega(n)} \cdot \frac{\gamma(n)}{2}\right], \tag{2}$$

where $\omega(n)$ denotes the number of distinct prime factors of n and $\gamma(n)$ is the product of distinct prime divisors of n. In fact, in Proposition 9 of [4] we should have (in place of $n^2\varphi(n)/2$ on the right-hand side)

$$\phi_2(n) \geq \frac{n^2\varphi(n)}{4}. \tag{3}$$

For a proof of (2), based on a simple argument of Hurwitz, see [2].

Since $\gamma(n) \leq n$, equality (2) implies the double inequality

$$\frac{n^2\varphi(n)}{3} - \frac{n\varphi(n)}{6} \leq \phi_2(n) \leq \frac{n^2\varphi(n)}{3} + \frac{n\varphi(n)}{6}, \text{ for any } n \geq 2. \tag{4}$$

Therefore, the order of $\phi_2(n)$ is $\frac{n^2\varphi(n)}{3}$, for $n \to \infty$.

From the fact that $\phi_1(n) = \frac{n\varphi(n)}{2}$ and using the Jensen's inequality, we obtain

$$\frac{n(n-1)^{j-1}\varphi(n)}{2} \geq \phi_j(n) \geq \left(\frac{n}{2}\right)^j \cdot \varphi(n), \tag{5}$$

for any $j \geq 1$ and $n \geq 2$.

In 1851 J. Binet proved that (see [3], [8])

$$\phi_k(n) \equiv 0 \pmod{n} \text{ if } k \text{ is odd and } n \geq 3, \tag{6}$$

which for $k = 1$ gives Proposition 3 of [4].

For many other congruences and identities related to Thacker's function, see [2], [3], [8].

2 Some improvements

Proposition 11 of [4] states that

$$n\varphi(n) + 2\sigma(n) \leq n^2 + n + 2, \quad n \geq 1, \tag{7}$$

where $\sigma(n)$ denotes the sum of divisors of n.

As for $n =$ prime one has $\varphi(n) = n - 1$ and $\sigma(n) = n + 1$, in (7) we get equality for $n =$ prime.

On the other hand, when n is composite, inequality (7) can be improved.

Lemma 1. *Let $d(n)$ denote the number of divisors of n. Then*

$$\frac{\sigma(n)}{d(n)} \leq \frac{n+1}{2}, \quad n \geq 2 \tag{8}$$

with equality only for $n =$ prime.

One has also the inequality

$$\varphi(n) + d(n) \leq n + 1, \quad n \geq 2 \tag{9}$$

with equality only for $n =$ prime or $n = 4$.

Proof. For the proof of (8), let $d_1, \ldots, d_r$ be the distinct divisors of n, where $r = d(n)$. Then as $\{d_1, \ldots, d_r\} = \left\{\frac{n}{d_1}, \ldots, \frac{n}{d_r}\right\}$, clearly,

$$\frac{d_1 + \ldots + d_r}{r} = \frac{\left(d_1 + \frac{n}{d_1}\right) + \ldots + \left(d_r + \frac{n}{d_r}\right)}{2r}. \tag{$*$}$$

Now remark that

$$d+\frac{n}{d}\le n+1 \tag{10}$$

for any $1\le d\le n$, since (10) may be written also as $(d-1)(d-n)\le 0$.

Since by (10) $d_i+\dfrac{n}{d_i}\le n+1$, by identity $(*)$ we get at once (8). There is equality only when n has two divisors; namely $d_1=1$ and $d_2=n$, when n is prime.

For the proof of (9), let us remark first that when $d>1$ is a divisor of n, then clearly $(d,n)>1$; so d cannot be a totative of n. Therefore, the set of divisors and the set of totatives has a single common element, namely 1.

When n is prime, then any $1\le k<n$ is a totative, and there are only two divisors: 1 and n, so $\varphi(n)+d(n)=n+1$. This is true also when $n=4$, as any number in the set

$$1<2<3<4$$

or is a divisor of 4, or a totative of n.

Lemma 2. *Assume that $n\ne$ prime and $n\ne 4$. Then there exists an $a\in\{1,2,\dots,n\}$, such that $a\nmid n$, but $(a,n)>1$ (where $a\nmid n$ means that a doesn't divide n).*

Proof. If n is composite, and odd, let $n=N\cdot M$, where $N,M>1$. Put $a=N\cdot m$, where $m<M$ and $(m,M)=1$. Then clearly $a\nmid n$, but $N|a$, $N|n$, so $(n,a)>1$.

If n is even, remark that $n-2$ is even, too, and $n-2\nmid n$, if $n\ne 4$. Thus $a=n-2$ is acceptable, as $2|a$, $2|n$.

This finishes the proof of Lemma 2.

By this lemma, the number of divisors, and the number of totients cannot be in sum greater than n, proving (9) for $n\ne$ prime and $n\ne 4$.

Remarks. 1) The inequality $\varphi(n)+d(n)\le n$ for $n\ne$ prime, $n\ne 4$ may be satisfied with equality. For example, $n=6$ is a solution.

2) As $n+1\le\sigma(n)$, by (9) we get

$$\varphi(n)+d(n)\le n+1\le\sigma(n). \tag{11}$$

This improves the inequality $\varphi(n)+d(n) \le \sigma(n)$ by H.D. Bagchi and M. Gupta (see [1], [9]).

Theorem 1. *If* $n \neq 4$ *is composite, then*

$$n\varphi(n) + 2\sigma(n) < n^2 + d(n). \tag{12}$$

Proof. By (8) we get $2\sigma(n) < (n+1)d(n)$, and by (9), $\varphi(n) + d(n) \le n$ for composite n, therefore, we can write

$$n\varphi(n)+2\sigma(n) < n\varphi(n)+(n+1)d(n) = n[\varphi(n)+d(n)]+d(n) \le n^2+d(n),$$

so (12) follows.

Remark 3. As $d(n) < 2\sqrt{n}$ (see e.g. [7], [9]), clearly (12) is an improvement of (7) for composite n.

Theorem 2. *If* $n \ge 74$, *is composite, then the following inequalities are true:*

$$n\varphi\left(n\right) + 2\sigma\left(n\right) < n^2 + 2n\log n - n\sqrt{n} < n^2 + 3 \le n^2 + d\left(n\right). \tag{13}$$

Proof. By applying the inequality $\sigma\left(n\right) < n\log n$, for $n \ge 7$ (see e.g.[5]), and combining it with Sierpinski's inequality $\varphi\left(n\right) \le n - \sqrt{n}$ for composite n, we get

$$n\varphi\left(n\right) + 2\sigma\left(n\right) < n^2 + 2n\log n - n\sqrt{n}. \tag{14}$$

But, we prove that

$$n\sqrt{n} - 2n\log n + 3 > 0, \text{ for any } n \ge 74. \tag{15}$$

Now, we consider the function $f : [74, +\infty) \to \mathbb{R}$ defined by $f\left(x\right) = \sqrt{x} - 2\log x + \frac{3}{x}$.

It is easy to see that $f\left(x\right) > 0$, for any $x \ge 74$,so inequality (15) is true.

Since n is composite, we deduce $d(n) \geq 3$, so we have

$$n^2 + 3 \leq n^2 + d(n) \tag{16}$$

From relations $(14), (15)$ and (16) we obtain the sequence of inequalities of the statement.

Remark 4. By using the Ivić's inequality $[8]$,

$$\sigma(n) < 2.59n \log\log n, \text{ for any } n \geq 7,$$

we can improve inequality (13), thus

$$\begin{aligned} n\varphi(n) + 2\sigma(n) &< n^2 + 5.18n \log\log n - n\sqrt{n} < \\ &< n^2 + 2n \log n - n\sqrt{n} < n^2 + 3 \leq n^2 + d(n), \end{aligned} \tag{17}$$

for any n composite, $n \geq 74$.

Theorem 3. *If $n \geq 8$ is composite, then the following inequality holds true:*

$$n\sqrt{n} + 2n + 2\sqrt{n} < n\varphi(n) + 2\sigma(n). \tag{18}$$

Proof. Since n is composite, we apply the Sierpinski's inequality, $\sigma(n) > n + \sqrt{n}$, from [9] and the Vaidya's inequality, $\varphi(n) > \sqrt{n}$, for $n \geq 7$, from [9], it follows that

$$n\varphi(n) + 2\sigma(n) > n\sqrt{n} + 2\left(n + \sqrt{n}\right) = n\sqrt{n} + 2n + 2\sqrt{n}.$$

Remark 5. According to inequalities $(7), (17)$ and (18) we can deduce the following sequence of inequalities:

$$\begin{aligned} n\sqrt{n} + 2n + 2\sqrt{n} &< n\varphi(n) + 2\sigma(n) < n^2 + 5.18n \log\log n - n\sqrt{n} < \\ &< n^2 + 2n \log n - n\sqrt{n} < n^2 + 3 \leq n^2 + d(n) \leq n^2 + n + 2, \end{aligned} \tag{19}$$

for any $n \geq 74$,composite integer.

Lemma 3. [10]

$$\sigma(n) \geq n + 1 + \sqrt{n} \cdot (d(n) - 2),\ \textit{for}\ n \geq 2 \tag{20}$$

with equality only for $n =$ *prime or square of a prime.*

Proof. When $n = p$ (prime), then $d(n) = 2$ and $\sigma(n) = n + 1$; so there is equality in (11). For $n = p^2$ (square of a prime), as $\sigma(n) = p^2 + p + 1$ and $d(n) = 3$, we get again equality in (11).

Let $1 = d_1 < d_2 < \ldots < d_i < \ldots < d_k = n$ be the distinct divisors of n. As $n = \frac{n}{d_1} > \frac{n}{d_2} > \ldots > \frac{n}{d_k} = 1$ are also the divisors of n, then

$$\sigma(n) = n + 1 + \frac{1}{2} \sum_{k=1}^{n-1} \left(d_k + \frac{n}{d_k} \right) \tag{21}$$

By the arithmetic mean - geometric mean inequality one has

$$d_k + \frac{n}{d_k} \geq 2\sqrt{d_k \cdot \frac{n}{d_k}} = 2\sqrt{n},$$

with equality only if $d_k = \frac{n}{d_k}$.

Now, if $n \neq$ prime, the sum in (20) is not empty, and inequality (11) follows, as there are $d(n) - 2$ terms in the sum. There is equality only when $n = d_k^2$, and this is possibly only when $k = 2$ and d_2 is prime (since d_2 is the single divisor $d_2 < n$).

Theorem 4. *For all* $n \geq 2$ *one has*

$$\sigma(n) + 1 \geq n + 2 + \sqrt{n}(d(n) - 2) \geq n + d(n) \tag{22}$$

Proof. The first inequality follows by relation (11). When n is prime, as $d(n) = 2$ and $\sigma(n) = n + 1$, there is equality in each sides of (22). When $n > 1$ is composite, as $d(n) - 2 \geq 1$ and $\sqrt{n} > 1$, we have

$$n + 2 + \sqrt{n}(d(n) - 2) > n + 2 + (d(n) - 2) = n + d(n),$$

so the right side of (22) holds true with strict inequality.
Remarks. 6) Inequality $\sigma(n) + 1 \geq n + d(n)$ is Proposition 14 of [4].

7) Many other arithmetic inequalities are included also in the recent book [11].

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Adams operator in the dihedral homology of operator algebras [1]

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Abstract

The operator theory in an interesting subject in homology theory, Adams and Steenrod operators are famous examples. There are some results of these operations in Hochschild and cyclic homology [7] and [3]. In this article we are interested in Adams operator in Dihedral homology operator algebras.

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It is known that there are (co)homology operators (operations) on (co)homologies of any topological space, which is called Steenrod operations and have means in algebraic topology. Adams operations on algebraic cobordism with rational coefficients are defined and shown to descend to the oriented cohomology theories with rational coefficients which are universal with respect to their group law [8]. Adams operations in Hochschild and cyclic homology of de Rham algebra and free loop spaces have been studied in[7].

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Adam's operator ψ^k in the cyclic homology from[7] and [5]. Let A be a unital commutative Banach algebra, and K is a field with characteristic zero. Let $\lambda^k = \wedge^k(1_n - n)$ be the k^{th} exterior dimension representation of the lie Banach algebra $gL_n(k)$ and n is the direct sum of the one dimensional representation (n-argument). Following [6] the ring $R(gL_n(k))$ is isometric to the ring of polynomial $K\left[\lambda^1,...,\lambda^n\right]$, let $R(g^r(k)) = \varprojlim R(gL_n(k))$. Consider for an arbitrary representation ρ of an Banach algebra $gL_n(k)$ the following sequance:

(1)

$$CC_\infty(A)\underrightarrow{S}\wedge^n (gL(k))_{gL(k)}\underrightarrow{\widehat{\rho}}\wedge^k (gL(k))_{gL(k)}\underrightarrow{\varphi}CC_n(M_\infty(A))\underrightarrow{Tr}CC_\infty(A)$$

Where $\wedge^*(gL(k))_{gL(k)}$ is the coinvariant complex of Cherilley-Eilenberg complex $\wedge(gL(k)$ see[3] , $M_\infty(A) = \varinjlim M_n(A)$ is the $(n \times n)$ matrix with coefficients in A. The composition maps in 1 are denoted by α_n , where $\alpha = \varprojlim \alpha_n$. The morphism S is given by:

(2) $$S(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = E_{12}a_1 \wedge E_{23}a_2 \wedge \cdots \wedge E_{n-1\ n}a_{n-1} \wedge E_{n1} \cdot a_n$$

where E_{ij} is the matrix, whose only non zero elements are the identity element $1 \in k$. The map $\widehat{\rho}$ is given by:

(3) $$\widehat{\rho}(X_1a_1 \wedge \cdots \wedge X_na_n) = \rho(x_1)a_1 \in ... \in \rho(x_n)a_n, \qquad x_i \in gL_n(k)$$

(4)

$$\varphi(Z_0 \wedge \cdots \wedge Z_n) = \sum_\sigma sgn(\sigma)(-1)^n Z_{\sigma(0)} \otimes Z_{\sigma(1)} \otimes \cdots \otimes Z_{(n)}, Z_i \in gL_n(k)$$

$\rho : gL_n(k) \to gL_n(k)$ and Tr is the trace map defined by:

(5) $$Tr(X_1a_1 \otimes \cdots \otimes X_na_n) = Tr(X_1 \cdots .X_n)a_1 \otimes \cdots \otimes a_n \cdot$$

We can check that $\alpha(\rho + \tau) = \alpha(\otimes)$ where ρ and τ are representations of $gL(k)$. From the above discussion we have the homomorphism: $\alpha : R(gL(k)) \to End(CC.(A))$.

Clearly for any $f \in k\left[\lambda^1 \cdots \lambda^n \ldots\right]$the homomorphism $\alpha(f)$ coinides with the homomorphism α [4]. Suppose that Q_k $k \geq 1$ is the Newton polynomial which is given by the symmetrc function $\sum_{i=1}^{k}(u_i)^k$ such that $\sigma_r = \sum_{i_1 \prec i_2 \prec \cdots \prec i_r}^{k} u_{i_1} \cdots u_{i_r}$ $1 \leq r \leq k$.

By acting with the morphism α on the Newton polynomial we get the Adams operators $\psi^k = \alpha(Q_k) = \alpha((-1)^k \,.k\ \lambda^k)$ since $(-1)^k.k\ \lambda^k$ is the linear part of K–Newton polynomial. Consider the chain complex $(CH_\bullet(A)\ b'_\bullet)$ and the Connes–Tsygan bicomplex (see[3])

(6)

$$\begin{array}{lllllll}
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
b_\bullet \downarrow & b'_\bullet \downarrow & b_\bullet \downarrow & b'_\bullet \downarrow & b_\bullet \downarrow & & \\
CH_2(A) \xleftarrow{1-t} & CH_2(A) \xleftarrow{N} & CH_2(A) \xleftarrow{1-t} & CH_2(A) \xleftarrow{N} & CH_2(A) \longleftarrow & \ldots & \\
b_\bullet \downarrow & b'_\bullet \downarrow & & b_\bullet \downarrow & b'_\bullet \downarrow & & b_\bullet \downarrow \\
CH_1(A) \xleftarrow{1-t} & CH_1(A) \xleftarrow{N} & CH_1(A) \xleftarrow{1-t} & CH_1(A) \xleftarrow{N} & CH_1(A) \longleftarrow & \ldots & \\
b_\bullet \downarrow & b'_\bullet \downarrow & b_\bullet \downarrow & b'_\bullet \downarrow & b_\bullet \downarrow & & \\
CH_0(A) \xleftarrow{1-t} & CH_0(A) \xleftarrow{N} & CH_0(A) \xleftarrow{1-t} & CH_0(A) \xleftarrow{N} & CH_0(A) \longleftarrow & \ldots &
\end{array}$$

then the subcomplex $(ker(1-t.)\ b'_\bullet) \subset (CH.(A)\ b'_\bullet)$ has the same homology as the compiex $(CC.(A)\ b_\bullet)$that is

$$H_\bullet(CC_\bullet(A)) = H_\bullet((CH_\bullet(A) b_\bullet)/\mathrm{Im}(1-t_\bullet)) = H_\bullet((CH_\bullet(A)\ b_\bullet)/KerN)$$

$$= H_\bullet(\mathrm{Im}\, N\ b'_\bullet) = H_\bullet(Ker(1-t_\bullet),\ b'_\bullet)$$

where $CH_n(A) = A^{\otimes n+1} = A \otimes \cdots \otimes A(n+1 \text{ times}) b_n$, $b'_\bullet : CH_n(A) \longrightarrow CH_{n-1}(A)$ such that $b'_\bullet(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$, $b_n(a_0 \otimes \cdots \otimes a_n) = b'_\bullet + (-1)^n(a_n a \otimes \cdots \otimes a_{n-1})$, $t_n : CH_n(A) \longrightarrow CH_n(A)$ such that $t_n(a_0 \otimes \cdots \otimes a_n) = (-1)^n(a_n \otimes a_0 \otimes \cdots \otimes a_{n-1})$ and $N_n = 1 = t_n^1 + \cdots + t_n^n$.

Therefore the complex $(Ker(1-t_\bullet)\ b'_\bullet)$ is isomorphic to complex $(CC_\bullet(A) b_\bullet)$. The isomorphism between them is given by the operator $N_\bullet : CC_\bullet(A) \longrightarrow (Ker(1-t_\bullet)\ b'_\bullet)$.

Consequently the action of the group $Z/2$ on the complex $CC_\bullet(A)$ by means of the operator ${}^\varepsilon h$ is equal to action of $Z/2$ on the complex

$(Ker(1-t_\bullet)\ b^{`}_\bullet)$ by means of the operator

$$(7) \qquad {}^{\varepsilon}h : a_0 \otimes a_1 \otimes \ldots \otimes a_n \longrightarrow (-1)^{\frac{n(n+1)}{2}} \varepsilon a_n^* \otimes a_{n-1}^* \otimes \ldots \otimes a_0^*$$

where a^* is the image of element $a \in A$ under involution $* : A \longrightarrow A$ $\varepsilon = \pm 1$. Since ${}^{\varepsilon}h_\bullet t_\bullet = t_\bullet^{-1\varepsilon} h_\bullet$. Hence $N_\bullet({}^{\varepsilon}h_\bullet)N_\bullet$.

On the other hand since ${}^{\varepsilon}r_\bullet = t_\bullet^{\varepsilon} h_\bullet$ then ${}^{\varepsilon}h_\bullet N_\bullet = N_\bullet^{\varepsilon} h_\bullet = (N_\bullet t_\bullet)^{\varepsilon} h_\bullet = N_\bullet(t_\bullet^{\varepsilon} h_\bullet) = N_\bullet^{\varepsilon} r_\bullet$. So the dihedral homology of A is given by formula

$$(8) \qquad {}^{\varepsilon}HD_\bullet(A) = H_\bullet(\ker(1-t_\bullet)\,/(\mathrm{Im}(1-^{\varepsilon} h_\bullet) \cap \ker(1-t_\bullet))).$$

Assume that the complex $CC_\bullet(A)$ is a subcomplex of $(CH_\bullet(A)\ b^{`}_\bullet)$ then the direct calculation of homomorphism $\alpha((-1)^k k\lambda^k)$ gives the Adam‘s operator Ψ^k in additive algebraic $K-theory$ (see[7]) that is $\Psi(a_0 \otimes ... \otimes a_n) = \sum\limits_I sgn(\sigma_{I(0)}) a_{\sigma_{I(n)}}$, where I is the division of the set $\{0\ 12\ \ldots\ n\}$ into non–empty intersected subsets that is $I = I_0 \cup \ldots \cup I_{k-1}$, and $\sigma_I \in \sum\limits_{n-1}$ is the permutation of the set $\{0\ 1 \ldots\ n\}$ such that:

(i) If $i_1 \in I_{p1}$, $i_2 \in I_{p2}$, $p_1 \prec p_2$ then $\sigma_I(i_1) \succ \sigma_I(i_2)$,

(ii) For any p $I_p = \{i_0 \ldots\ i_q\}(i_1 \prec i_2 \prec \ldots \prec i_q)$.

