A note on a diophantine equation

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Abstract

We prove, without using Catalan’s equation that, the only solution in positive integers of the equation $5^a - 2^b = 1$ is $a = 1, b = 2$. This shows a completely elementary method of solution of an equation from [1].

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Introduction

1. In paper [1] it is shown that all solutions to the equation

\begin{equation}
2^x + 5^y = z^2
\end{equation}

in nonnegative integers are provided by $x = 3, y = 0, z = 3$ and $x = 2, y = 1, z = 3$. 

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When \( x \geq 1, \ y \geq 1 \), in the proof offered in [1] we are led to the following equation

\[
5^y - 2^{k+1} = 1.
\]  

Then, the author uses the strong conjecture of Catalan, and proved recently by P. Mihăilescu ([2]), that the only solution to the equation

\[
a^b - c^d = 1
\]

in positive integers \( \geq 2 \) is offered by \( a = 3, \ b = 2, \ c = 2, \ d = 3 \). Since \( k + 1 \geq 2 \), clearly, we cannot have \( y \geq 2 \). When \( y = 1 \), however, we get \( k = 1 \).

In what follows, we shall prove elementarily this fact (i.e., without using the theory of equation (3)).

2. The proof

First we prove that \( k + 1 \) is even. If \( k + 1 = b \) is odd, then \( 2^b + 1 \) is divisible by \( 2 + 1 = 3 \), which is impossible, as 3 doesn’t divide \( 5^y \). Put \( b = 2B \). If \( B = 1 \), then we are done, as then \( y = 1 \), etc.

Let \( B > 1 \). Then we get the equation

\[
5^y - 4^B = 1.
\]

If \( y = 2A \) is even, then \( 5^{2A} - 1 = 4^B \), and as \( 5^{2A} - 1 \) is divisible by \( 5^2 - 1 = 24 \), which is divisible by 3, we get a contradiction, as \( 4^B \) cannot be divisible by 3. Thus \( y \) is odd; put \( y = 2A + 1 \). If \( A = 0 \), then \( y = 1 \); so we may suppose \( A \geq 1 \). Then (4) implies \( 5 \cdot 5^{2A} = 4^B + 1 \).

As \( 5 = 8 - 3, \ 5^{2A} = 25^A = (8 \cdot 3 + 1)^A \equiv 1(\mod 8) \), we get that

\[
5 \cdot 5^{2A} \equiv -3(\mod 8).
\]

On the other hand, if \( B \geq 2 \), clearly \( 4^B + 1 \equiv 1(\mod 8) \). As

\[-3(\mod 8) \not\equiv 1(\mod 8),
\]

the contradiction follows.

When \( B = 1 \) we get \( 5^{2A} = 1 \), which is impossible, since we have assumed \( A \geq 1 \).
References


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