The permutation σ_I satisfises the following condition:

$$(9) \qquad \sigma_I(i_q) = \sigma_I(i_{q-1}) + 1 = \ldots = \sigma_I(i_0) + q.$$

Lemma 1 *The following diagram is commutative:*

$$(10) \qquad \begin{array}{ccc} CC_\bullet(A) & \xrightarrow{\psi^k} & CC_\bullet(A) \\ \varepsilon_r \downarrow & & \downarrow \varepsilon_r \\ CC_\bullet(A) & \xrightarrow{\psi^k} & CC_\bullet(A) \end{array}$$

Proof. Assume that the complex $CC_\bullet(A)$ is a subcomplex of the complex $(CH_\bullet(A)\ b^{`}_\bullet)$ and the element $a_0 \otimes \ldots \otimes a_n \in \ker(1-t_n)$ then

$$\begin{aligned} {}^{\varepsilon}r\psi^k(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= {}^{\varepsilon}r \sum_I sgn(\sigma_I) a_{\sigma_I(0)} \otimes \ldots \otimes a_{\sigma_I(n)} \\ (11) \qquad &= (-1)^{\frac{n(n+1)}{2}} \varepsilon \sum_I sgn(\sigma_I) a^*_{\sigma_I(n)} \otimes \ldots \otimes a^*_{\sigma_{I(0)}.} \end{aligned}$$

On the other hand

$$\psi^k({}^{\varepsilon}r)(a_0 \otimes \ldots \otimes a_n) = (-1)^{\frac{n(n+1}{2}} \varepsilon\psi^k(a_n^* \otimes \ldots \otimes a_0^*)$$

$$= (-1)^{\frac{n(n+1}{2}} \varepsilon \sum_J sgn(gj) a^*_{g_j(n)} \otimes \ldots \otimes a^*_{g_j(0)}, \qquad (12)$$

where g_j is the permutation of the ordered set $\{nn-1 \ \ldots \ 0\}$ satisfies the conditions (i) (ii) and J is the division of the ordered set $\{n \ n-1 \ldots \ 0\}$. Note that in general the permutation g_J of ordered set $\{0 \ 1 \ \ldots n\}$ satisfies the following conditions:

$i)$' If $i_1 \in J_{p_1}$ $i_2 \in J_{p_2} p_1 \prec p_2$ then $g_J(i_1) \succ g_J(i_2)$

$ii)$" For any p $g_{_J} = \{i_1 \ldots \ i_0\} i_q \succ \ldots \succ i_0$ we have

$$g_{_J}(i_0) = g_{_J}(i_1) - 1 = \ldots = g_{_J}(i_q) - q. \qquad (13)$$

Note that the decreasing (by one) of the elements set $\{01 \ \ldots \ n\}$ met the increasing of elements (also by one) in the set $\{n \ n-1 \ \ldots 0\}$. Suppose that the arguments of the summation in 11 correspond the permutation σ_I . The permutation g_J of the set $\{n \ n-1 \ \ldots \ 0\}$ where $g_J \ (i) = \sigma_I(i)$ will correspond to the division $J = I^*_{k-1} \cup \ldots \cup I^*_0$, where

$$I^*_i = \left\{p^i_{q_i} \ \ldots p^i_0\right\} (I = \left\{p^i_0 \ \ldots \ p^i_{q_i}\right\} \qquad p^i_0 \prec \ldots \prec p^i_{q_i}). \qquad (14)$$

We can easily check for any p and $I^*_p = \left\{i^p_{q_p} \ldots i^p_0\right\}$, $i^p_{qp} < ... < i^p_0$, that $g_{_J}(i^p_0) = g_{_J}(i^p_1) - 1 = \ldots = g_{_J}(i^p_{qp}) - q_p$ If $i_1 \in I^*_{p1}, i_2 \in I^*_{p2}$, $p_1 < p_2$, then $g_{_J}(i_1) > g_{_J}(i_2)$.

From the definition of σ_I and g_J we have ${}^{\varepsilon}r\psi^k = \psi^k({}^{\varepsilon}h)$ in $ker((1-t)$, $b'_\bullet)$ and, hence ${}^{\varepsilon}r\psi^k = \psi^k({}^{\varepsilon}r)$ in $(CC_\bullet(A), b_\bullet)$.

Clearly the inverse of the isomorphism $(CC_\bullet(A)) \longrightarrow \ker(1-t)$ is $\frac{1}{n}id$: $(\ker(1-t), b'_\bullet) \longrightarrow (CC_\bullet(A), b_\bullet)$. The operator ψ^k in $CC_\bullet(A)$ in given by $\frac{1}{n}\psi^k N$, where ψ^k is an operator in $(\ker(1-t), \ b'_\bullet)$. Since the operator ψ^k on $CC_\bullet(A)$ commutes with the operator ${}^{\varepsilon}r$ then we have the Adam's operator ${}^{\varepsilon}\psi^k$ in the dihedral homology. Following [6] the multiplication in thedihedral homology of the Banach algebra A is given as follows

$$\cup : HC_p(A) \otimes HC_q(A) \longrightarrow HC_{p+q+1}(A), \tag{15}$$

such that

$$\cup : TotB(A) \otimes TotB(A) \longrightarrow TotB(A), \tag{16}$$

(17)
$$xuy = \begin{cases} (x)T(\beta y), \; r = 0 \\ 0 \qquad\qquad , \; r \neq 0 \end{cases} \Bigg] \in B(A)_{\ell+r,m+s+I},\; x \in B(A)_{l,m} = A \otimes \overset{-\otimes(m-\ell)}{A}$$

$y \in B(A)_{r,s} = A \otimes \overset{-\otimes(s-r)}{A}$, where T is a product map [7] $TotB(A)$ is the total complex of the biccomplex $B(A)$, β is the Connes's operator. The group $Z/2$ acts on the column of the bicomplex $B(A)$ with the numbers $2\ell(n > 0)$ by means of the operator ${}^{\varepsilon}r$, on the the column with the numbers $(2\ell{+}1)$ by means of the operator $(-1)^{\in}r$, and on the complex $Tot^{\varepsilon}B(A) \otimes Tot^{\delta}B(A)$ by means of $\overset{\varepsilon\frown}{r} \otimes {}^{\delta}\overset{\frown}{r}$, where ${}^{\varepsilon}\overset{\frown}{r}$ is the action of $Z/2$ on $Tot^{\varepsilon}B(A)$ induced by the action $Z/2$ on $Tot^{\varepsilon}B(A)$. Since the action of the group $Z/2$ the complex $Z/2$ on $Tot^{\varepsilon}B(A) \otimes Tot^{\delta}B(A)$ commutes with the multiplicaion in the cyclic homology [3], then

$$\overset{\varepsilon\frown}{r} \otimes \overset{\delta\frown}{r}(a \otimes b) = \overset{E\frown}{r}(a) \otimes \overset{\delta\frown}{r}(b) \overset{\smile}{\longrightarrow} \overset{\varepsilon\frown}{r}(a)T\beta(\overset{\delta\frown}{r}(b)), \tag{18}$$

$a \in Tot^{\varepsilon}B(A)$,$b \in Tot^{\delta}B(A)$. On the other hand

$$\begin{aligned} (-(\overset{\varepsilon\frown}{r}(a)T\beta(\overset{\delta\frown}{r}(b))) &= \overset{\varepsilon\frown}{r}(a)T\beta(-\overset{\delta\frown}{r}(b)) = -(\overset{\varepsilon\frown}{r}(a)T(\overset{\delta\frown}{r}(\beta(b)) \\ &= \overset{(\varepsilon\delta)\frown}{r}(a)T\beta(-\overset{\delta\frown}{r}(a \cup b). \end{aligned} \tag{19}$$

Therefore ${}^{\varepsilon}r(a) \cup^{\delta} r(b) =^{-(\varepsilon\delta)} r(a \cup b)$.

From the above we have the multiplication in the aihedral homology

$$\cup :^{\varepsilon} HD_p(a) \otimes^{\delta} HD_q(A) \longrightarrow^{-(\varepsilon\delta)} HD_{p+q+1}(A).$$

It is well known that ([6, 3]) the dihedral homology can be considered as the hyperhomology of the group $Z/2$ with the coefficient in $Tot^{\varepsilon}B(A)$, the

$$H_{\bullet}(Z/2, Tot^{\varepsilon}B(A)) \otimes H_{\bullet}(Z/2, Tot^{\delta}B(A)) \longrightarrow H_{\bullet}(Z/2, Tot^{\varepsilon}B(A) \otimes Tot^{\delta}B(A)) \longrightarrow H_{\bullet}(Z/2, Tot^{-(\varepsilon\delta)}B(A). \quad (20)$$

Consider the Adam's operator properties in the cyclic homology [7] Since the Adam's operator ψ^k commutes with the action of the group $Z/2$ and the multiplicaton $\cup$ in the cyclic homology anti-commutes with the action of group $Z/2$, weget the following theorem .

Theorem 2 *Assume that A is involutive Banach algebra . The Adam's operator ψ^k has the following properties:*

1- ${}^{\varepsilon}\psi^k \circ^{\varepsilon}\psi^l =^{\varepsilon} \psi^{kl}$,
2- ${}^{\varepsilon}\psi^k(\alpha) \cup^{\delta} \psi^k(\beta) =^{-(\varepsilon\delta)} \psi^k(\alpha \cup \beta)$, where $\alpha, \beta \in HD_{\bullet}(A)$.

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Preserving properties of subclasses p-valent and estimating the coefficients by operator $L_{a,p}$ [1]

Radu Diaconu

Abstract

In this paper we study the preserving properties of subclasses of p-valent and estimate coefficients by $L_{a,p}$ integral operator.

2010 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: p-valent functions, Alexander type integral operator, Bernardi type integral operator, $I_{c+\delta,p}$ integral operator, $D_{\lambda}^{n,p}$ type operator, $L_{a,p}$ type operator .

1 Introduction

Let $\mathcal{T}_p(j)$ be the class of analytic functions f of the form:

$$f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k, \; (a_k \geq 0, \; j, \; p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \qquad (1)$$

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defined in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Definition 1.1 *[1] Let $I_{A,p}$ be a Alexander type integral operator defined as:*

$$I_{A,p} : \mathcal{A}_p \to \mathcal{A}_p, I_{A,p}(F) = f, p \in \mathbb{N}, \text{ where}$$

$$f(z) = p \int_0^z \frac{F(t)}{t} dt. \tag{2}$$

Definition 1.2 *[1] Let $I_{a,p}$ be a Bernardi type integral operator defined as:*

$$I_{a,p} : \mathcal{A}_p \to \mathcal{A}_p, I_{a,p}(F) = f, a = 1, 2, 3, \ldots, p \in \mathbb{N}, \text{ where}$$

$$f(z) = \frac{p+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{3}$$

Definition 1.3 *[1] Let $L_{a,p}$ be a generalization of the previously integral operator defined as:*

$$L_{a,p} : \mathcal{A}_p \to \mathcal{A}_p, L_{a,p}(F) = f, a \in \mathbb{R}, a \geq 0, p \in \mathbb{N}, \text{ where}$$

$$f(z) = \frac{p+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{4}$$

Definition 1.4 *[1] Let $I_{c+\delta,p}$ be the integral operator defined as: $I_{c+\delta,p}$: $\mathcal{A}_p \to \mathcal{A}_p, 0 < u \leq 1\,,\ 1 \leq \delta < \infty\,,\ 0 < c < \infty\,,$*

$$f(z) = I_{c+\delta,p}(F)(z) = (c+\delta+p-1) \int_0^1 u^{c+\delta-2} F(uz) du\,. \tag{5}$$

Remark 1.1 *For $\delta = 1$ and c=1,2,..., from the integral operator $I_{c+\delta,p}$ we obtain the Bernardi integral operator defined by (3).*

2 Preliminary results

Definition 2.1 *[4] Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}, \lambda \geq 0$. We define the operator $D_\lambda^{n,p} : \mathcal{T}_p(j) \to \mathcal{T}_p(j)$ is defined as:*

$$D_\lambda^{0,p} f(z) = f(z),$$

$$D_{\lambda}^{1,p} f(z) = (1-\lambda) f(z) + \frac{\lambda}{p} z f'(z) = D_{\lambda} f(z),$$

$$\vdots$$

$$D_{\lambda}^{n,p} f(z) = D_{\lambda}\left(D_{\lambda}^{n-1,p} f(z)\right).$$

Further, if $f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k$, *then we have,*

$$D_{\lambda}^{n,p} f(z) = z^p - \sum_{k=j+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\lambda\right]^n a_k z^k. \tag{6}$$

Definition 2.2 *[2] A function* $f \in \mathcal{T}_p(j)$ *is said to be in the class* $\mathcal{T}_j(n, m, p, A, B, \lambda)$ *if*

$$\frac{D_{\lambda}^{n+m,p} f(z)}{D_{\lambda}^{n,p} f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U}, \tag{7}$$

where $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, m \in \mathbb{N}, \lambda \geq 1$ *and* $-1 \leq B < A \leq 1$.

Theorem 2.1 *[3] A function* $f \in \mathcal{T}_p(j)$ *belongs to the class* $\mathcal{T}_j(n, m, p, A, B, \lambda)$ *if and only if*

$$\sum_{k=j+p}^{\infty} [1 + (\frac{k}{p} - 1)\lambda]^n \left\{(1-B)[1 + (\frac{k}{p} - 1)\lambda]^m - (1-A)\right\} a_k \leq A - B, \tag{8}$$

for $n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m \in \mathbb{N}, \lambda \geq 1$ *and* $-1 \leq B < A \leq 1$.

Corollary 2.1 *[3] Let* $f(z)$ *defined by the relation (1), be in the class* $\mathcal{T}_j(n, m, p, A, B, \lambda)$.*Then*

$$a_k \leq \frac{A-B}{[1 + (\frac{k}{p} - 1)\lambda]^n \left\{(1-B)[1 + (\frac{k}{p} - 1)\lambda]^m - (1-A)\right\}}. \tag{9}$$

Definition 2.3 *[2] A function* $f \in \mathcal{T}_p(j)$ *is said to be in the class* $\mathcal{R}_j(n, p, A, B, \lambda)$ *if it satisfies*

$$(D_{\lambda}^{n,p} f(z))' \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U},$$

where $-1 \le B < A \le 1, \lambda \ge 1, n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *.*

Definition 2.4 *[2] A function* $f \in \mathcal{T}_p(j)$ *is said to be in the class* $\mathcal{P}_j(n,p,A,B,\lambda)$ *if it satisfies*

$$\frac{D_\lambda^{n,p} f(z)}{z} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U},$$

where $-1 \le B < A \le 1, \lambda \ge 1, n \in \mathbb{N}_0,\ \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *.*

Theorem 2.2 *[4] A function* $f \in \mathcal{T}_p(j)$ *belongs to the class* $\mathcal{R}_j(n,p,A,B,\lambda)$ *if and only if*

$$\sum_{k=j+p}^{\infty} (1-B)[1+(\frac{k}{p}-1)\lambda]^{n+1} a_k \le A-B.$$

Theorem 2.3 *[4] A function* $f \in \mathcal{T}_p(j)$ *belongs to the class* $\mathcal{P}_j(n,p,A,B,\lambda)$ *if and only if*

$$\sum_{k=j+p}^{\infty} (1-B)[1+(\frac{k}{p}-1)\lambda]^{n} a_k \le A-B.$$

3 Main results

Theorem 3.1 *The Alexander type integral operator defined by (2) preserves the class* $\mathcal{T}_j(n,m,p,A,B,\lambda)$*, that is: If* $F(z) \in \mathcal{T}_j(n,m,p,A,B,\lambda)$*, then* $f(z) = I_{A,p}F(z) \in \mathcal{T}_j(n,m,p,A,B,\lambda)$*, for* $F(z) = z^p - \sum\limits_{k=j+p}^{\infty} a_k z^k$*,* $(a_k \ge 0,\ j,\ p \in \mathbb{N} = \{1,2,3,\ldots\})$*.*

Proof. Let $F(z) \in \mathcal{T}_j(n,m,p,A,B,\lambda) \subset \mathcal{T}_p(j)$, $F(z) = z^p - \sum\limits_{k=j+p}^{\infty} a_k z^k$, $a_k \ge 0$. Then

$$f(z) = I_{A,p}F(z) = p\int_0^z \frac{F(t)}{t}dt = p\int_0^z \frac{1}{t}\left(t^p - \sum_{k=j+p}^{\infty} a_k\, t^k\right) dt$$

$$= p\left(\frac{z^p}{p} - \sum_{k=j+p}^{\infty} \frac{a_k}{k} z^k\right) = z^p - \sum_{k=j+p}^{\infty} b_k z^k, \text{ with}$$

$b_k = p\dfrac{a_k}{k} \geq 0, k \geq j+p$. It follows that $f \in \mathcal{T}_p(j)$. We have now to prove that $f \in \mathcal{T}_j(n, m, p, A, B, \lambda)$. Using Theorem 2.1 we need to prove that

$$\sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A)\right\} b_k \leq A-B, \tag{10}$$

for $n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m \in \mathbb{N}, \lambda \geq 1$ and $-1 \leq B < A \leq 1$.
This means

$$\sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A)\right\} p\frac{a_k}{k} \leq A-B. \tag{11}$$

But we have $p\dfrac{a_k}{k} \leq a_k, \text{for k} \geq \text{j + p}$, and by using (8) and (11), we observe that inequality (10) is fulfilled. This means that $f(z) \in \mathcal{T}_j(n, m, p, A, B, \lambda)$.

Theorem 3.2 *The integral operator $I_{c+\delta,p}$ defined by (5) preserves the class $\mathcal{T}_j(n, m, p, A, B, \lambda)$, that is: If $F(z) \in \mathcal{T}_j(n, m, p, A, B, \lambda)$, then $f(z) = I_{c+\delta,p}(F)(z) \in \mathcal{T}_j(n, m, p, A, B, \lambda)$, for $F(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k$, $(a_k \geq 0,\ j,\ p \in \mathbb{N} = \{1, 2, 3, \ldots\})$.*

Proof. Let $F(z) \in \mathcal{T}_j(n, m, p, A, B, \lambda), F(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k, a_k \geq 0$.
We have, from Theorem 2.1

$$\sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A)\right\} a_k \leq A-B. \tag{12}$$

From (5) we obtain

$$f(z) = I_{c+\delta,p}(F)(z) = z^p - \sum_{k=j+p}^{\infty} \frac{c+\delta+p-1}{c+k+\delta-1} a_k z^k,$$

where $0 < c < \infty$, $1 \leq \delta < \infty$.
We also remark that for $0 < c < \infty, k \geq j + p \, and \, 1 \leq \delta < \infty, we \, have$

$$0 < \frac{c + \delta + p - 1}{c + k + \delta - 1} < 1. \tag{13}$$

Thus $f(z) \in T_p(j)$ and by using Theorem 2.1 we have only to prove that.

$$\sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A)\right\} \frac{c+\delta+p-1}{c+k+\delta-1} a_k \leq A - B, \tag{14}$$

where $n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m \in \mathbb{N}, \lambda \geq 1$, $-1 \leq B < A \leq 1$, $0 < c < \infty$ and $1 \leq \delta < \infty$.
By using the relation (13) we have

$$\frac{c + \delta + p - 1}{c + k + \delta - 1} \cdot a_k < a_k,$$

for $0 < c < \infty, k \geq j + p, 1 \leq \delta < \infty$, and thus from (12) we conclude that the condition (14) take place and thus the proof it is complete.
The following theorem is proved similarly (see Remark 1.1):
Theorem 3.3 *The Bernardi type integral operator defined by (3) preserves the class* $\mathcal{T}_j(n, m, p, A, B, \lambda)$, *that is: If* $F(z) \in \mathcal{T}_j(n, m, p, A, B, \lambda)$, *then* $f(z) = I_{a,p}F(z) \in \mathcal{T}_j(n, m, p, A, B, \lambda)$, *for* $F(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k$, $(a_k \geq 0,\ j,\ p \in \mathbb{N} = \{1, 2, 3, \ldots\})$.
Similarly, we can prove the preserving properties for the subclasses $\mathcal{R}_j(n, p, A, B, \lambda)$ and $\mathcal{P}_j(n, p, A, B, \lambda)$, using the integral operators defined by (2), (3) and (5).
Theorem 3.4 *The Alexander type integral operator defined by (2) preserves the class* $\mathcal{R}_j(n, p, A, B, \lambda)$, *that is: If* $F(z) \in \mathcal{R}_j(n, p, A, B, \lambda)$, *then* $f(z) = I_{A,p}F(z) \in \mathcal{R}_j(n, p, A, B, \lambda)$, *for* $F(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k$, $(a_k \geq 0,\ j,\ p \in \mathbb{N} = \{1, 2, 3, \ldots\})$.

Theorem 3.5 *The Alexander type integral operator defined by (2) preserves the class* $\mathcal{P}_j(n,p,A,B,\lambda)$*, that is: If* $F(z) \in \mathcal{P}_j(n,p,A,B,\lambda)$*, then* $f(z) = I_{A,p}F(z) \in \mathcal{P}_j(n,p,A,B,\lambda)$*, for* $F(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k$*,* $(a_k \geq 0,\ j,\ p \in \mathbb{N} = \{1,2,3,\ldots\})$.

Theorem 3.6 *The integral operator* $I_{c+\delta,p}$ *defined by (5) preserves the class* $\mathcal{R}_j(n,p,A,B,\lambda)$*, that is: If* $F(z) \in \mathcal{R}_j(n,p,A,B,\lambda)$*, then* $f(z) = I_{c+\delta,p}(F)(z) \in \mathcal{R}_j(n,p,A,B,\lambda)$*, for* $F(z) = z^p - \sum_{k=j+p}^{\infty} a_k\, z^k$*,* $(a_k \geq 0,\ j,\ p \in \mathbb{N} = \{1,2,3,\ldots\})$.

Theorem 3.7 *The integral operator* $I_{c+\delta,p}$ *defined by (5) preserves the class* $\mathcal{P}_j(n,p,A,B,\lambda)$*, that is: If* $F(z) \in \mathcal{P}_j(n,p,A,B,\lambda)$*, then* $f(z) = I_{c+\delta,p}(F)(z) \in \mathcal{P}_j(n,p,A,B,\lambda)$*, for* $F(z) = z^p - \sum_{k=j+p}^{\infty} a_k\, z^k$*,* $(a_k \geq 0,\ j,\ p \in \mathbb{N} = \{1,2,3,\ldots\})$.

Theorem 3.8 *The Bernardi type integral operator defined by (3) preserves the class* $\mathcal{R}_j(n,p,A,B,\lambda)$*, that is: If* $F(z) \in \mathcal{R}_j(n,p,A,B,\lambda)$*, then* $f(z) = I_{a,p}F(z) \in \mathcal{R}_j(n,p,A,B,\lambda)$*, for* $F(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k$*,* $(a_k \geq 0,\ j,\ p \in \mathbb{N} = \{1,2,3,\ldots\})$.

Theorem 3.9 *The Bernardi type integral operator defined by (3) preserves the class* $\mathcal{P}_j(n,p,A,B,\lambda)$*, that is: If* $F(z) \in \mathcal{P}_j(n,p,A,B,\lambda)$*, then* $f(z) = I_{a,p}F(z) \in \mathcal{P}_j(n,p,A,B,\lambda)$*, for* $F(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k$*,* $(a_k \geq 0,\ j,\ p \in \mathbb{N} = \{1,2,3,\ldots\})$.

Theorem 3.10 *Let* $F(z) \in \mathcal{T}_j(n,m,p,A,B,\lambda)$ *with* $n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m \in \mathbb{N}, \lambda \geq 1$ *and* $-1 \leq B < A \leq 1$ *,* $F(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k,\ b_k \geq 0$. *For* $f(z) = L_{a,p}(F)(z)$*,* $f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k, a_k \geq 0, z \in \mathbb{U}$*, where* *the integral operator $L_{a,p}$ it is defined by (4), we have:*

$$a_k \le \frac{A-B}{[1+(\frac{k}{p}-1)\lambda]^n\left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m-(1-A)\right\}}\cdot\frac{a+p}{a+k},\quad k\ge j+p.$$

Proof. For $f = L_{a,p}(F)(z)$ with $F(z)=z^p-\sum_{k=j+p}^{\infty}b_kz^k$ and $f(z)=z^p-\sum_{k=j+p}^{\infty}a_kz^k$ we have

$$a_k=b_k\cdot\frac{a+p}{a+k},$$

where $a\in\mathbb{R}, a\ge 0, k\ge j+p$.
The coefficient bounds for the functions belonging to the class $\mathcal{T}_j\,(n,m,p,A,B,\lambda)$ are

$$b_k \le \frac{A-B}{[1+(\frac{k}{p}-1)\lambda]^n\left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m-(1-A)\right\}}.$$

For $k\ge j+p$ we obtain

$$a_k=b_k\cdot\frac{a+p}{a+k}\le\frac{A-B}{[1+(\frac{k}{p}-1)\lambda]^n\left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m-(1-A)\right\}}\cdot\frac{a+p}{a+k}.$$

Hence the theorem is proved.
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The Cesàro space of double gai sequences [1]

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Abstract

FK-space inclusion, weak convergence, extreme points of the unit disc and other properties of Cesàro space of double gai sequences are discussed.

2000 Mathematics Subject Classification:40A05,40C05,40D05

Key words and phrases: Sectional sequence spaces, double gai sequences, double analytic, Cesàro space, dual

1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [8], Moricz [12], Moricz and Rhoades [13], Basarir and Solankan [2], Tripathy [20], Colak and Turkmenoglu [6], Turkmenoglu [22], and many others.

Let us define the following sets of double sequences:

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$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1\, for\, some\, l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27,28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta) -$ duals of the spaces $\mathcal{CS}_{bp}$ and

$\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]). Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a 1 in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

If X is a sequence space, we give the following definitions:

(i)X'= the continuous dual of X;

(ii)$X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\right\};$

(iii)$X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convegent, for each } x \in X\right\};$

(iv)$X^\gamma = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\};$

(v)let X be an FK-space $\supset \phi$; then $X^f = \left\{f(\Im_{mn}) : f \in X'\right\};$

(vi)$X^\delta = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\right\};$

$X^\alpha, X^\beta, X^\gamma$ are called $\alpha -$ (or Köthe-Toeplitz)dual of $X, \beta-$ (or generalized-Köthe-Toeplitz) dual of $X, \gamma -$ dual of X, $\delta -$ dual of X respectively. X^α is defined by Gupta and Kamptan [24]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\alpha \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [36] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_∞ denote the classes of all, convergent,null and bounded sclar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

2 Definitions and Preliminaries

A double sequence $x = (x_{mn})$ is called convergent (with limit L) if and only if for every $\epsilon > 0$ there exists a positive integer $n_0 = n_0(\epsilon)$ such that $|x_{mn} - L| < \epsilon$, for all $m, n \geq n_0$. We write $x_{mn} \to L$ or $lim_{m,n\to\infty} x_{mn} = L$ if (x_{mn}) is convergent to L. The limit L is called double limit or Pringsheim sense limit.

A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all Pringsheim sense double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called Pringsheim sense double entire sequence if $|x_{mn}|^{1/m+n} \to 0 \,\text{as}\, m, n \to \infty$. The double entire sequences will be denoted by Γ^2. The space $\Lambda^2 \,\text{and}\, \Gamma^2$ is a metric space with the metric

$$d(x, y) = sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, \cdots \right\} \tag{2}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2.

A sequence $x = (x_{mn})$ is called Pringsheim sense double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. The space $\chi^2$is a metric space with the metric

$$(3)\quad \widetilde{d}(x,y) = sup_{mn}\left\{((m+n)!\,|x_{mn} - y_{mn}|)^{1/m+n} : m,n : 1,2,3,\cdots\right\}$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in χ^2.

Let

$$\chi_s^2 = \{x = (x_{mn}) : \xi = ((m+n)!\xi_{mn}) \in \chi^2\}$$

where

$$(m+n)!\xi_{mn} = \\ (2!x_{11} + \ldots + (1+n)!x_{1n}) + \ldots + ((m+1!)!x_{m1} + \ldots + (m+n)!x_{mn})$$

Further, let

$$\Lambda_s^2 = \{y = (\eta_{mn}) \in \Lambda^2\},$$

where

$$\eta_{mn} = (y_{11} + y_{12} + \ldots + y_{1n}) + \ldots + (y_{m1} + y_{m2} + \ldots + y_{mn})$$

Then χ_s^2 is a metric space with the metric

$$(4)\quad \widetilde{d}\,(x,y) = sup_{mn}\left\{((m+n)!\,|\xi_{mn} - \eta_{mn}|)^{1/m+n} : m,n = 1,2,\ldots\right\}.$$

Let $\sigma\left(\chi^2\right)$ denote the vector space of all sequences $x = (x_{mn})$ such that the Cesàro transform $\left(\frac{(m+n)!\xi_{mn}}{mn}\right)$ is a double gai sequence, with metric

$$(5)\quad \widetilde{d}\,(x,y) = sup_{mn}\left\{\left(\frac{(m+n)!\,|\xi_{mn} - \eta_{mn}|}{mn}\right)^{1/m+n} : m,n = 1,2,\ldots\right\}$$

Let ϕ denote the set of all finite sequences. Let X be an FK-space containing ϕ. Then $F^+\left(X\right)$ will be the set of all those sequences z such that the series $\sum_{m,n=1}^{\infty} z_{mn} f\left(\zeta_{mn}\right)$ converges for every $f \in X^{'}$. We note that $\sigma\left(\chi^2\right)$ is normal and hence it has monotone metric. Therefore,

$$\left[\sigma\left(\chi^2\right)\right]^{\alpha} = \left[\sigma\left(\chi^2\right)\right]^{\beta} = \left[\sigma\left(\chi^2\right)\right]^{\gamma} \tag{6}$$

Also, $\sigma\left(\chi^2\right)$ has monotone metric, by Theorem 10.3.12 of [23] it has AB-property. Consequently

$$\left[\sigma\left(\chi^2\right)\right]^{f} = \left[\sigma\left(\chi^2\right)\right]^{\gamma} \tag{7}$$

Lemma 2.1 *Let X be an FK-space containing ϕ. Then (i)$X^{\gamma} \subset X^{f}$; (ii) If X has AK, $X^{\beta} = X^{f}$; (iii)If X has AD, $X^{\beta} = X^{f}$.*

3 Main Results

Proposition 3.1 *$\sigma\left(\chi^2\right) \subset \chi^2$ and the inclusion is strict.*

Proof. Let $x \in \sigma\left(\chi^2\right)$ Then $\xi \in \chi^2$. Hence

$$\left(\frac{1}{mn}(m+n)!\left|\xi_{mn}\right|\right)^{\frac{1}{m+n}} \to 0, \text{ as } m, n \to \infty.$$

But as $(mn)^{1/m+n} \to 1, as\, m, n \to \infty$, $\left((m+n)!\left|\xi_{mn}\right|\right)^{1/m+n} \to 0$, $as\, m, n \to \infty$.
Further, we have $(m+n)!x_{mn} = ((m+n)!\xi_{mn} - (m+n-1)!\xi_{m-1\,n}) - ((m+n-1)!\xi_{mn-1} - (m+n-2)!\xi_{m-1\,n-1})$
Hence

$$\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n} \leq \left((m+n)!\left|\xi_{mn}\right|\right)^{1/m+n} + \left((m+n-1)!\left|\xi_{m-1\,n}\right|\right)^{1/m+n} +$$

$$\left((m+n-1)!\left|\xi_{mn-1}\right|\right)^{1/m+n} + \left((m+m-2)!\left|\xi_{m-1\,n-1}\right|\right)^{1/m+n} \to 0$$

as $m, n \to \infty$.
As such $x \in \chi^2$. Hence $\sigma\left(\chi^2\right) \subseteq \chi^2$.

Next we show that the inclusion is strict. For this we show that there is at least one element in χ^2 which is not in $\sigma\left(\chi^2\right)$. Let us consider the sequence

$$\zeta_{11} = \begin{pmatrix} \frac{1}{2!} & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . \\ . \\ . \\ 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

with $\frac{1}{2!}$ in the $(1,1)^{th}$ position and zero elsewhere. Then we have
$2!\xi_{11} = 2!x_{11} = 1$
$4!\xi_{22} = 2!x_{11} + 3!x_{12} + 3!x_{21} + 4!x_{22} = 1 + 0 + 0 + 0 = 1$
$\vdots$
$(m+n)!\xi_{mn} = 1 + 0 + 0 + \ldots + 0 = 1$
Now $\left(\frac{1}{mn}(m+n)!\,|\xi_{mn}|\right)^{\frac{1}{m+n}} = 1$, for all m, n. Hence does not tend to zero as $m, n \to \infty$. So $\zeta_{11} \notin \sigma\left(\chi^2\right)$. Thus the inclusion is strict. This completes the proof.

Proposition 3.2 $\sigma\left(\chi^2\right)$ *has AK-property.*

Proof. Let $x = (x_{mn}) \in \sigma\left(\chi^2\right)$. Then

$$d\left(x, x^{[r,s]}\right) = sup_{mn}\left\{((m+n)!\,|x_{mn}|)^{1/m+n} : m \geq r+1, n \geq s+1\right\} \to 0 \text{ as } m, n \to \infty.$$

Therefore, $x^{[r,s]} \to x$ as $r, s \to \infty$ in $\sigma\left(\chi^2\right)$. Thus $\sigma\left(\chi^2\right)$ has AK. This completes the proof.

Proposition 3.3 $\sigma\left(\chi^2\right)$ *is solid.*

Proof. Let $|x_{mn}| \leq |y_{mn}|$ with $y = (y_{mn}) \in \sigma\left(\chi^2\right)$. Then, $|\xi_{mn}| \leq |\eta_{mn}|$ with $\eta = (\eta_{mn}) \in \chi^2$. But as χ^2 is solid, $\xi = (\xi_{mn}) \in \chi^2$. Hence $x = (x_{mn}) \in \sigma\left(\chi^2\right)$ and consequently $\sigma\left(\chi^2\right)$ is solid. This completes the proof of the proposition.

Proposition 3.4 *The $\beta-$ dual space of* $\sigma\left(\chi^2\right)$ *is* Λ^2

Proof. Step1:Let $y = (y_{mn})$ be an arbitrary point in $\left(\sigma\left(\chi^2\right)\right)^\beta$. Then either $y \in \Lambda^2$ or $y \notin \Lambda^2$. If y is not in Λ^2, then for each natural number p, we can find indices m_p, n_q such that

$$\left|y_{m_p n_q}\right|^{1/m_p+n_q} > p \quad (m_p+n_q)!, (p=1,2,3,\ldots)$$

Let us define $x=(x_{mn})$ by

$$x_{mn} = \begin{cases} \frac{1}{((m_p+n_q)! \;\; p)^{m_p+n_q}}, & \text{for } (m,n)=(m_p, n_q) \; forall\, p, q \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Then clearly x is in χ^2. But for sufficiently large mn

$$|y_{mn}x_{mn}| > 1 \tag{8}$$

Consider the sequence $z=\{z_{mn}\}$, where

$$z_{11} = x_{11} - s \text{ with } s = \sum x_{mn}$$

$$z_{mn} = x_{mn}\, otherwise.$$

Then clearly z is in $\sigma(\chi^2)$. But by (7) $\sum z_{mn}x_{mn}$ does not converge. Hence $y \notin (\sigma(\chi^2))^\beta$. Therefore

$$\left(\sigma\left(\chi^2\right)\right)^\beta \subset \Lambda^2 \tag{9}$$

Next, by Proposition-3.1, we have $\sigma(\chi^2) \subset \Lambda^2$. Hence $(\chi^2)^\beta \subset (\sigma(\chi^2))^\beta$. But as $(\chi^2)^\beta \overset{\subset}{\neq} \Lambda^2$,

$$\Lambda^2 \subset \left(\chi^2\right)^\beta. \tag{10}$$

From (8) and (9) it follows that the $\beta-$ dual space of $\sigma(\chi^2)$ is Λ^2. This completes the proof.

Proposition 3.5 $(\sigma(\chi^2))^\mu = \Lambda^2$ *for* $\mu = \alpha, \beta, \gamma, f$

Proof. We have, by Proposition 3.2, $\sigma(\chi^2)$ has AK-property. Hence by (ii)of lemma -2.1 $(\sigma(\chi^2))^\beta = (\sigma(\chi^2))^f$. But as $(\sigma(\chi^2))^\beta = \Lambda^2$, $(\sigma(\chi^2))^f = \Lambda^2$.

Further as AK implies AD, by (iii) of lemma-2.1, we have $(\sigma(\chi^2))^\beta = (\sigma(\chi^2))^\gamma$. Hence $(\sigma(\chi^2))^\gamma = \Lambda^2$

Finally, as $\sigma(\chi^2)$ is normal by Proposition 3.3, by Proposition 2.7 of [24] we get $(\sigma(\chi^2))^\alpha = (\sigma(\chi^2))^\gamma = \Lambda^2$. This completes the proof.

Proposition 3.6 *In* $\sigma(\chi^2)$ *weak convergence does not imply strong convergence.*

Proof. Let us suppose that weak convergence implies strong convergence $\sigma(\chi^2)$. Then we have $[\sigma(\chi^2)]^{\beta\beta} = \sigma(\chi^2)$ (See[23]). Then $[\sigma(\chi^2)]^{\beta\beta} = [\Lambda^2]^{\beta} \overset{\subset}{\neq} \chi^2$. But as $\sigma(\chi^2)$ is a proper subspace of χ^2, $[\sigma(\chi^2)]^{\beta\beta} \neq \sigma(\chi^2)$. Hence weak convergence does not imply strong convergence in $\sigma(\chi^2)$. This completes the proof.

Proposition 3.7 $\sigma(\chi^2)$ *has monotone metric.*

Proof. We know that

$$d(x,y) = sup_{mn}\left\{\left(\frac{1}{mn}(m+n)!\,|\xi_{mn}-\eta_{mn}|\right)^{1/m+n} : m,n = 1,2,3,\dots\right\}$$

Then we have

$$d(x^n,y^n) = sup_{n,n}\left\{\left(\tfrac{1}{n^2}(2n)!\,|\xi_{mn}-\eta_{mn}|\right)^{1/2n}\right\}$$

and

$$d(x^m,y^m) = sup_{m,m}\left\{\left(\tfrac{1}{m^2}(2m)!\,|\xi_{mm}-\eta_{mm}|\right)^{1/2m}\right\}$$

Let $m > n$. Then

$$\sup_{m,m}\left\{\left(\frac{1}{m^2}(2m)!\,|\xi_{mm}-\eta_{mm}|\right)^{1/2m}\right\} \geq \sup_{n,n}\left\{\left(\frac{1}{n^2}(2n)!\,|\xi_{mn}-\eta_{mn}|\right)^{1/2n}\right\}$$

$$\Rightarrow \quad d(x^m,y^m) \geq d(x^n,y^n) \; for\, m > n. \tag{11}$$

Thus $\{d(x^n,y^n)\}_{n=1}^{\infty}$ is monotonically increasing. Further as

$$\lim_{n\to\infty} d(x^n,y^n) = d(x,y)$$

the matrix as defined is a monotone metric for $\sigma(\chi^2)$. This completes the proof.

Proposition 3.8 $(\chi^2)^\beta \overset{\subset}{\neq} \Lambda^2$.

Proof. Let $y = (y_{mn})$ be an arbitrary element of $(\chi^2)^\beta$. Then either $y \in \Lambda^2$ or $y \notin \Lambda^2$. In the former case $(\chi^2)^\beta \subset \Lambda^2$. If y is not in Λ^2, then, proceeding in the lines of the proof of Proposition 3.4, we show that $(\chi^2)^\beta \subset \Lambda^2$.

Next we choose $y_{1n} = x_{1n} = 1$ and $y_{mn} = x_{mn} = 0\,(m > 1)$ for all n, then clearly $x \in \chi^2$ and $y \in \Lambda^2$, but

$$\sum_{m,n=1}^{\infty} x_{mn} y_{mn} = \infty.$$

As such $y \notin (\chi^2)^\beta$ This completes the proof.

Proposition 3.9 $(\Lambda^2)^\beta \overset{\subset}{\neq} \chi^2$.

Proof. Let $(x_{mn}) \in (\Lambda^2)^\beta$

$$\sum_{m,n=1}^{\infty} |x_{mn} y_{mn}| < \infty \forall\, (y_{mn}) \in \Lambda^2 \tag{12}$$

Assume that $(x_{mn}) \notin \chi^2$. Then there exist a sequence positive integers $(m_p + n_p)$ strictly increasing such that

$$\left|x_{(m_p+n_p)}\right| > \frac{1}{((m_p+n_p)!\ \ 2)^{(m_p+n_p)}}, (p = 1, 2, 3, \ldots)$$

Take

$$y_{(m_p+n_p)} = \begin{cases} (2\ \ (m_p + n_p)!)^{(m_p+n_p)}, & \text{if } (p = 1, 2, 3, \ldots) \\ 0, & \text{otherwise} \end{cases}$$

Then $(y_{mn}) \in \Lambda^2$. But $\sum_{m,n=1}^{\infty} |x_{mn} y_{mn}| = \sum_{p=1}^{\infty} \left|x_{(m_p n_p)} y_{(m_p n_p)}\right| > 1 + 1 \ldots \cdot$
We kn ow that the infinite series $1{+}1{+}1{+}\ldots$ diverges. Hence $\sum_{m,n=1}^{\infty} |x_{mn} y_{mn}|$ diverges. This contradicts (11). Hence $(x_{mn}) \in \chi^2$. Therefore

$$\left(\Lambda^2\right)^\beta \subset \chi^2 \tag{13}$$

If we now choose $y_{1n} = x_{1n} = 1$ and $y_{mn} = x_{mn} = 0\,(m > 1)$ for all n, then obviously $x \in \chi^2$ and $y \in \Lambda^2$, but $\sum_{m,n=1}^{\infty} x_{mn}y_{mn} = \infty$, hence

$$y \notin \left(\Lambda^2\right)^\beta \tag{14}$$

From (12) and (13) we are granted $(\Lambda^2)^\beta \overset{\subset}{\neq} \chi^2$. This completes the proof.

Proposition 3.10 *Let X be an FK-space containing ϕ. Then X contains $\sigma(\chi^2)$ if and only if the sequence $\{f(\zeta^{mn})\}$ belongs to Λ^2 for every f in X'.*

Proof. First, $\sigma(\chi^2)$ has AK. Hence it has AD. Therefore, by theorem 8.6.1 of [23]. $\sigma(\chi^2) \subset X \Leftrightarrow X^f \subset [\sigma(\chi^2)]^f = \Lambda^2 \Leftrightarrow \{f(\zeta^{mn})\} \in \Lambda^2$ for every f in X'. This completes the proof.

Proposition 3.11 *Let X be an FK-space containing ϕ. Then X contains $\sigma(\chi^2)$ if and only if $F^+(X) \overset{\subset}{\neq} \chi^2$*

Proof. Suppose that X contains $\sigma(\chi^2)$. Then $F^+(X)$ contains $F^+(\sigma(\chi^2))$. But, by Theorem 10.4.2 of [23] we have $F^+(\sigma(\chi^2)) = [\sigma(\chi^2)]^{f\beta} = (\Lambda^2)^\beta \overset{\subset}{\neq} \chi^2$. Hence $F^+(X) \overset{\subset}{\neq} \chi^2$.

Conversely: Suppose that $F^+(X) \overset{\subset}{\neq} \chi^2$. Then $[\chi^2]^\beta \overset{\subset}{\neq} [F^+(X)]^\beta$. But $[\chi^2]^\beta \overset{\subset}{\neq} \Lambda^2$ Therefore, $[\Lambda^2] \overset{\subset}{\neq} [F^+(X)]^\beta$. Also $X^f \subset X^{f\beta\beta} = [F^+(X)]^\beta$. Thus, $[\Lambda^2] \overset{\subset}{\neq} X^f$. But then, since Λ^2 has AD, it follows that $\sigma(\chi^2) \overset{\subset}{\neq} \chi^2 = (\Lambda^2)^f \overset{\subset}{\neq} X^{ff} \overset{\subset}{\neq} X$. This completes the proof.

Proposition 3.12 *χ^2 is dense in $\sigma(\chi^2)$.*

Proof. Let X be any element $\sigma(\chi^2)$. Take $(m,n)^{th}$ section $x = x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\zeta_{ij}$ for all $m, n \in \mathbb{N}$, Now the sequence of sequences $\left\{x^{[m,n]}\right\}$ is in χ^2. Because $\sigma(\chi^2)$ has AK. $x^{[m,n]} \to x$ in $\sigma(\chi^2)$ as $m, n \to \infty$ Therefore, x belongs to the closure of χ^2. Hence χ^2. is dense in $\sigma(\chi^2)$. This completes the proof.

Proposition 3.13 *$\sigma(\chi^2)$. is the largest AD-space X such that $X^{\beta\beta} \overset{\subset}{\neq} \chi^2$.*

Proof. Let Y be an arbitrary AD-space such that $\Lambda^2 = Y^\beta \subset Y^f$ so that $Y^{\beta\beta} = (\Lambda^2)^\beta \overset{\subset}{\neq} (\chi^2)$. But $[\sigma(\chi^2)]^\beta = \Lambda^2$. Also Y has AD. Therefore, by Theorem 8.6.1 of [23], we have $[\sigma(\chi^2)]^f \subset Y^f$ implies that $Y \subset [\sigma(\chi^2)]$. This completes the proof.

Proposition 3.14 *The unit disc (Closed unit sphere)D in $\sigma(\chi^2)$ has no extreme points.*

Proof. Let $z \in D$. Let $(m+n)!\gamma_{mn} = (2!z_{11} + \ldots + (1+n)!z_{1n}) + \ldots + ((m+1)!z_{m1} + \ldots + (m+n)!z_{mn})$
Let us consider $\left\{\frac{1}{mn}(m+n)!\,|\gamma_{mn}|\right\}^{1/m+n} < 1$ for some $mn = m_0n_0$. This is possible, because the sequence $\left\{\frac{1}{mn}(m+n)!\,|\gamma_{mn}|\right\}^{1/m+n}$ is a double gai sequence. Let $\epsilon > 0$ be defined by $\epsilon < 1 - \left\{\frac{1}{mn}(m+n)!\,|\gamma_{mn}|\right\}^{1/m+n}$. In case $mn = m_0n_0$ we take

$$\xi = \gamma + \epsilon\zeta^{m_0n_0}$$
$$\eta = \gamma - \epsilon\zeta^{m_0n_0}$$

But then

$$\left\{\frac{1}{m_0n_0}(m+n)!\,|\gamma_{mn}|\right\}^{1/m_0+n_0} = \left\{\left\{\frac{1}{mn}(m+n)!\,|\gamma_{mn}|\right\}^{1/m+n} + \epsilon\right\}$$
$$< \left\{\frac{1}{m_0n_0}(m+n)!\,|\gamma_{mn}|\right\}^{1/m_0+n_0} + \epsilon < 1$$

so that ϵ is in D. Similarly it can be shown that η is in D. In case $mn \neq m_0n_0$, we take

$$\xi = \gamma + \epsilon\left(\zeta^{m_0,n_0} - \zeta^{m_0+1,n_0+1}\right)$$

$$\eta = \gamma - \epsilon\left(\zeta^{m_0,n_0} - \zeta^{m_0+1,n_0+1}\right)$$

so that $\left\{\frac{1}{mn}(m+n)!\,|\xi_{mn}|\right\}^{1/m+n} \leq d(z,0) \leq 1$ for $mn \neq m_0n_0$. Therefore, $d(x,0) \leq 1$, so that $x \in D$. A similar argument shows that $y \in D$. In either case $z = \frac{x+y}{2}$. So, z is not an extreme point of D. Thus, D has no extreme points in $\sigma(\chi^2)$. This completes the proof.

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Inclusion properties of certain subclasses of analytic functions associated with the Dziok-Raina operator [1]

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Abstract

The purpose of the present paper is to study various inclusion properties for several new classes of analytic univalent functions which are defined here by means of a linear operator.

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1 Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

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which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g.,[3] and [18]):

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in A$ given by (1.1) and $g(z) \in A$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For $0 \leq \eta, \gamma < 1$, we denote by $S(\eta),\ K(\eta)$ and $C(\eta, \gamma)$, the subclasses of A consisting of all analytic functions which are, respectively, starlike of order η, convex of order η and close-to-convex functions of order γ type η in U. For various other interesting developments involving functions in the class A, see the work of Owa and Srivastava [24].

Let S be the class of functions ϕ which are analytic and univalent in U and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\operatorname{Re}\{\phi(z)\} > 0,\ z \in U$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $S(\eta; \phi)$, $K(\eta; \phi)$ and $C(\eta, \gamma; \phi, \psi)$ of A, $0 \leq \eta, \gamma < 1$ and $\phi, \psi \in S$ (cf. [4] and [16]), which are defined by:

$$S(\eta; \phi) = \left\{ f \in A : \frac{1}{1-\eta} \left(\frac{zf'(z)}{f(z)} - \eta \right) \prec \phi(z),\quad z \in U \right\};$$

$$K(\eta; \phi) = \left\{ f \in A : \frac{1}{1-\eta} \left(1 + \frac{zf''(z)}{f'(z)} - \eta \right) \prec \phi(z),\quad z \in U \right\};$$

and

$$C(\eta,\gamma;\phi,\psi) = \left\{ f \in A : \exists\ g \in S(\eta;\phi) \text{ s.t.} \tfrac{1}{1-\gamma} \left(\tfrac{zf'(z)}{g(z)} - \gamma \right) \prec \psi(z),\ z \in U \right\}.$$

We note that the classes mentioned above are the familiar classes which have been used widely on the space of analytic and univalent functions in U, and for special choices for the functions ϕ and ψ involved in these definitions, we can obtain the well-known subclasses of A. For examples, we have

$$S\left(\eta; \frac{1+z}{1-z}\right) = S(\eta), \ K\left(\eta; \frac{1+z}{1-z}\right) = K(\eta),$$

and

$$C\left(\eta, \gamma; \frac{1+z}{1-z}, \frac{1+z}{1-z}\right) = C(\eta, \gamma).$$

Let $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s$ $(q, s \in \mathbb{N} = \{1, 2, ...\})$ be positive real parameters such that

$$1 + \sum_{n=1}^{s} B_n - \sum_{n=1}^{q} A_n \geq 0.$$

The Wright generalized hypergeometric function [25] (see also [26])

$${}_q\Psi_s[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s); z]$$

$$=_q \Psi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z],$$

is defined by

$${}_q\Psi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z]$$

$$= \sum_{k=0}^{\infty} \{\prod_{n=1}^{q} \Gamma(\alpha_n + kA_n)\}\{\prod_{n=1}^{s} \Gamma(\beta_n + kB_n)\}^{-1} \frac{z^k}{k!} \quad (z \in U).$$

If $A_n = 1$ $(n = 1, ..., q)$ and $B_n = 1$ $(n = 1, ..., s)$, we have the relationship:

$$\Omega_q\Psi_s[(\alpha_n, 1)_{1,q}; (\beta_n, 1)_{1,s}; z] =_q F_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

where ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is the generalized hypergeometric function (see for details [6], [7], [8], [10] and [14]) and

$$\Omega = (\prod_{n=1}^{q} \Gamma(\alpha_n))^{-1} (\prod_{n=1}^{s} \Gamma(\beta_n)). \tag{1.2}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [5], [6], [7], [19], [20] and [21]).

By using the generalized hypergeometric function Dziok and Srivastava [7] introduced a linear operator. In [5] Dziok and Raina and in [1] Aouf and Dziok extended the linear operator by using the Wright generalized hypergeometric function.

First we define a function ${}_q\Phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z]$ by

(1.3)
$$ {}_q\Phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] = \Omega\ z\ {}_q\Psi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z], $$

and consider a linear operator $\theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}] : A \to A$ defined by the following Hadamard product:

(1.4)
$$ \theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}]f(z) =_q \Phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] * f(z). $$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$ \theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}]f(z) = z + \sum_{k=2}^{\infty} \Omega \sigma_k a_k z^k, $$

where Ω is given by (1.2) and σ_k is defined by

$$ \sigma_k = \frac{\Gamma(\alpha_1 + (k-1)A_1)...\Gamma(\alpha_q + (k-1)A_q)}{\Gamma(\beta_1 + (k-1)B_1)...\Gamma(\beta_s + (k-1)B_s)(k-1)!}. $$

Corresponding to a function ${}_q\Phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z]$ defined by (1.3), we introduce a function $\Phi_{\lambda,q,s}[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z]$ given by

$$ {}_q\Phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] * \Phi_{\lambda,q,s}[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] $$

(1.5)
$$ = \frac{z}{(1-z)^{\lambda+1}} \quad (\lambda > -1). $$

Analogous to $\theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}]$ defined by (1.4), we now define the linear operator $\theta_\lambda[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}] : A \to A$ as follows:

$$\theta_\lambda[(\alpha_n, A_n)_{1,q};(\beta_n, B_n)_{1,s}]f(z)=\Phi_{\lambda,q,s}[(\alpha_n, A_n)_{1,q};(\beta_n, B_n)_{1,s};z] * f(z)$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(\lambda+k) \prod_{n=1}^{s} \Gamma(\beta_n + (k-1)B_n) \prod_{n=1}^{q} \Gamma(\alpha_n)}{\Gamma(\lambda+1) \prod_{n=1}^{q} \Gamma(\alpha_n + (k-1)A_n) \prod_{n=1}^{s} \Gamma(\beta_n)} a_k z^k \qquad (1.6)$$

$$(\lambda > -1;\ z \in U;\ f \in A).$$

For convenience, we write

$$\theta_{\lambda,q,s}(\alpha_1; A_1, B_1) = \theta_\lambda[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s)].$$

We note that for $A_n = 1$ $(n = 1, ..., q)$ and $B_n = 1$ $(n = 1, ..., s)$, we have:

(i) $\theta_{\lambda-1,2,1}(\delta+1, 1, 1; 1, 1)f(z) = I_{\delta,\lambda}f(z)$ $(\lambda > 0,\ \delta > -1)$, where $I_{\delta,\lambda}$ was introduced by Choi et. al. [4];

(ii) $\theta_{1,2,1}(\delta+1, 1, 1; 1, 1)f(z) = I_\delta f(z)$ $(\delta \in \mathbb{N} \cup \{0\})$, where I_δ is the Noor integral operator of δth order of f studied by Liu [13], Noor [22] and Noor and Noor[23];

(iii) $\theta_{\lambda-1,q,s}(\alpha_1; 1, 1)f(z) = H_{\lambda,q,s}(\alpha_1)f(z)$ $(\lambda > 0)$, where $H_{\lambda,q,s}(\alpha_1)$ was introduced and studied by Kwon and Cho [11].

It is easily verified from the definition (1.6) that

$$A_1 z(\theta_{\lambda,q,s}(\alpha_1+1; A_1, B_1)f(z))' = \alpha_1\theta_{\lambda,q,s}(\alpha_1, A_1, B_1)f(z) -$$

$$(\alpha_1 - A_1)\theta_{\lambda,q,s}(\alpha_1+1; A_1, B_1)f(z)\ (A_1 > 0) \qquad (1.7)$$

and

$$z(\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f(z))' = (\lambda+1)\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)f(z) -$$

$$\lambda\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f(z). \tag{1.8}$$

Next, by using the operator $\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)$, we introduce the following classes of analytic functions for $\phi, \psi \in S,\ \lambda > -1,\ 0 \leq \eta, \gamma < 1$:

$$S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi) := \{f \in A : \theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f \in S(\eta; \phi)\},$$

$$K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi) := \{f \in A : \theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f \in K(\eta; \phi)\},$$

and

$$C_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta, \gamma; \phi, \psi) := \{f \in A : \theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f \in C(\eta, \gamma; \phi, \psi)\}.$$

We also note that

$$f(z) \in K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \Leftrightarrow zf^{'}(z) \in S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi). \tag{1.9}$$

In particular, we set

$$S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \frac{1+Az}{1+Bz}) = S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; A, B)\ \ (-1 < B < A \leq 1)$$

and

$$K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \frac{1+Az}{1+Bz}) = K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; A, B)\ \ (-1 < B < A \leq 1).$$

In this paper, we investigate several inclusion properties of the classes $S_{\lambda,q,s}\ (\alpha_1; A_1, B_1; \eta; \phi)$, $K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi)$ and $C_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta, \gamma; \phi, \psi)$ associated with the operator $\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)$. Some applications involving integral operators are also considered.

2 Inclusion properties involving the operator $\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)$

To prove our main results, we need the following lemmas.

Lemma 1 [9]. *Let ϕ be convex univalent in U with $\phi(0) = 1$ and* $\mathrm{Re}\{\kappa\phi(z) + \nu\} > 0$ $(\kappa, \nu \in C)$. *If p is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\kappa p(z) + \nu} \prec \phi(z) \quad (z \in U),$$

implies

$$p(z) \prec \phi(z) \quad (z \in U).$$

Lemma 2 [17]. *Let ϕ be convex univalent in U and ω be analytic in U with* $\mathrm{Re}\{\omega(z)\} \geq 0$. *If p is analytic in U and $p(0) = \phi(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec \phi(z) \quad (z \in U),$$

implies

$$p(z) \prec \phi(z) \quad (z \in U).$$

Theorem 1. *Let $\frac{\alpha_1}{A_1} > 1$, $\lambda > 0$ and $\phi \in S$. Then we have*

$$S_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \subset S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \subset S_{\lambda,q,s}(\alpha_1 + 1; A_1, B_1; \eta; \phi).$$

Proof. To prove the first part of Theorem 1, let $f \in S_{\lambda+1,q,s}(\alpha_1; \eta; \phi)$ and set

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f(z))'}{\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f(z)} - \eta\right), \tag{2.1}$$

where p is analytic in U with $p(0) = 1$. Applying (1.8) in (2.1), we obtain

$$\frac{1}{1-\eta}\left(\frac{z(\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)f(z))'}{\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)f(z)} - \eta\right)$$

$$= p(z) + \frac{zp'(z)}{(1-\eta)p(z) + \lambda + \eta} \quad (z \in U). \tag{2.2}$$

Since $\lambda > 0$ and $\phi \in S$, we see that

$$\mathrm{Re}\{(1-\eta)\phi(z) + \lambda + \eta\} > 0 \quad (z \in U).$$

Applying Lemma 1 to (2.2), it follows that $p \prec \phi$, that is, $f \in S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi)$. Moreover, by using the arguments similar to those detailed above with (1.7), we can prove the second part of Theorem 1. Therefore we complete the proof of Theorem 1.

Theorem 2. *Let* $\frac{\alpha_1}{A_1} > 1$, $\lambda > 0$ *and* $\phi \in S$ *. Then we have*

$$K_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \subset K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \subset K_{\lambda,q,s}(\alpha_1+1; A_1, B_1; \eta; \phi).$$

Proof. Applying (1.9) and Theorem 1, we observe that

$$\begin{gathered} f(z) \in K_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \Leftrightarrow zf'(z) \in S_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \\ \Rightarrow zf'(z) \in S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \Leftrightarrow f(z) \in K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi), \\ f(z) \in K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \Leftrightarrow zf'(z) \in S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \\ \Rightarrow zf'(z) \in S_{\lambda,q,s}(\alpha_1+1; A_1, B_1; \eta; \phi) \Leftrightarrow f(z) \in K_{\lambda,q,s}(\alpha_1+1; A_1, B_1; \eta; \phi), \end{gathered}$$

which evidently proves Theorem 2.

Taking

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 < B < A \le 1;\ z \in U),$$

in Theorems 1 and 2, we have

Corollary 1. *Let* $\frac{\alpha_1}{A_1} > 1$ *and* $\lambda > 0$*. Then we have*

$$\begin{aligned} S_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta; A, B) &\subset S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; A, B) \\ &\subset S_{\lambda,q,s}(\alpha_1+1; A_1, B_1; \eta; A, B), \end{aligned}$$

and

$$\begin{aligned} K_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta; A, B) &\subset K_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; A, B) \\ &\subset K_{\lambda,q,s}(\alpha_1+1; A_1, B_1; \eta; A, B). \end{aligned}$$

Next, by using Lemma 2, we obtain the following inclusion relation for the class $C_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta, \gamma; \phi, \psi)$.

Theorem 3. *Let* $\frac{\alpha_1}{A_1} > 1$, $\lambda > 0$ *and* $\phi, \psi \in S$. *Then we have*

$$\begin{aligned} C_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta, \gamma; \phi, \psi) &\subset C_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta, \gamma; \phi, \psi) \\ &\subset C_{\lambda,q,s}(\alpha_1 + 1; A_1, B_1; \eta, \gamma; \phi, \psi). \end{aligned}$$

Proof. To prove the first inclusion of Theorem 3, let $f \in C_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta, \gamma; \phi, \psi)$. Then, from the definition of $C_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta, \gamma; \phi, \psi)$, there exists a function $g \in S_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta; \phi)$ such that

$$\frac{1}{1-\gamma}\left(\frac{z(\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)f(z))'}{\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)g(z)} - \gamma\right) \prec \psi(z) \quad (z \in U).$$

Now, let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f(z))'}{\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)g(z)} - \gamma\right), \tag{2.3}$$

where p is analytic in U with $p(0) = 1$. Using (1.8), we obtain

$$(1-\gamma)zp'(z)\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)g(z) + [(1-\gamma)p(z)+\gamma]z(\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)g(z))'$$

$$= (\lambda+1)z(\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)f(z))' - \lambda z(\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f(z))'. \tag{2.4}$$

Since $g \in S_{\lambda+1,q,s}(\alpha_1; A_1, B_1; \eta; \phi) \subset S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi)$, by Theorem 1, we set

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)g(z))'}{\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)g(z)} - \eta\right), \tag{2.5}$$

where $q \prec \phi$ in U . Then, using (1.8) once again, we have

$$(\lambda+1)\frac{\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)g(z)}{\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)g(z)} = (1-\eta)q(z) + \lambda + \eta. \tag{2.6}$$

From (2.4) and (2.6), we obtain

$$\begin{aligned} &\frac{1}{1-\gamma}\left(\frac{z(\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)f(z))'}{\theta_{\lambda+1,q,s}(\alpha_1; A_1, B_1)g(z)} - \gamma\right) \\ &= p(z) + \frac{zp'(z)}{(1-\eta)q(z) + \lambda + \eta} \prec \psi(z) \quad (z \in U). \end{aligned} \tag{2.7}$$

Since $\lambda > 0$ and $q \prec \phi$ in U,

$$\mathrm{Re}\{(1-\eta)q(z)+\lambda+\eta\} > 0 \quad (z \in U).$$

Hence, applying Lemma 2, we can show that $p \prec \psi$ in U, so that $f \in C_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta, \gamma; \phi, \psi)$. Moreover, we have the second inclusion by using arguments similar to those detailed above with (1.7), we obtain $C_{\lambda,q,s}(\alpha_1; A_1,\ B_1; \eta, \gamma; \phi, \psi) \subset C_{\lambda,q,s}(\alpha_1+1; A_1, B_1; \eta, \gamma; \phi, \psi)$. Therefore we complete the proof of Theorem 3.

3 Inclusion properties involving the integral operator F_μ

In this section, we consider the generalized Bernardi–Libera–Livingston linear integral operator $F_\mu(f)(z)$ (see [2],[12] and [15]) defined by

$$F_\mu(f) = F_\mu(f)(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt \quad (f \in A;\ \mu > -1). \tag{3.1}$$

We first prove the following.

Theorem 4. *If* $f \in S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi)$, *then* $F_\mu(f) \in S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi)$ $(\mu \geq 0)$.

Proof. Let $f \in S_{\lambda,q,s}(\alpha_1; A_1, B_1; \eta; \phi)$ and set

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)F_\mu(f)(z))'}{\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)F_\mu(f)(z)} - \eta\right), \tag{3.2}$$

where p is analytic in U with $p(0) = 1$. From (3.1), we have

$$\begin{aligned} & z(\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)F_\mu(f)(z))' \\ = \ & (\mu+1)\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f(z) - \mu\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)F_\mu(f)(z). \end{aligned} \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$(\mu+1)\frac{\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)f(z)}{\theta_{\lambda,q,s}(\alpha_1; A_1, B_1)F_\mu(f)(z)} = (1-\eta)p(z) + \mu + \eta. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$\frac{1}{1-\eta}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)f(z))'}{\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)f(z)}-\eta\right)=p(z)+\frac{zp'(z)}{(1-\eta)p(z)+\mu+\eta}\ (z\in U).$$

Hence, by virtue of Lemma 1, we conclude that $p\prec\phi$ in U, which implies that $F_\mu(f)\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)$.

Next, we derive an inclusion property involving F_μ, which is given by the following.

Theorem 5. *If* $f\in K_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)$, *then* $F_\mu(f)\in K_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)$ $(\mu\geq 0)$.

Proof. By applying Theorem 4, it follows that

$$\begin{aligned}
f(z) &\in K_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)\Leftrightarrow zf'(z)\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)\\
&\Rightarrow F_\mu(zf'(z))\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)\\
&\Leftrightarrow z(F_\mu(f)(z))'\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)\\
&\Leftrightarrow F_\mu(f)(z)\in K_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi),
\end{aligned}$$

which proves Theorem 5.

From Theorems 4 and 5,we have

Corollary 2. *If* $f\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;A,B)$ *(or* $K_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;A,B)$*), then* $F_\mu(f)\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;A,B)$ *(or* $K_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;A,B)$*)* $(\mu\geq 0)$.

Finally, we prove.

Theorem 6. *If* $f\in C_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta,\gamma;\phi,\psi)$, *then* $F_\mu(f)\in C_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta,\gamma;\phi,\psi)$ $(\mu\geq 0)$.

Proof. Let $f\in C_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta,\gamma;\phi,\psi)$. Then, from the definition of $C_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta,\gamma;\phi,\psi)$, there exists a function $g\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)$ such that

$$\frac{1}{1-\gamma}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)f(z))'}{\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)g(z)}-\gamma\right)\prec\psi(z)\quad(z\in U).$$

Thus, we set

$$p(z)=\frac{1}{1-\gamma}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)F_\mu(f)(z))'}{\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)F_\mu(g)(z)}-\gamma\right),$$

where p is analytic in U with $p(0)=1$. Since $g\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)$, we see from Theorem 1 that $F_\mu(g)\in S_{\lambda,q,s}(\alpha_1;A_1,B_1;\eta;\phi)$. using (3.3), we have

$$[(1-\gamma)p(z)+\gamma]\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)F_\mu(g)(z)+\mu\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)F_\mu(f)(z)$$
$$=(\mu+1)\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)f(z).$$

Then, by a simple calculation, we get

$$(\mu+1)\frac{z(\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)f(z))'}{\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)F_\mu(g)(z)}=[(1-\eta)p(z)+\eta][(1-\eta)q(z)+\mu+\eta].$$

where

$$q(z)=\frac{1}{1-\eta}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)F_\mu(g)(z))'}{\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)F_\mu(g)(z)}-\eta\right).$$

Hence, we have

$$\frac{1}{1-\eta}\left(\frac{z(\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)f(z))'}{\theta_{\lambda,q,s}(\alpha_1;A_1,B_1)g(z)}-\eta\right)=p(z)+\frac{zp'(z)}{(1-\eta)q(z)+\mu+\eta}.$$

The remaining part of the proof in Theorem 6 is similar to that of Theorem 3 and so we omit it.

Remark 1. *If we take* $A_n=1$ $(n=1,...,q)$ *and* $B_n=1$ $(n=1,...,s)$ *and replaced* λ *with* $\lambda-1$ *in the above results of this paper, we obtain the results of Kwon and Cho [11].*

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The fuzzy join and extension hyperoperations obtained from a fuzzy binary relation [1]

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Abstract

On a set (H, R) equipped with a fuzzy binary relation we introduce a family of join hyperoperations $*_{R_p}$ where $p \in [0,1]$. As a result we obtain a family of semihypergroup $(H, *_{R_p})$. We show that for every $a, b \in H$ the family $\{a *_{R_p} b\}_{p\in[0,1]}$ can be considered as the p-cuts of a fuzzy set $a *_R b$ and in this manner we synthesize a fuzzy hyperoperation $*_R$. Similarly, we synthesize a fuzzy hyperoperation $/_R$ from $/_{R_p}$ which is derived from $*_{R_p}$.

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1 Introduction and Preliminaries

Hyperstructures and binary relations have been studied by many reachers(For instance [1, 2, 4, 5, 6, 7, 9, 10, 18, 20, 23, 22]). J. Chvalina [1, 2] and D. Hort [18] use ordered structures for the construction of semihypergroups and hypergroups. Stefanos Spartalis [23] constructs H_v-semigroup, semihypergroup and H_v-group by the use of a binary relation. P. Corsini and V. Leoreanu [6] study hypergroups and binary relations. Feng [11, 12, 13, 14, 15, 16, 17] researches hyperoperations, hypergraphs and binary relations.

A partial hypergroupoid $(H, *)$ is a nonempty set H with a function from $H \times H$ to the set of subsets of H.

A hypergroupoid is a nonempty set H, endowed with a hyperoperation, that is a function from $H \times H$ to the set of nonempty subsets of H.

If $A, B \in \mathbf{P}(H) - \{\emptyset\}$, then we define $A * B = \cup\{a * b | a \in A, b \in B\}$, $x * B = \{x\} * B$ and $A * y = A * \{y\}$.

A hypergoupoid $(H, *)$ is called H_v-semigroup [24, 8], if $x * (y * z) \cap (x * y) * z \neq \emptyset$, for all $x, y, z \in H$. $(H, *)$ is called semihypergoup in the case that the equivalent hold.

Let $(H, *)$, $(H, \circ)$ be two hypergroupoids defined on the same set. We write $* \leq \circ$ [24] iff there exist an $f \in Aut(H, \circ)$, such that $x * y \subseteq f(x \circ y)$, for all $x, y \in H$.

The collection of all fuzzy sets of H is denoted by $\mathbf{F}(H)$.

A fuzzy hyperoperation is a function $* : H \times H \to \mathbf{F}(H)$.

Given a fuzzy set $M : H \to [0, 1]$, the p-cut of M is denoted by M_p and defined by $M_p \doteq \{x \in H | M(x) \geq p\}$.

For all $M, N \in \mathbf{P}(H)$ we write $M \sim N$ iff $\exists x \in M \cap N$. For all $M, N \in \mathbf{F}(H)$ and $p \in [0, 1]$ we write $M \sim_p N$ iff $\exists x \in H : M(x) \wedge N(x) \geq p$.

For all $M, N \in \mathbf{F}(H)$ we write $M \subseteq N$ iff $M(x) \leq N(x)$, for all $x \in H$.

A fuzzy binary relation on a set H is a map $R : H \times H \rightarrow [0, 1]$.

Proposition 1 ([21, 19]) *Take any* $M \in \mathbf{F}(H)$ *with* p*-cuts* $\{M_p\}_{p\in[0,1]}$*. Then we have the following.*

(i) For all $p, q \in [0, 1]$ *we have:* $p \leq q \Rightarrow M_q \subseteq M_p$*.*

(ii) For all $P \in [0, 1]$ *we have:* $\cap_{p\in P} M_p = M_{\vee P}$*.*

(iii) $M_0 = H$*.*

Proposition 2 ([21, 19]) *Consider a family of sets* $\{\widetilde{M_p}\}_{p\in[0,1]}$ *which satisfy the following.*

(i) For all $p, q \in [0, 1]$ *we have:* $p \leq q \Rightarrow \widetilde{M_q} \subseteq \widetilde{M_p}$*.*

(ii) For all $P \in [0, 1]$ *we have:* $\cap_{p\in P} \widetilde{M_p} = \widetilde{M}_{\vee P}$*.*

(iii) $\widetilde{M_0} = H$*.*

Define the fuzzy set M *as follows: for all* $x \in H$ *define* $M(x) \doteq \vee\{p | x \in \widetilde{M_p}\}$*. Then for all* $p \in [0, 1]$ *we have* $M_p = \widetilde{M_p}$*.*

Proposition 3 ([21, 19]) *For any fuzzy sets* $M, N \in \mathbf{F}(H)$ *we have:* $M = N \Leftrightarrow (\forall p \in [0, 1]$ *we have* $M_p = N_p)$*.*

The readers can consult [3] to learn more about hyperstructure and [21] to know more about fuzzy set.

In this paper we do the following. On a set (H, R) equipped with a fuzzy binary relation we generalize the $\circ$ hyperoperation(introduced by I.G. Rosenberg[22] and studied by P. corsini and V. Leoreanu[5]) and introduce a family of join hyperoperations $*_{R_p}$ where $p \in [0, 1]$. As a result we obtain a family of semihypergroup $(H, *_{R_p})$. We show that for every $a, b \in H$ the family $\{a *_{R_p} b\}_{p\in[0,1]}$ can be considered as the p-cuts of a fuzzy set $a *_R b$ and in this manner we synthesize a fuzzy hyperoperation $*_R$. Similarly, we synthesize a fuzzy hyperoperation $/_R$ from $/_{R_p}$ which is derived from $*_{R_p}$.

2 The construction of $*_{R_p}$ and $*_R$

Firstly, we define a family of join hyperoperations and then we synthesize a fuzzy hyperoperation in this section.

Suppose (H, R) is a nonempty set equipped with a fuzzy binary relation and $x \in H$. We set

$R_p(x) = \{y \in H | R(x, y) \geq p\}$ and $\overline{R}_p(x) = \{z \in H | R(z, x) \geq p\}$.

For every $p \in [0, 1]$ we define the p-join hyperoperation $*_{R_p}$ as follows.

$$*_{R_p} : H \times H \to \mathbf{P}(H) : \quad (x, y) \to R_p(x) \cup R_p(y).$$

Then, $(H, *_{R_p})$ is a partial hypergoupoid. Similarly, a hyperoperation $\overline{*}_{R_p}$ is defined by the way $x \overline{*}_{R_p} y = \overline{R}(x) \cup \overline{R}(y)$, for all $x, y \in H$.

In what follows we focus on the hyperoperation $*_{R_p}$ and notice that the results for the $\overline{*}_{R_p}$ are similar.

It is easy to see that $a *_{R_p} b = b *_{R_p} a$ for all $a, b \in H$, $p \in [0, 1]$. So $(H, *_{R_p})$ obtained from (H, R) with a fuzzy binary relation is commutative .

Remark 1 *If R is a binary relation and we set $p \neq 0$, then we recover the $\circ$ hyperoperation [5, 22].*

Let (H, R, S) be a nonempty set with two fuzzy binary relations R, S. The following example shows that $R \neq S \not\Rightarrow (H, *_{R_p}) \neq (H, *_{S_p})$.

Example 1 *Suppose $p = 0.5$ and fuzzy binary relations tables for (H, R, S) where $H = \{e, a\}$ is*

R	e	a
e	0.5	0.8
a	0.5	0.5

S	e	a
e	0.5	1
a	0.5	0.5

Then

$*_{R_{0.5}}$	e	a
e	$\{e,a\}$	$\{e,a\}$
a	$\{e,a\}$	$\{e,a\}$

$*_{S_{0.5}}$	e	a
e	$\{e,a\}$	$\{e,a\}$
a	$\{e,a\}$	$\{e,a\}$

Remark 2 *Suppose (H, R, S) is a nonempty set equipped with two fuzzy binary relations. Then $R \subseteq S \Rightarrow *_{R_p} \leq *_{S_p}$ for any $p \in [0, 1]$.*

Indeed, let $a, b \in H$, $p \in [0, 1]$. Since $R \subseteq S$ it follows that $a *_{R_p} b = R_p(a) \cup R_p(b) \subseteq S_p(a) \cup S_p(b) = a *_{S_p} b$.

The following proposition shows that $a *_{R_p} b$ viewed as function of p behave like p-cuts of a fuzzy set.

Proposition 4 *Let (H, R) be a nonempty set equipped with a fuzzy binary relation. For all $a, b \in H$, $p \in [0, 1]$ we have:*

(i) For all $p, q \in [0, 1] : p \leq q \Rightarrow a *_{R_q} b \subseteq a *_{R_p} b$;

(ii) For all $P \subseteq [0, 1] : \cap_{p \in P} a *_{R_p} b = a *_{R_{\vee P}} b$;

(iii) $a *_{R_0} b = H$.

Proof. Take any $a, b \in H$ and $p \in [0, 1]$.

$a *_{R_p} b = R_p(x) \cup R_p(y) = \{x \in H | R(a, x) \geq p\} \cup \{y \in H | R(b, y) \geq p\} = \{x \in H | R(a, x) \geq p$, or $R(b, x) \geq p\} \cong R_p$ where R is restricted to $\{a, b\} \times H$. Since R is a fuzzy set of $H \times H$, it is a fuzzy set of $\{a, b\} \times H$. Apply Proposition 1.1 and we complete the proof.

We will now construct the fuzzy hyperoperation $*_R$.

Definition 1 *For all $a, b \in H$ define the fuzzy set $a *_R b$ by defining (for any $x \in H$)*

$$(a *_R b)(x) \doteq \vee\{q | x \in a *_{R_q} b\}.$$

Proposition 5 *For all $a, b \in H$ and $p \in [0, 1]$ we have $(a *_R b)_p = a *_{R_p} b$.*

Proof. The required result follows from the previous definition, Proposition 2.4 and Proposition 1.2.

Proposition 6 *For all $a, b, c \in H$, $p \in [0,1]$ we have*

$$a *_R b = b *_R a.$$

Proof. For all $p \in [0,1]$ we have $(a *_R b)_p = a *_{R_p} b = b *_{R_p} a = (b *_R a)_p$. Since $a *_R b$ and $b *_R a$ have the same cuts, they are identical(see Proposition 1.3).

3 The construction of $/_{R_p}$ and $/_R$

We will introduce the p-extension hyperoperation which is derived from the p-join hyperoperations.

Definition 2 *For every $p \in [0,1]$ and for all $a, b \in H$, the p-extension hyperoperation is denoted by $a/_{R_p}b$ and is defined by:*
$a/_{R_p}b \doteq \{x | a \in x *_{R_p} b\} = \{x | R(x,a) \geq p$ *or* $R(b,a) \geq p\}$.

The following example shows the p-extension hyperoperation does not enjoy the join property.

Example 2 *Suppose that the fuzzy binary relation table for (H, R) where $H = \{e, a, b\}$ is*

R	e	a	b
e	0	0	0
a	1	0.5	0
b	0	0	0

Then $e/_{R_{0.5}}b = \{a\}$ and $a/_{R_{0.5}}b = \{a\}$. But $e *_{R_{0.5}} b = \emptyset$ and $a *_{R_{0.5}} b = \{e, a\}$.

Proposition 7 *If R is reflexive(i.e. $R(x,x) = 1$, $\forall x, y \in H$) and symmetry(i.e. $R(x,y) = R(y,x)$, $\forall x, y \in H$), then $a/_{R_p}b \sim c/_{R_p}d \Rightarrow a *_{R_p} d \sim b *_{R_p} c$, for all $a, b \in H$, $p \in [0,1]$.*

Proof. Take any $x \in a/_{R_p}b \cap c/_{R_p}d$. Then

$x \in a/_{R_p}b \Rightarrow R(b,a) \geq p$ or $R(x,a) \geq p$,

$x \in c/_{R_p}d \Rightarrow R(d,c) \geq p$ or $R(x,c) \geq p$.

Case 1. If $R(b,a) \geq p$ and $R(d,c) \geq p$, then $\{a,b,c,d\} \in a *_{R_p} b$ and $\{a,b,c,d\} \in b *_{R_p} c$. So $a *_{R_p} d \sim b *_{R_p} c$.

Case 2. If $R(b,a) \geq p$ and $R(x,c) \geq p$, then $\{a,b\} \in a *_{R_p} b$ and $\{a,b,c,x\} \in b *_{R_p} c$. So $a *_{R_p} d \sim b *_{R_p} c$.

Case 3. If $R(x,a) \geq p$ and $R(d,c) \geq p$, then $\{a,c,d,x\} \in a *_{R_p} b$ and $\{b,c,d\} \in b *_{R_p} c$. So $a *_{R_p} d \sim b *_{R_p} c$.

Case 4. If $R(x,a) \geq p$ and $R(x,c) \geq p$, then $\{a,d,x\} \in a *_{R_p} b$ and $\{b,c,x\} \in b *_{R_p} c$. So $a *_{R_p} d \sim b *_{R_p} c$.

Proposition 8 *If R is reflexive and symmetry, then $(H, *_{R_p}, /_{R_p})$ is a join space.*

Proof. Obviously, $a *_{R_p} H \subseteq H$, for any $a \in H$.

Conversely, for all $x \in H$, Since R is reflexive, $x \in x *_{R_p} x$. So $x \in x *_{R_p} x \cup a *_{R_p} a = a *_{R_p} x \subseteq a *_{R_p} H$. Hence $H \subseteq a *_{R_p} H$.

Therefore $H = a *_{R_p} H$. And so $(H, *_{R_p})$ is a commutative hypergroup.

Apply the above proposition and the proof is finished.

Proposition 9 *Suppose (H, R, S) is a nonempty set with two fuzzy binary relations. Then $R \subseteq S \Rightarrow /_{R_p} \leq /_{S_p}$, for all $p \in [0,1]$.*

Proof. Take any $a, b \in H$, $p \in [0,1]$. Suppose $x \in a/_{R_p}b$. Then $R(b,a) \geq p$ or $R(x,a) \geq p$. So $S(b,a) \geq p$ or $S(x,a) \geq p$, for $R \subseteq S$. Hence $x \in a/_{S_p}b$. Thus $a/_{R_p}b \subseteq a/_{S_p}b$.

Proposition 10 *Let (H, R) be a nonempty set equipped with a fuzzy binary relation. For all $a, b \in H$, $p \in [0,1]$ we have:*

(i) For all $p, q \in [0,1] : p \leq q \Rightarrow a/_{R_q}b \subseteq a/_{R_p}b$;

(ii) For all $P \subseteq [0,1] : \cap_{p \in P} a/_{R_p}b = a/_{R_{\vee P}}b$;

(iii) $a/_{R_0}b = H$.

Proof. Take any $a, b \in H$, $p \in [0,1]$.

Definition 3 *For all* $a, b \in H$ *define the fuzzy set* $a/_R b$ *by defining (for any* $x \in H$*)*
$(a/_R b)(x) \doteq \vee\{p | x \in a/_{R_p} b\}$.

Proposition 11 *For all* $p \in [0,1]$ *and* $a, b \in H$ *we have* $(a/_R b)_p = a/_{R_p} b$.

Proposition 12 *For all* $a, b, c, d \in H$ *and* $p \in [0,1]$ *we have:*
(i) $a *_R b \sim_p c *_R d \Leftrightarrow a *_{R_p} b \sim c *_{R_p} d$.
(ii)$a/_R b \sim_p c/_R d \Leftrightarrow a/_{R_p} b \sim c/_{R_p} d$.

Proof. Similar to the proof of Proposition 4.11 of [19].

Proposition 13 *If* R *is reflexive and symmetry, then for all* $a, b, c, d \in H$ *and* $p \in [0,1]$ *we have:* $a/_R b \sim_p c/_R d \Rightarrow a *_R d \sim_p b *_R c$.

Proof. Similar to the proof of Proposition 4.12 of [19].

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Saturation theorem for an iterative combination of Bernstein-Durrmeyer polynomials [1]

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Abstract

The Bernstein-Durrmeyer polynomial

$$M_n(f;t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 p_{n,k}(u)f(u)\,du,$$

where $p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}$, $t \in [0,1]$ defined on $L_B[0,1]$, the space of bounded and integrable functions on $[0,1]$ were introduced by Durrmeyer[5] and extensively studied by Derriennic[3] and other researchers (see[1]-[3],[5],[6],
[9]). It turns out that the order of approximation by these operators is, at best, $O(n^{-1})$ however smooth the function may be. In order to improve the rate of approximation we consider an iterative combination $T_{n,k}(f;t)$ of the operators $M_n(f;t)$. This technique was given by Micchelli[10] who first used it to improve the order of approximation by Bernstein polynomials $B_n(f;t)$. In our paper [12] we obtained direct theorems in ordinary approximation in the L_p norm by the operators $T_{n,k}$. The object of the

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present paper is to study the corresponding saturation theorem in L_p- approximation.

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1 Introduction

For $f \in L_p[0,1], 1 \leqslant p < \infty$ the operators M_n can be expressed as

$$M_n(f;t) = \int_0^1 W_n(u,t) f(u)\, du,$$

where $W_n(u,t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t) p_{n,k}(u)$ is the kernel of the operators. For $m \in N^0$ (the set of non-negative integers), the mth order moment for the operators M_n is defined as

$$\mu_{n,m}(t) = M_n\left((u-t)^m;t\right).$$

The iterative combination $T_{n,k} : L_p[0,1] \to C^\infty(-\infty,+\infty)$ of the operators is defined as

$$T_{n,k}(f;t) = \left(I - (I-M_n)^k\right)(f;t) = \sum_{r=1}^{k} (-1)^{r+1}\binom{k}{r} M_n^r(f;t),\ \ k \in N,$$

where $M_n^0 \equiv I$ and $M_n^r \equiv M_n(M_n^{r-1})$ for $r \in N$.
In what follows, we suppose that $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < 1$. Also, $AC[a,b]$ and $BV[a,b]$ denote the classe of absolutely continuous functions and functions of bounded variation, respectively in the interval $[a,b]$. Further C is a constant not always the same.
In [12], we obtained following direct theorem:

Theorem 1. *If* $p \geqslant 1, f \in L_p[0,1]$. *Then for all* n *sufficiently large there holds*

$$\|T_{n,k}(f;.) - f\|_{L_p[a_2,b_2]} \leqslant C_k \left(\omega_{2k}\Big(f, \frac{1}{\sqrt{n}}, p, [a_1,b_1]\Big) + n^{-k}\|f\|_{L_p[0,1]}\right),$$

where C_k *is a constant independent of* f *and* n.

The aim of this paper is to establish a corresponding saturation theorem for the operators $T_{n,k}(f,t)$ in the L_p-norm i.e. the characterization of the class of functions for which $\|T_{n,k}(f,.) - f\|_{L_p[a_2,b_2]} = O(n^{-\alpha/2})$ as $n \to \infty$, where $0 < \alpha < 2k$.
Thus we prove the following theorem (*saturation theorem*):

Theorem 2. *Let* $f \in L_p[0,\infty), p \geqslant 1$. *Then, in the following statement, the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and* (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) *hold:*

(i) $\|T_{n,k}(f,.) - f(.)\|_{L_p[a_1,b_1]} = O(n^{-k})$ *as* $n \to \infty$.;

(ii) f *coincides almost everywhere with a function* F *on* $[a_2,b_2]$ *having* $2k$ *derivatives such that:*

(a) *when* $p > 1$, $F^{(2k-1)} \in AC[a_2,b_2]$ *and* $F^{(2k)} \in L_p[a_2,b_2]$,

(b) *when* $p = 1$, $F^{(2k-2)} \in AC[a_2,b_2]$ *and* $F^{(2k-1)} \in BV[a_2,b_2]$;

(iii) $\|T_{n,k}(f,.) - f(.)\|_{L_p[a_3,b_3]} = O(n^{-k})$ *as* $n \to \infty$;

(iv) $\|T_{n,k}(f,.) - f(.)\|_{L_p[a_3,b_3]} = o(n^{-k})$ *as* $n \to \infty$;

(v) f *coincides almost everywhere with a function* F *on* $[a_2,b_2]$, *where* F *is* $2k$ *times continuously differentiable on* $[a_2,b_2]$ *and satisfies* $\sum_{j=1}^{2k} p(j,k,x)F^{(j)}(x) = 0$,*where* $p(j,k,x)$ *are the polynomials occurring in Theorem 3;*

(vi) $\|T_{n,k}(f,.) - f(.)\|_{L_p[a_1,b_1]} = o(n^{-k})$ *as* $n \to \infty$.

2 Preliminaries

In this section we give some results which are useful in establishing our main theorem.

Theorem 3. *[1] Let $f \in L_p[0,1]$ admit a continuous derivative of order $2k$ at a point $t \in [0,1]$. Then*

$$\lim_{n\to\infty} n^k \left[T_{n,k}(f;t) - f(t)\right] = \sum_{\nu=1}^{2k} \frac{f^{(\nu)}(t)}{\nu!} Q(\nu,k,t)$$

and

$$\lim_{n\to\infty} n^k \left[T_{n,k+1}(f;t) - f(t)\right] = 0,$$

where $Q(\nu,k,t)$ are certain polynomials in t of degree ν. Further the above limits hold uniformly in $[0,1]$ if $f^{(2k)}(t)$ is continuous in $[0,1]$.

Proof. The proof follows along the lines parallel to Theorem 1[1].

Theorem 4. *[1]If $p > 1, f \in L_p[0,1], f$ has derivatives of order $2k$ on $[a_1,b_1]$ with $f^{(2k-1)} \in AC[a_1,b_1)$ and $f^{(2k)} \in L_p[a_1,b_1]$, then for sufficiently large n*

$$\|T_{n,k}(f;.) - f(.)\|_{L_p[a_2,b_2]} \leqslant C_1 n^{-k} \left[\left\|f^{(2k)}\right\|_{L_p[a_1,b_1]} + \|f\|_{L_p[0,1]}\right].$$

Moreover,if $f \in L_1[0,1], f$ has derivatives upto the order $(2k-1)$ on $[a_1,b_1]$ with $f^{(2k-2)} \in AC[a_1,b_1]$ and $f^{(2k-1)} \in BV[a_1,b_1]$, then for sufficiently large n there holds

$$\|T_{n,k}(f;.) - f(.)\|_{L_1[a_2,b_2]} \leqslant C_2 n^{-k} \Big[\left\|f^{(2k-1)}\right\|_{BV[a_1,b_1]} + \left\|f^{(2k-1)}\right\|_{L_1[a_2,b_2]} + \|f\|_{L_1[a_2,b_2]}\Big]$$

where C_1 and C_2 are certain constants independent of f and n.

Recently [13], corresponding to above direct theorem, following inverse theorem was established:

Theorem 5. *Let* $f \in L_p[0,1], p \geqslant 1,\ 0 < \alpha < 2k$ *and* $\|T_{n,k}(f,.) - f\|_{L_p[a_1,b_1]} = O(n^{-\alpha/2})$ *as* $n \to \infty$. *Then,* $\omega_{2k}(f,\tau,p,[a_2,b_2]) = O(\tau^{\alpha})$ *as* $\tau \to 0$.

Let $f \in L_p[a,b]$, $1 \leqslant p < \infty$ and $[a_1,b_1] \subset (a,b)$. Then for sufficiently small $\eta > 0$ the Steklov mean $f_{\eta,m}$ of m th order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} ... \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \Big(f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^m t_i} f(t) \Big) \prod_{i=1}^{m} dt_i, t \in [a_1,b_1],$$

where Δ_h^m is the forward difference operator with step length h.

Lemma 1. *Let* $f \in L_p[a,b]$, $1 \leqslant p < \infty$ and $[a_1,b_1] \subset (a,b)$. *Then for the function* $f_{\eta,m}$, *we have*

(a) $f_{\eta,m}$ *has derivatives up to order* m *over* I_1*;*

(b) $\|f^{(r)}_{\eta,m}\|_{L_p[a_1,b_1]} \leqslant C_r\ \eta^{-r}\ \omega_r(f,\eta,[a,b]), r = 1,2,...,m;$

(c) $\|f - f_{\eta,m}\|_{L_p[a_1,b_1]} \leqslant C_{m+1}\ \omega_m(f,\eta,[a,b])$*;*

(d) $\|f_{\eta,m}\|_{L_p[a_1,b_1]} \leqslant C_{m+2}\ \|f\|_{L_p[a,b]}$;

(e) $\|f^{(m)}_{\eta,m}\|_{L_p[a_1,b_1]} \leqslant C_{m+3}\ \eta^{-m} \|f\|_{L_p[a,b]}$

where $C_i's$ *are certain constants that depend on* i *but are independent of* f *and* η.

Following [7] Theorem 18.17 or [14], pp.163-165, the proof of the above lemma easily follows hence the details are omitted.

Remark 1. *Observing the density of* $C[a_1,b_1]$ *in* $L_p[a_1,b_1]$, *we obtain* $f_{\eta,m} \to f$ *for all* $m = 1(1)2k$ *as* $\eta \to 0$.

Lemma 2. *Let $h \in L_p[0,1], p \geqslant 1$, then for any function $g \in C_0^{2k}$, with supp $g \in (a_1, b_1)$ there holds,*

$$|\langle T_{n,k}(h,t) - h(t),\, g(t)\rangle| \leqslant \frac{C}{n^k}\|h\|_{L_p[0,1]}. \tag{1}$$

Proof. We have

$$\begin{aligned}
\big\langle T_{n,k}(h,t),\, g(t)\big\rangle &= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\big\langle M_n^r(h,t),\, g(t)\big\rangle \\
&= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\big\langle h(t),\, M_n^r(g,t)\big\rangle \\
&= \big\langle h(t),\, T_{n,k}(g,t)\big\rangle \qquad (2)
\end{aligned}$$

Using smoothness of g from the direct theorem, we obtain

$$\big\langle T_{n,k}(h,t),\, g(t)\big\rangle = \big\langle h,\, g\big\rangle + Cn^{-k}\|h\|_{L_p[0,1]}.$$

This proves the lemma.

Lemma 3. *[14] Let $1 \leqslant p < \infty$, $f \in L_p[a,b]$ and there holds*

$$\omega_m(f, \tau, p, [a,b]) = O(\tau^{r+\alpha}), (\tau \to 0),$$

where $m, r \in N$ and $0 < \alpha < 1$. Then f coincides a.e. on $[c,d] \subset (a,b)$ with a function F possessing an absolutely continuous derivative $F^{(r-1)}$, the rth derivative $F^{(r)} \in L_p[c,d]$, and there holds $\omega(F^{(r)}, \tau, p, [c,d]) = O(\tau^\alpha), (\tau \to 0)$.

3 Proof of The Main Theorem

Proof. Assume (i). Then it follows from inverse theorem and Theorem that for $a_1 < c, d < b_1, f$ coincides almost a.e. on $[c,d]$ with a function F possessing an absolutely continuous derivative $F^{(2k-2)}$, and a $(2k-1)$th derivative $F^{(2k-1)}$, which belongs to $L_p[c,d]$. Moreover, for any integer k, there holds for $0<\beta<1$

$$\omega_k\Big(F^{(2k-1)},\tau,p,[c,d]\Big)=O(\tau^{\beta}),\ \ (\tau\to 0). \tag{3}$$

we choose points $x_i, y_i, i=1,2$, such that $a_1<x_1<x_2<a_2<b_2<y_2<y_1<b_1$. Let $q\in C_0^{2k}$ with $\operatorname{support} q\subset(a_1,b_1)$ and $q(t)=1$ for $t\in[x_1,y_1]$. Let we define a function $\mathcal{F}(u)=F(u)q(u)$, $u\in[0,1]$. Then

$$\begin{aligned}\|T_{n,k}(\mathcal{F},t)-\mathcal{F}(t)\|_{L_p[x_2,y_2]} &\leqslant \|T_{n,k}(f,t)-f(t)\|_{L_p[x_2,y_2]}\\ &+ \|T_{n,k}(\mathcal{F}-f,t)\|_{L_p[x_2,y_2]}\end{aligned}$$

It follows from the constant preserving property of $T_{n,k}$ that

$$\|T_{n,k}(\mathcal{F}-f,t)\|_{L_p[x_2,y_2]}=O(n^{-k}).$$

This alongwith the hypothesis that (i) holds, implies

$$\|T_{n,k}(\mathcal{F},t)-\mathcal{F}(t)\|_{L_p[x_2,y_2]}=O(n^{-k}). \tag{4}$$

Now, if $p>1$, by Alaoglu's theorem that there exists a function $H\in L_p[x_2,y_2]$, such that for every subsequence n_j and $g\in C_0^{2k}$ with $\operatorname{supp} g\in(0,1)$, we have

$$\lim_{n_j\to\infty} n_j^k\big\langle T_{n_j,k}(\mathcal{F},t)-\mathcal{F}(t),\, g(t)\big\rangle=\big\langle H(t),\, g(t)\big\rangle. \tag{5}$$

When $p=1$, the functions ϕ_n defined by:

$$\phi_n(u)\ =\int_{x_2}^{u} n^k\big\{T_{n,k}(\mathcal{F},t)-\mathcal{F}(t)\big\}\,dt \tag{6}$$

are uniformly bounded and are of uniformly bounded variation. Making use of Alaoglu's theorem, it follows that there exists a function $\phi_0\in BV[x_2,y_2]$ such that for some subsequence $\{n_j\}$ for all $g\in C_0^{2k}$ with $\operatorname{supp} g\in(x_2,y_2)$

$$\int_{x_2}^{y_2} g(t)d\left(\phi_{n_j}(t)-\phi_0(t)\right)\to 0,\ \ (n_j\to\infty). \tag{7}$$

Now,

$$\int_{x_2}^{y_2} g(t)d\left(\phi_{n_j}(t)-\phi_0(t)\right)=\int_{x_2}^{y_2} g(t)d\phi_{n_j}(t)-\int_{x_2}^{y_2} g(t)d\phi_0(t).$$

It follows from (6), Theorem 17.17 of [7] and the fact that $\operatorname{supp} g \in (x_2, y_2)$, we get

$$\begin{aligned}&\int_{x_2}^{y_2} g(t)d\left(\phi_{n_j}(t)-\phi_0(t)\right)\\ &= n_j^k\int_{x_2}^{y_2} g(t)\left[T_{n_j,k}(\mathcal{F},t)-\mathcal{F}(t)\right]\,dt+\int_{x_2}^{y_2} g'(t)\phi_0(t)\,dt\end{aligned}$$

This together with (7) implies that

$$\lim_{n_j\to\infty} n_j^k\left\langle T_{n_j,k}(\mathcal{F};t)-\mathcal{F}(t),\, g\right\rangle \;=\; -\langle\phi_0(t),\, g'(t)\rangle. \tag{8}$$

As the steklov means $\mathcal{F}_{\eta,2k}$ for $\mathcal{F}$ have continuous derivatives of order upto $2k$, there holds

$$\|\mathcal{F}^{(i)}_{\eta,2k}-\mathcal{F}^{(i)}\|_{L_p[a_1,b_1]}\to 0,\quad (\eta\to 0). \tag{9}$$

Now, by Theorem 3

$$T_{n_j,k}(\mathcal{F}_{\eta,2k};t)-\mathcal{F}_{\eta,2k}(t)=\frac{1}{n_j^k}(P_{2k}D)\mathcal{F}_{\eta,2k}(t)+o\Big(\frac{1}{n_j^k}\Big). \tag{10}$$

Hence, if $P^*_{2k}(D)$ denotes the differential operator adjoint to $P_{2k}D \equiv \sum_{i=1}^{2k}\frac{Q(i,k,x)}{i!}D^i$, we have

$$\begin{aligned}\langle\mathcal{F}_{\eta,2k}(t),P^*_{2k}(D)g(t)\rangle &= \langle P_{2k}(D)\mathcal{F}_{\eta,2k}(t),g(t)\rangle\\ &= \lim_{n_j\to\infty} n_j^k\left\langle T_{n_j,k}(\mathcal{F}_{\eta,2k},t)-\mathcal{F}_{\eta,2k}(t),\, g(t)\right\rangle\\ &= \lim_{n_j\to\infty} n_j^k\left\langle T_{n_j,k}(\mathcal{F}_{\eta,2k}-\mathcal{F},t)-(\mathcal{F}_{\eta,2k}(t)-\mathcal{F}(t)),g(t)\right\rangle\\ &+\lim_{n_j\to\infty} n_j^k\left\langle T_{n_j,k}(\mathcal{F},t)-\mathcal{F}(t),\, g(t)\right\rangle\end{aligned}$$

i.e.

$$\begin{aligned}&\langle \mathcal{F}_{\eta,2k}(t), P^*_{2k}(D)g(t)\rangle - \lim_{n_j\to\infty} n_j^k \big\langle T_{n_j,k}(\mathcal{F},t) - \mathcal{F}(t),\, g(t)\big\rangle \\ &= \lim_{n_j\to\infty} n_j^k \big\langle T_{n_j,k}(\mathcal{F}_{\eta,2k} - \mathcal{F},t) - (\mathcal{F}_{\eta,2k}(t) - \mathcal{F}(t)),\, g(t)\big\rangle\end{aligned}$$

Hence, by Lemma 2

$$\begin{aligned}&\langle \mathcal{F}_{\eta,2k}(t), P^*_{2k}(D)g(t)\rangle - \lim_{n_j\to\infty} n_j^k \big\langle T_{n_j,k}(\mathcal{F},t) - \mathcal{F}(t),\, g(t)\big\rangle \\ &\leqslant \frac{C}{n^k}\,\|\mathcal{F}_{\eta,2k}(t) - \mathcal{F}(t)\|_{L_p[0,1]} \qquad (11)\end{aligned}$$

Taking limit as $\eta \to 0$ in (11) and using (9), we obtain

$$\langle \mathcal{F}(t), P^*_{2k}(D)g(t)\rangle = \lim_{n_j\to\infty} n_j^k \big\langle T_{n_j,k}(\mathcal{F},t) - \mathcal{F}(t),\, g(t)\big\rangle. \qquad (12)$$

Comparing (12) with (5) and (8), we have

$$\langle \mathcal{F}(t), P^*_{2k}(D)g(t)\rangle = \begin{cases}\big\langle H(t),\, g(t)\big\rangle, & \text{if } p > 1;\\ -\langle \phi_0(t),\, g'(t)\rangle, & \text{if } p = 1;\end{cases} \qquad (13)$$

Using integration by parts it easily follows that

$$\langle \mathcal{F}(t), P^*_{2k}(D)g(t)\rangle = \langle Q(2k,t)\mathcal{F}(t) + \sum_{i=1}^{2k} I_i(b_iG)(t),\, g^{(2k)}(t)\rangle, \qquad (14)$$

where $b_i(t)$ are certain polynomials in t and I_i denotes the ith iterated indefinite integral operator, namely

$$I_i = \overbrace{\int_0^t \ldots \int_0^t}^{i\,\text{times}} dt \ldots dt$$

Similarly,

$$\big\langle H(t),\, g(t)\big\rangle = \big\langle I_{2k}H(t),\, g^{(2k)}(t)\big\rangle. \qquad (15)$$

When $p > 1$, from (14) and (15) we have

$$\int_0^1 \left[Q(2k,t)\mathcal{F}(t) + \sum_{i=1}^{2k} I_i(b_iG)(t) - I_{2k}H(t)\right] g^{(2k)}(t)\,dt = 0. \tag{16}$$

It follows from Theorem 3 and Lemma 1.5.1 [9] that $Q(2k,t) = C_k(t\overline{1-t})^k$, where C_k is a non-zero constant.
This implies by Lemma 1.1.1 [11] and the assumed smoothness for f that $\mathcal{F}^{(2k-1)} \in AC[x_2, y_2]$ and $\mathcal{F}^{(2k)} \in L_p[x_2, y_2]$. Since $\mathcal{F}(u) = F(u)$ for $u \in [x_1, y_1]$, we have $F^{(2k-1)} \in AC[a_2, b_2]$ and $F^{(2k)} \in L_p[a_2, b_2]$.
When $p = 1$, proceeding similarly, we obtain $F^{(2k-1)} \in BV[a_2, b_2]$. This completes the proof of the implication "$(i) \Rightarrow (ii)$".
The implication "$(ii) \Rightarrow (iii)$" follows from Theorem 4.
Assuming (iv) and proceeding as in the proof of the implication "$(i) \Rightarrow (ii)$", we first find that H and ϕ are zero functions. This does imply that F is $2k$ times continuously differentiable function and that $P_{2k}(D)F(t) = 0$.
Finally "$(v) \Rightarrow (vi)$" holds by Theorem 3.
This completes the proof.

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Some interpolation schemes on a triangle with one curved side [1]

Alina Baboş

Abstract

On of the quite simple procedures for constructing multidimensional approximation operators consist in the composition of univariate approximation operators, using tensor product and boolean sum operators. In this paper, we will construct such interpolation operators for functions defined on a triangle with on curved side.

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Key words and phrases: Triangle, curved side, multidimensional, approximation operators.

1 Introduction

Let $T_h = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \geq h\}$ be the standard triangle.

In [5] the authors consider a standard triangle,$\tilde{T}_h$, having the vertices $V_1 = (h, 0), V_2 = (0, h)$ and $V_3 = (0, 0)$, two straight sides Γ_1, Γ_2, along the coordinate axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e. $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$ and F a real-valued function defined on $\tilde{T}_h$.

For generating interpolation formulas on this triangle, we will use Lagrange and Hermite univariate interpolation operators.

In the paper [5] the authors construct certain Lagrange and Hermite type

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operators, which interpolate a given function and some of its derivatives on the border of a triangle with one curved side, as well as some of their product and Boolean sum operators.

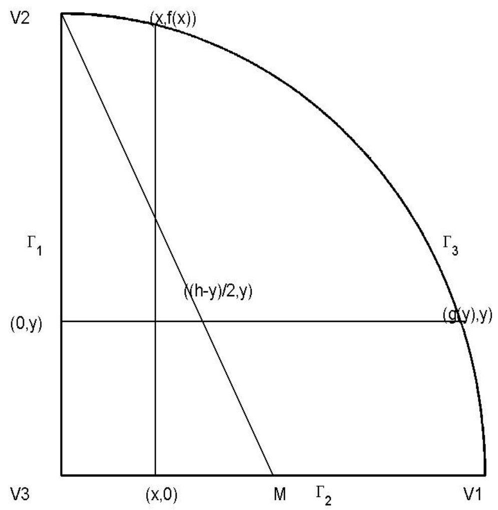

Figure 1: Triangle $\tilde{T}_h$

The Lagrange interpolation operators were defined by:

$$(1)\qquad \begin{aligned} (L_1F)(x,y) &= \frac{g(y)-x}{g(y)}F(0,y)+\frac{x}{g(y)}F(g(y),y),\\ (L_2F)(x,y) &= \frac{f(x)-y}{f(x)}F(x,0)+\frac{y}{f(x)}F(x,f(x)),\\ (L_3F)(x,y) &= \frac{x}{x+y}F(x+y,0)+\frac{y}{x+y}F(0,x+y). \end{aligned}$$

The product of the operators L_i and L_j, i.e., $P_{ij}=L_iL_j, i,j=1,2,3, i\neq j$ verifies the interpolation properties

$$\begin{aligned} P_{12}^L F &= F,\ on\,\Gamma_3\cup V_3,\\ P_{13}^L F &= F,\ on\,\Gamma_1\cup V_1,\\ P_{23}^L F &= F,\ on\,\Gamma_2\cup V_2, \end{aligned}$$

and $dex(P_{ij})^L = 1, i, j = 1, 2, 3, i \neq j$. For the remainders $R^P_{ij}F$, of the interpolation formulas

$$F = P^L_{ij}F + R^P_{ij}F,\ i, j = 1, 2, 3, i \neq j,$$

was proved that for $F \in B_{11}(0, 0)$

$$\begin{aligned}(R^{LP}_{12}F)(x, y) &= \frac{x[x - g(y)]}{2}F^{(2,0)}(\xi, 0) + \frac{y(y - h)[g(y) - x]}{2g(y)}F^{(0,2)}(0, \eta)\\ &+ \frac{xy[g(y) - x]}{g(y)}[F^{(1,1)}(\xi_1, \eta_1) - F^{(1,1)}(\xi_2, \eta_2)],\end{aligned}$$

where $\xi, \eta \in [0, h], (\xi_1, \eta_1) \in [0, x] \times [0, y]$ and $(\xi_2, \eta_2) \in [x, g(y)] \times [0, y]$, respectively

$$|(R^{LP}_{12}F)(x, y)| \leq \frac{h^2}{8}[\|F^{(2,0)}(\cdot, 0)\|_\infty + \|F^{(0,2)}(0, \cdot)\|_\infty + \|F^{(1,1)})\|_\infty]. \tag{2}$$

Also, the boolean sum of the operators L_i and L_j, i.e. $S^L_{ij} = L_i \oplus L_j, i, j = 1, 2, 3, i < j$ verifies the interpolation proprieties: $S^L_{ij}F = F, i, j = 1, 2, 3, i < j$ on $\partial\tilde{T}_h$, and

$$\begin{aligned}dex(S^L_{12}) &= 1,\\ dex(S^L_{13}) &= dex(S^L_{23}) = 2.\end{aligned} \tag{3}$$

For the remainders $R^{LS}_{ij}F$, of the interpolation formulas

$$F = S^L_{ij}F + R^{LS}_{ij}F,\ i, j = 1, 2, 3, i < j$$

was proved that for $F \in B_{11}(0, 0)$

$$\begin{aligned}|(R^{LS}_{12}F)(x, y)| &\leq \|F^{(2,0)}(\cdot, 0)\|_\infty \int_0^h |K_{02}(x, y, t)|dt\\ &+ \|F^{(1,1)}\|_\infty \iint_{\tilde{T}_h} |K_{11}(x, y, s, t)|dsdt\end{aligned}$$

The Hermite interpolation operators were defined by:

$$\begin{aligned}(H_1F)(x, y) &= \frac{[x - g(y)]^2}{g^2(y)}F(0, y) + \frac{x[2g(y) - x]}{g^2(y)}F(g(y), y)\\ &+ \frac{x[x - g(y)]}{g(y)}F^{(1,0)}(g(y), y),\\ (H_2F)(x, y) &= \frac{[y - f(x)]^2}{f^2(x)}F(x, 0) + \frac{y[2f(x) - y]}{f^2(x)}F(x, f(x))\\ &+ \frac{y[y - f(x)]}{f(x)}F^{(0,1)}(x, f(x)).\end{aligned} \tag{4}$$

The product of the operators H_1 and H_2 verifies the interpolation properties

$$\begin{array}{rcl} P_{12}^H F & = & F,\ on\, \Gamma_3 \cup V_3, \\ (P_{12}^L F)^{(1,0)} & = & F^{(1,0)},\ on\, \Gamma_3, \\ (P_{12}^L F)^{(0,1)} & = & F^{(0,1)},\ on\, \Gamma_3, \end{array}$$

and $dex(P_{12})^H = 2$. For the remainder of the corresponding interpolation formula

$$F = P_{12}^H F + R_{12}^{HP} F,$$

was proved that for $F \in B_{12}(0,0)$

$$\begin{array}{rcl} |(R_{12}^{HP} F)(x,y)| & \leq & \dfrac{x[g(y)-x]^2}{6} \|F^{(3,0)}(\cdot,0)\|_\infty + \dfrac{xy[g(y)-x]^2}{2g(y)-x} \|F^{(2,1)}(\cdot,0)\|_\infty \\ & + & \dfrac{y[g(y)-x]^2(h-y)}{6g^2(y)} \|F^{(0,3)}(0,\cdot)\|_\infty \\ & + & \dfrac{xy[g(y)-x][3g(y)-2x])}{g^2(y)} \|F^{(1,2)}(\cdot,\cdot)\|_\infty. \end{array} \tag{5}$$

Also, the boolean sum of the operators H_1 and H_2 verifies the interpolation properties

$$\begin{array}{rcl} S_{12}^H F & = & F,\ on\, \partial\tilde{T}_h, \\ (S_{12}^H F)^{(1,0)} & = & F^{(1,0)},\ on\, \Gamma_3, \\ (S_{12}^L F)^{(0,1)} & = & F^{(0,1)},\ on\, \Gamma_3, \end{array}$$

and $dex(S_{12})^H = 2$. For the remainder of the corresponding interpolation formula

$$F = S_{12}^H F + R_{12}^{HS} F,$$

was proved that for $F \in B_{12}(0,0)$

$$\begin{array}{rcl} |(R_{12}^{HS} F)(x,y)| & \leq & \|F^{(0,3)}(0,\cdot)\|_\infty \displaystyle\int_0^h |K_{03}(x,y,t)|dt \\ & + & \|F^{(1,2)}(\cdot,\cdot)\|_\infty \displaystyle\iint_{\tilde{T}_h} |K_{12}(x,y,s,t)|dsdt. \end{array}$$

In [1] we introduce an Lagrange operator L_2^x which interpolate the function, F, and some interior line of triangle $\tilde{T}_h$ and we consider the case when the interior line is a median (Figure 1). L_2^x interpolate the function F with respect to x in the points $(0,y)$, $\big(\frac{h-y}{2},y\big)$, $(g(y),y)$:

$$
\begin{aligned}
(L_2^x F)(x,y) &= \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}F(0,y)\\
&+ \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]}F\Big(\frac{h-y}{2},y\Big)\\
&+ \frac{x(2x-h+y)}{g(y)[2g(y)-h+y]}F(g(y),y). \qquad (6)
\end{aligned}
$$

2 Main Results

Next, we will build new interpolation operators for which we will determine the interpolation properties and degree of exactness. Also, the generated interpolation formulas will be studied.
Let us consider the Hermite operator H_2^y given in 4, i.e.

$$
\begin{aligned}
(H_2^y F)(x,y) &= \frac{[y-f(x)]^2}{f^2(x)}F(x,0) + \frac{y[2f(x)-y]}{f^2(x)}F(x,f(x))\\
&+ \frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x)),
\end{aligned}
$$

respectively the Lagrange operator L_2^x given in 6, i.e.

$$
\begin{aligned}
(L_2^x F)(x,y) &= \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}F(0,y)\\
&+ \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]}F\Big(\frac{h-y}{2},y\Big)\\
&+ \frac{x(2x-h+y)}{g(y)[2g(y)-h+y]}F(g(y),y). \qquad (7)
\end{aligned}
$$

2.1

Let P be

$$P := H_2^y L_2^x$$

and

$$F = PF + R_1 F \qquad (8)$$

approximation formula generated by P.

Theorem 1 *Let consider* $F : \tilde{T}_h \to \mathbb{R}$. *If there exist* $F^{(0,1)}$ *on the side* Γ_3 *then* P *verifies the interpolation properties:*

$$\begin{aligned} PF &= F, \, on \, \Gamma_2 \cup \Gamma_3 \\ (PF)^{(0,1)} &= F^{(1,0)}, \, on \, \Gamma_3 \end{aligned}$$

and $dex(P) = 2$.

Proof.

$$\begin{aligned} (PF)(x,y) &= \frac{[y-f(x)]^2}{f^2(x)}\left[\frac{(2x-h)(x-h)}{h^2}F(0,0) - \frac{4x(x-h)}{h^2}F\Big(\frac{h}{2},0\Big) + \right. \\ &+ \left. \frac{x(2x-h)}{h^2}F(h,0)\right] \\ &+ \frac{y[2f(x)-y]}{f^2(x)}F(x,f(x)) + \frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x)). \end{aligned} \tag{9}$$

Now, the interpolation properties are easy verified, by direct computation. We, also, have $Pe_{ij} = e_{ij}$, for $i,j \leq 2$ and $Pe_{03} \neq e_{03}$, where $e_{ij}(x,y) = x^i y^j$. It follows that $dex(P) = 2$.

Theorem 2 *If* $F \in B_{1,2}(0,0)$ *then the following inequality holds*

$$\begin{aligned} |(R_1F)(x,y)| &\leq \frac{x[y-f(x)]^2(h-2x)(h-x)}{12f^2(x)}\|F^{(3,0)}(\cdot,0)\|_\infty \\ &+ \frac{xy[y-f(x)]^2(2x-h)(x-h)}{f^2(x)(3h-2x)}\|F^{(2,1)}(\cdot,0)\|_\infty \\ &+ \frac{y[y-f(x)]^2}{6}\|F^{(0,3)}(0,\cdot)\|_\infty \\ &+ \frac{xy[f(x)-y]^2}{2f(x)-y}\|F^{(1,2)}(\cdot,\cdot)\|_\infty. \end{aligned} \tag{10}$$

Proof. As $dex(P) = 2$ it results, from Peano's theorem that

$$\begin{aligned} (R_1F)(x,y) &= \int_0^h K_{30}(x,y,s)F^{(3,0)}(s,0)ds \\ &+ \int_0^h K_{21}(x,y,s)F^{(2,1)}(s,0)ds \\ &+ \int_0^h K_{03}(x,y,t)F^{(0,3)}(0,t)dt \\ &+ \iint_{\tilde{T}_h} K_{12}(x,y,s,t)F^{(1,2)}(s,t)dtdsdt, \end{aligned}$$

with

$$\begin{aligned}
K_{30}(x,y,s) &= \frac{(x-s)_+^2}{2} \\
&- \frac{[y-f(x)]^2}{f^2(x)}\left[-\frac{2x(x-h)}{h^2}\left(\frac{h}{2}-s\right)_+^2+\frac{x(2x-h)}{h^2}\frac{(h-s)^2}{2}\right] \\
&- \frac{y[2f(x)-y]}{f^2(x)}\frac{(x-s)_+^2}{2} \\
K_{21}(x,y,s) &= y(x-s)_+ \\
&- \frac{[y-f(x)]^2}{f^2(x)}\left[-\frac{4xy(x-h)}{h^2}\left(\frac{h}{2}-s\right)_+ + \frac{xy(2x-h)}{h^2}(h-s)\right] \\
&- \frac{y^2[2f(x)-y]}{f^2(x)}(x-s)_+ \\
K_{03}(x,y,t) &= \frac{(y-t)_+^2}{2} - \frac{y[2f(x)-y]}{f^2(x)}\frac{[f(x)-t]_+^2}{2} \\
&- \frac{y[y-f(x)]}{f(x)}[f(x)-t]_+ \\
K_{12}(x,y,s,t) &= (x-s)_+^0(y-t)_+ - \frac{y[2f(x)-y]}{f^2(x)}(x-s)_+^0[f(x)-t]_+ \\
&- \frac{y[y-f(x)]}{f(x)}(x-s)_+^0[f(x)-t]_+^0
\end{aligned}$$

We have

$$K_{30}(x,y,s) = \begin{cases} \frac{s^2[y-f(x)]^2(h-x)(h-2x)}{2h^2f^2(x)} \geq 0, & \text{for } s\in[0,x)\,, \\ -\frac{x[y-f(x)]^2[h(h-s)(x-s)+s(h-x)(2s-h)]}{2h^2f^2(x)} \geq 0, & \text{for } s\in\left[x,\frac{h}{2}\right)\,, \\ -\frac{x[y-f(x)]^2(2x-h)(h-s)^2}{2h^2f^2(x)} \geq 0, & \text{for } s\in\left[\frac{h}{2},h\right), \end{cases}$$

$$K_{21}(x,y,s) = \begin{cases} -\frac{sy[y-f(x)]^2(x-h)(2x-h)}{h^2f^2(x)} \leq 0, & \text{for } s\in[0,x)\,, \\ -\frac{xy[y-f(x)]^2[h^2-s(3h-2x)]}{h^2f^2(x)} \leq 0, & \text{for } s\in\left[x,\frac{h^2}{3h-2x}\right)\,, \\ -\frac{xy[y-f(x)]^2[h^2-s(3h-2x)]}{h^2f^2(x)} \geq 0, & \text{for } s\in\left[\frac{h^2}{3h-2x},\frac{h}{2}\right)\,, \\ -\frac{xy[y-f(x)]^2(2x-h)(h-s)}{h^2f^2(x)} \geq 0, & \text{for } s\in\left[\frac{h}{2},h\right), \end{cases}$$

$$K_{03}(x,y,t) = \begin{cases} \frac{t^2[f(x)-y]^2}{2f^2(x)} \geq 0, & \text{for } t\in[0,y)\,, \\ -\frac{y[f(x)-t][f(x)(y-t)+t(y-f(x))]}{2f^2(x)} \geq 0, & \text{for } t\in[y,f(x))\,, \\ 0, & \text{for } t\in[f(x),h), \end{cases}$$

$$K_{12}(x,y,s,t) = \begin{cases} -\frac{t^2[f(x)-y]^2}{f^2(x)} \leq 0, & \text{for } (s,t) \in [0,x) \times [0,y)\,, \\ -\frac{y[f^2(x)-t(2f(x)-y)]}{f^2(x)} \leq 0, & \text{for } (s,t) \in [0,x) \times [y, \frac{f^2(x)}{2f(x)-y})\,, \\ -\frac{y[f^2(x)-t(2f(x)-y)]}{f^2(x)} \geq 0, & \text{for } (s,t) \in [0,x) \times [\frac{f^2(x)}{2f(x)-y}, f(x))\,, \\ 0, \text{ for } (s,t) \in ([x,h] \times [0,f(x)] \cup [0,x] \times [f(x),h]) \cap \tilde{T}_h. & \end{cases}$$

We obtain that

$$\begin{aligned} |(R_1F)(x,y)| &\leq \|F^{(3,0)}(\cdot,0)\|_\infty \int_0^h K_{30}(x,y,s)ds \\ &+ \|F^{(2,1)}(\cdot,0)\|_\infty \int_0^h |K_{21}(x,y,s)|ds \\ &+ \|F^{(0,3)}(0,\cdot)\|_\infty \int_0^h K_{03}(x,y,t)dt \\ &+ \|F^{(1,2)}(\cdot,\cdot)\|_\infty \iint_{\tilde{T}_h} |K_{12}(x,y,s,t)|dsdt, \end{aligned}$$

whence, after some computation, we get 10.

2.2

Let S be

$$S := H_2^y \oplus L_2^x$$

and

$$F = SF + R_2F \tag{11}$$

approximation formula generated by S.

Theorem 3 *Let consider* $F : \tilde{T}_h \to \mathbb{R}$ *then :*

1. $SF = F$, *on* $\partial\tilde{T}_h$.
2. $dex(S) = 2$.

Proof.

$$(SF)(x,y) = \frac{[y-f(x)]^2}{f^2(x)}F(x,0) + \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}F(0,y)$$
$$+ \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]}F\Big(\frac{h-y}{2},y\Big) + \frac{x(2x-h+y)}{g(y)[2g(y)-h+y]}F(g(y),y)$$
$$- \frac{[y-f(x)]^2}{f^2(x)}\left[\frac{(2x-h)(x-h)}{h^2}F(0,0) - \frac{4x(x-h)}{h^2}F\Big(\frac{h}{2},0\Big) + \frac{x(2x-h)}{h^2}F(h,0)\right]$$

The first statement results by a direct computation.
Also by direct computation, we obtain $Se_{ij} = e_{ij}$, for $i, j \leq 2$ and $Se_{30} \neq e_{30}$,wich implies that $dex(S) = 2$.

Theorem 4 *If $F \in B_{1,2}(0,0)$ then*

$$
\begin{aligned}
(R_2F)(x,y) &= \int_0^h K_{30}(x,y,s)F^{(3,0)}(s,0)ds + \\
&+ \int_0^h K_{21}(x,y,s)F^{(2,1)}(s,0)ds + \\
&+ \int_0^h K_{03}(x,y,t)F^{(0,3)}(0,t)dt + \\
&+ \iint_{\tilde{T}_h} K_{12}(x,y,s,t)F^{(1,2)}(s,t)dsdt,
\end{aligned} \tag{12}
$$

with the Peano's kernels

$$
\begin{aligned}
K_{30}(x,y,s) &= \frac{(x-s)_+^2}{2} - \frac{[y-f(x)]^2}{f^2(x)} \cdot \frac{(x-s)_+^2}{2} \\
&- \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]} \cdot \frac{\left(\frac{h-y}{2}-s\right)_+^2}{2} - \frac{x(2x-h-y)}{g(y)[2g(y)-h+y]} \cdot \frac{[g(y)-s]_+^2}{2} \\
&+ \frac{[y-f(x)]^2}{f^2(x)} \cdot \left[-\frac{4x(x-h)}{h^2} \cdot \frac{\left(\frac{h}{2}-s\right)_+^2}{2} + \frac{x(2x-h)}{h^2} \cdot \frac{(h-s)^2}{2} \right] \\
K_{21}(x,y,s) &= y(x-s)_+ - \frac{y[y-f(x)]^2}{f^2(x)}(x-s)_+ \\
&- \frac{4xy[x-g(y)]}{(h-y)[h-y-2g(y)]}\left(\frac{h-y}{2}-s\right)_+ - \frac{xy(2x-h-y)}{g(y)[2g(y)-h+y]}[g(y)-s]_+ \\
&+ \frac{[y-f(x)]^2 y}{f^2(x)} \left[-\frac{4x(x-h)}{h^2}\left(\frac{h}{2}-s\right)_+ + \frac{x(2x-h)}{h^2}(h-s) \right] \\
K_{03}(x,y,t) &= 0 \\
K_{12}(x,y,s,t) &= (y-t)_+ \left[(x-s)_+^0 - \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]}\left(\frac{h-y}{2}-s\right)_+^0 \right. \\
&- \left. \frac{x(2x-h-y)}{g(y)[2g(y)-h-y]}[g(y)-s]_+^0 \right]
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|(R_2F)(x,y)| &\leq \|F^{(3,0)}(\cdot,0)\|_\infty \int_0^h |K_{30}(x,y,s)|ds \\
&+ \|F^{(2,1)}(\cdot,0)\|_\infty \int_0^h |K_{21}(x,y,s)|ds \\
&+ \|F^{(1,2)}(\cdot,\cdot)\|_\infty \iint_{\tilde{T}_h} |K_{12}(x,y,s,t)|dsdt,
\end{aligned} \tag{13}
$$

Proof. As $dex(S) = 2$, applying the Peano's theorem we get the form 12 of the remainder and the inequality 13.

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Mathematical model, discrete model, informatics model and C++ source program for solving partial differential parabolic equations with imposed conditions [1]

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Abstract

A partial differential equation of second order PDE2 with the parabolic type is given. The unknown function is $u(x,t)$, where $x \in [0,L], t \geq 0, t \in [0,T]$. The PDE2 is related with the heating transport on a linear conductor. The work presents: the mathematical model, the discrete mathematical model, the informatics model, the C++ source program, the numerical results and print screen.

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1 The mathematical foundations. Notations

We denote by PDE2 a partial differential equation of second order. In this work we deal with the PDE2 having the form

(1)
$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{a^2}\frac{\partial u(x,t)}{\partial t} + F(x,t) = 0,\ a > 0,\ u(x,t) \text{ is unknown function}$$

The equation (1) has the parabolic type and it corresponds to heat transportation in a linear conductor. The variable x marks the point on the conductor and t represents the time when the heat is measured. The value $u(x,t)$ is the heat level in the point x at the time t.

1.1 The mathematical model

The model is related with a **finite linear conductor** having the length L. The mathematical model contains the PDE2 and two types of conditions: initial condition and conditions at the heads of the conductor. So we obtain the mathematical model

(2) $$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{a^2}\frac{\partial u(x,t)}{\partial t} + F(x,t) = 0,\ a > 0,\ 0 \le x \le L;\ t \ge 0$$

(3) $$u(x,0) = f(x),\ 0 \le x \le L \text{ (initial condition)}$$

(4) $$u(0,t) = \varphi(t),\ u(L,t) = \psi(t),\ t \ge 0$$

(5) $$f(0) = \varphi(0),\ f(L) = \psi(0) \text{ (the compatibility conditions)}$$

The above model (problem) is denoted PXXX (problem P with 3 conditions X).

Remark 1 *In the future we take into account only the fixed order (2),(3), (4),(5). The functions F, f, g, φ, ψ could be null functions or non-null functions. When one function is null function we denote it by the letter N and when it is non-null function we denote it by O. So we obtain 7 different mathematical models [2], 1996, page 284 (the model POOO is excluded):*

$$POON, PONO, PONN, PNOO, PNON, PNNO, PNNN.$$

We can assimilate this order with the binary numbers 001, 010, 011, 100, 101, 110, 111.

All these 7 problems have an analytical solution in the form of series of functions [2]. To use in technical problems the analytical solutions is difficult. That is why we look for the numerical solutions. This means we transform the **mathematical model** *in* **discrete model**.

1.2 The discrete mathematical model

In the discrete model we use the variable t only for $0 \leq t \leq T$, where the value T is given by the user. So, the functions $\varphi(t), \psi(t)$ are known only for $0 \leq t \leq T$.

The interval $[0, L]$ has the division $\triangle x$ with $M + 1$ the points $x_i, i = 0, 1, ..., M$, (or $i = \overline{0, M}$); the interval $[0, T]$ has the division $\triangle t$ with $N + 1$ the points t_j, $j = 0, 1, ..., N$ (or $j = \overline{0, N}$).

If we denote the step for x by $h = L/M$, then $x_i = ih$, $i = 0, M$. For $\triangle t$ we denote the step by $p = T/N$ and $t_j = jp, j = \overline{0, N}$.

The input known data are: $a, F, f, g, \varphi, \psi; L, T; M, N$.

The unknown function $u(x, t)$ must be computed in all points (x_i, t_j), for $i = \overline{0, M}$ and $j = \overline{0, N}$. We denote $u(x_i, t_j) = u_{ij}$. In an orthogonal system of axes we arrange the points x_i on the vertical axis and t_j on the horizontal axis.

Now it is possible to define the partial derivatives of (2)

$$(6)\quad \frac{\partial^2 u}{\partial x^2}(x_i,t_j)=\frac{1}{h^2}\Big(u_{i-1j}-2u_{ij}+u_{i+1j}\Big);\ \frac{\partial u}{\partial t}(x_i,t_j)=\frac{1}{p}\Big(u_{ij+1}-u_{ij}\Big);$$

The initial condition $u(x,0)=f(x)$ generates the values $u(x_i,0)=f(x_i)$, $i=\overline{0,M}$

$$u(x_0,0)=u_{00},u(x_1,0)=u_{10},...,u(x_i,0)=u_{i0},...,u(x_{M-1},0)=u_{M-10},$$
$$u(x_M,0)=u_{M0}$$

(the column number 0 in the matrix).

The initial condition $u(0,t)=\varphi(t)$ generates the values $u(0,t_j)=\varphi(t_j)$, $j=\overline{0,N}$

$$u(0,t_0)=u_{00},u(0,t_1)=u_{01},...,u(0,t_j)=u_{0j},...,u(0,t_{N-1})=u_{0,N-1},$$
$$u(0,t_N)=u_{0N}$$

(the line number 0 in the matrix).

The initial condition $u(L,t)=\psi(t)$ generates the values $u(L,t_j)=\psi(t_j)$, $j=\overline{0,N}$

$$u(L,t_0)=u_{M0},u(L,t_1)=u_{M1},...,u(L,t_j)=u_{Mj},...,$$
$$u(L,t_{N-1})=u_{M,N-1},u(L,t_N)=u_{MN}$$

(the line number M in the matrix).

The matrix U with the elements u_{ij} has the type $(M+1)\times(N+1)$; $M+2N+1$ elements are known elements; $MN-N$ elements are unknown elements.

If we put the relations (6) in equation (2) we obtain the **unknowns elements**

$$(7)\qquad u_{ij+1}=u_{ij}+\frac{a^2p}{h^2}\Big(u_{i-1j}-2u_{ij}+u_{i+1j}\Big)+a^2pF(x_i,t_j)$$

for the fixed value $j, j=\overline{0,N-1}$ and all $i=\overline{1,M-1}$.

Finally we print the numerical results arranged in the matrix U with $x_i, i = \overline{0, M}$ on the column and $t_j, j = \overline{0, N}$ on lines.

$u00 \quad u01 \quad u02........u0j........u0 \quad N-1 \quad u0N; \quad u(0,t) = \varphi(t)$

$u10$

.... unknown elements .

$ui0$

....

$uM-10$

$uM0 \quad uM1 \quad uM2.......uMj.......uMN-1 \quad uMN; \quad u(L,t) = \psi(t)$

$u(x, 0 = f(x).$

Remark 2 *As we have mentioned the functions F, f, φ, ψ are given by the user. We propose the following functions*

$$f(x) = a_1x^2 + a_2x + a_3; \varphi(t) = b_1t^2 + b_2t + b_3; \psi(t) = c_1t^2 + c_2t + c_3$$

$$F(x,t) = (d_1x^2 + d_2x + d_3)(e_1t^2 + e_2t + e_3)$$

where the polynomial coefficients are given by the user.

1.3 The informatics model

We construct a table containing the mathematical notations and the corresponding variable identifiers for the source C++ program.

No	Mathematical notations	Informatics notations
1	M	M
2	N	N
3	h	hxi
4	p	ptj
5	f	f
6	φ	fi
7	ψ	psi
8	F	Fxitj
9	x_i	xi
10	t_j	tj
11	a_k	af[]
12	b_k	bfi[]
13	c_k	cpsi[]
14	d_k	dF[]
15	e_k	eF[]
16	u_{ij}	uij[]
17	i=0, M	i=0, M
18	j=0, N	j=0, N
19	a	a

1.4 The solved problems and numerical results

Problem 1. $L = 9, M = 9$; $T = 9, N = 9$; $a = 1$.

$f(x) = x + 1, \varphi(t) = 2t + 1, \psi(t) = -t + 10$; $F(x, t) = 0$ (the null function)

The conditions are compatible.

Problem 2. $L = 9, M = 0$; $T = 9, N = 9$; $a = 1$.

$f(x) = x+1, \varphi(t) = 2t+1, \psi(t) = -t+10$; $F(x, t) = (x-1)(t^2+1)$

The conditions are compatible.

Problem 3. $L = 9, M = 9$; $T = 9, N = 9$; $a = 1$.

$f(x) = x+1, \varphi(t) = 2t+1, \psi(t) = -t+5$; $F(x,t) = (-x+1)(t^2+1)$

The conditions are incompatible.

Problem 4. $L = 10, M = 10$; $T = 10, N = 10$; $a = 1$.

$f(x) = x + 1, \varphi(t) = 2t + 1, \psi(t) = -t + 11$; $F(x,t) = 0$ (the null function)

The conditions are compatible.

Problem 5. $L = 10, M = 10$; $T = 10, N = 10$; $a = 1$.

$f(x) = x+1, \varphi(t) = 2t+1, \psi(t) = -t+11$; $F(x,t) = (x-1)(t^2+1)$

The conditions are compatible.

Problem 6. $L = 10, M = 10$; $T = 9, N = 5$; $a = 1$.

$f(x) = x+2, \varphi(t) = 3t+2, \psi(t) = t+12$; $F(x,t) = (2x+1)(2t-1)$

The conditions are compatible.

Problem 7. $L = 10, M = 10$; $T = 9, N = 5$; $a = 1$.

$f(x) = x + 3, \varphi(t) = 3t + 2, \psi(t) = t + 5$; $F(x,t) = 0$ (the null function)

The conditions are incompatible.

Problem 8.[NP 1996]. $L = 10, M = 6$; $T = 9, N = 9$; $a = 1$.

$$f(x) = \begin{cases} x, & 0 \le x \le \dfrac{L}{2} \\ L - x, & \dfrac{L}{2} \le x \le L \end{cases}$$

$\varphi(t) = 0, \psi(t) = 0; F(x,t) = 00$ (the null functions).

The conditions are compatible.

2 The source C++ program and print screen

Here we present a description of the source C++ program, print screen and numerical results.

2.1 The structure of the C++ source program

The C++ source program has au unitary structure composed of 7 sections.

Section 1. The program description.
Section 2. The C++ files.
Section 3. The variables type description.
Section 4. The procedures or functions.
Section 5. Start program.
Section 6. The input data in the program.
Section 7. Print the final results. .

A great importance has the section 1 where all the main aspects of the problem are described and illustrated: mathematical model, discrete model, informatics model etc.

Then, the sections 5, 6, 7 contain many steps inside the C++ source program. Example: step 1-general information about the program; step 2-input the data; step 3-the compatibility conditions; step 4-the incompatibility case; step 5-computing the border values from given conditions; step 6-construct the partial matrix; step 7-compute all the unknown values of the matrix; step 8-print the partial matrix line by line; step 9-print the entire final matrix of numerical results.

2.2 Description of C++ source program

```
//
    // Program Cpp C++ 60 NP
    // Partial Differential Equations of Second Order ; parabolic type
    // Heating transport on a linear conductor
    // Discrete problem

        // Section 1. The program description
    // The continuous Mathematical Model is
```

// $d\hat{}2u/dx\hat{}2 - (1/a\hat{}2)d^2u/dt + F(x,t) = 0$

// d is the partial differential operator

// $u = u(x,t)$ is unknown function

// $0 <= x <= L; 0 <= t <= T$

// Initial conditions : $u(x,0) = f(x)$

// Heads conditions : $u(0,t) = fi(t); u(L,t) = psi(t)$

// The numerical values of L, T and the functions f, fi, psi and F are known

// The first compatibility conditions (for f, fi and psi) are

// $f(0) = fi(0), f(L) = psi(0)$

// The mathematical model is denoted EDP2 XXX

// X=O means the homogenous case; X=N means non-homogenous case

// There exist 7 possible problems [2] EDP2 XXX:

// OON,ONO,ONN,NOO,NON,NNO,NNN

// OON means $F = 0, f = 0, fi$ or psi non-null functions

// ONO means $F = 0, f$ non-null functions; ,fi and psi null functions

// ONN means $F = 0, f$ non-null, fi or psi non-null functions etc

// A null polynomial function has all the coefficients equal to zero

//

// The analytical solution of the mathematical model has a very laborious form

// See the bibliography: [2]

// In the source program for the functions $f(x), fi(t), psi(t)$ we use the polynomial functions having the degree $grad <= 2$

// $f(x) = a1^*x + a2^*x + a3$; $fi(t) = b1t^*t + b2^*t + b3$; $psi(t) = c1^*t + c2^*t + c3$

// For $F(x,t)$ we use $F(x,t) = (d1^*x^*x + d2^*x + d3)(e1^*t^*t + e2^*t + e3)$

// (The user could change the form of $F(x,t)$)

// The changings must be done in Procedure Function $Fxitj$ in Section 4

//

// The Discrete Mathematical Model

// The interval $[0, L]$ has a division given by points xi, situated on vertical axis Ox

// The interval $[0, T]$ has a division given by the points tj, situated on horizontal axis Ot

// We denote by $h = L/M$ the step for the values $xi, xi = ih; i = \overline{0, M}$

// $x0 = 0, x1 = h, x2 = 2h, ..., xi = ih, ..., xM = Mh = L$; xi from the interval $[0, L]$

// We denote by $p = T/N$ the step for the values $tj, tj = jp; j = \overline{0, N}$

// $t0 = 0, t1 = p, t2 = 2p, ..., tj = jp, ..., tN = Np = T; tj$ from the interval $[0, T]$

// The natural values M and N are at the users disposal

// We denote $uij = u(xi, tj)$. All computations are done in these points

// of the Cartesian discrete product $[0, L]X[0, T]; (xi, tj)$

// We define the following discrete operators $d\hat{}2u$ si $d\hat{}1u$ by the formulas

// $d\hat{}2u(xi, tj) = (ui - 1j - 2uij + ui + 1j)/h\hat{}2$; counter i runs on Ox vertical axis

// $d\hat{}2u(xi, tj) = (uij - 1 - uij)/p$; counter j runs on Ot axis

// Examples of functions: $f(x) = x+1; fi(t) = 2t+1; psi(t) = L+t+1$

// $f(0) = fi(0) = 1, f(L) = psi(0) = L + 1$

// The counters running is $i = \overline{0, M}; j = \overline{0, N}$

// Denote by u the matrix $u = (uij)$ having the type $(M+1)x(N+1)$

// $i = \overline{0, M}$ counts the lines; $xi; j = \overline{0, N}$ counts the columns

// The matrix u has are $MN + M + N + 1$ elements

//

// The Informatics Model for C++ language is described below

// The coefficients of f, fi, psi and F are respectively

// $a1f, a2f, a3f; b1fi, b2fi, b3fi; c1psi, c2psi, c3psi$

// $d1F, d2F, d3F, e1F, e2F, e3F$

// We use the vectors : $af[3]; bfi[3]; cpsi[3]$

// Example: $f(x) = x + 1; fi(t) = 2t + 1; psi(t) = t + L - 1$

// The compatibility prove: $f(0) = 1, fi(0) = 1, f(L) = psi(0)$

// $af[3] = (0; 1; 1); bfi[3] = (0; 2; 1); cpsi[3] = (0; 1; L + 1)$

// We compute $u(x, 0) = f(x), u(xi, 0) = f(xi); xi = ih; i = \overline{0, M}$ (on vertical axis Ox)

// $xi = i * h, i = 0, 1, 2, 3, ..., M - 1, M$

// $u(xi, 0) = a1f * xi * xi + a2f * xi + a3f$

// We obtain $u00, u10; u20; ...; ui0; ...; uM{-}10; uM0$ (1) ($M{+}1$ known values)

//

// We compute $u(0, t) = fi(t), u(0, tj) = fi(tj); tj = jp; j = \overline{0, N}$

// $tj = j * p, j = \overline{0, N}$

// $u(0, tj) = b1fi * tj * tj + b2fi * tj + b3$

// We obtain $u00; u01; u02; ...; u0j; ...; u0N{-}1; u0N$ (2) ($N{+}1$ known values)

//

// We compute $u(L, t) = psi(t), u(L, tj) = psi(tj); tj = jp; j = \overline{0, N}; L = xM = h * M$

// $tj = j * p, j = \overline{0, N}$

// $u(L, tj) = c1psi * tj * tj + c2psi * tj + c3$

// We obtain $uM0; uM1; uM2; ...; uMj; ...; uMN{-}1; uMN$ (3)($N{+}1$ known values)

// (1) and (2) have the common part the value uoo

// (1) and (3) have in common part the value $uM0$

//

// We compute $du/dt(xi, 0) = g(xi)i = \overline{1, M}$

// $g(xi) = (ui1 - ui0)/p$, the result is $ui1 = pg(xi) + ui0$ where uio is known

// We obtain $u01; u11; u21; u31; ...; ui1; ...; uM1$ (4) ($M+1$ known values)

// This is the column $j=1$ (after the column $j=0$) in the matrix uij

//

// Al the values $u(xi, tj)$ are deposed in the rectangular matrix

// uij having the type $(N+1)x(M+1)$; denoted $uij[i][j]$

// The matrix uij has $(M+1)(N+1) = MN+M+N+1$ elements

// Known elements of the matrix uij are $M+2N+1$ elements

// Unknown elements of the matrix uij are $MN-N$ elements

//

// The unknown elements of the matrix uij are computed by the formulas

// $uij+1 = uij+((a^*a^*p)/(h^*h))^*(ui-1j-2uij+ui+1j)+F(xi, tj)$ for

// j fixed ($j = 1, 2, ..., N-1$) and all $i = 1, 2, ..., M-1$

// Section 2. The C++ files

$\#include < iostream.h >$

$\#include < conio.h >$

$\#include < math.h >$

// Section 3.Variables declarations

$intprob, nrprobdorit, i, j, k, M, Mm1, N, Nm1, im1, ip1, jp1, jm1, fictiv;$

float $a, A, B, L, T, xi, hxi, tj, ptj, f0, fL, fi0, psi0;$

float $af[4], bfi[4], cpsi[4], dF[4], eF[4], fg[4], uij[50][50];$

float $ui0M[50], u0jN[50], uLMjN[50];$

float $Fxitj, F1xi, F2tj, Fxt;$

// Section 4.Procedures. Functions

// Function $F(xi, tj) = (d1x^2 + d2x + d3)(e1t^2 + e2t + e3)$

float Function $Fxitj(intI, intj, floatFxitj)$

$xi = i * hxi; tj = j * ptj;$

$F1xi = dF[1] * xi * xi + dF[2] * xi + dF[3];$

$F2tj = eF[1] * tj * tj + eF[2] * tj + eF[3];$

$Fxitj = F1xi * F2tj;$

return $Fxitj;$

// Sectiona 5.START PROGRAM

void main()

The C++ program could be obtained from the authors by a simple e-mail message.

It is very easy to use this C++ program because it asks for input data step by step. The program was validated on many numerical problems. The numerical results appear in the screen. The screen could be printed.

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