Subordination results of certain analytic functions

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Abstract

In this paper, we obtain some subordination results for two integral operators defined in the open unit disk.

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1 Introduction and definitions

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. For the functions $f$ and $g$ in $\mathcal{A}$, we say that $f$ is subordinate to $g$ in $\mathcal{U}$, and write $f \prec g$, if there exists a Schwarz function $w$ in $\mathcal{U}$ with $|w(z)| < 1$ and $w(0) = 0$ such that $f(z) = g(w(z))$ in $\mathcal{U}$ (see [9]).

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Breaz and Breaz [4] and Breaz et al. [8] introduced and studied the integral operators

\[
F_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} \, dt
\]

and

\[
F_{\alpha_1, \ldots, \alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \cdots (f_n'(t))^{\alpha_n} \, dt
\]

where \( f_i \in \mathcal{A} \) and \( \alpha_i > 0 \), for all \( i = 1, \ldots, n, \ n \in \mathbb{N} \) (see also [1, 2, 3, 5, 7, 12, 13]).

Breaz and Güney [6] considered the above integral operators and they obtained their properties on the classes \( \mathcal{S}^*(b) \) and \( \mathcal{C}_\alpha(b) \) of starlike and convex functions of complex order \( b \) and type \( \alpha \) introduced and studied by Frasin [10].

Recently, Frasin [11] obtained some sufficient conditions for the above integral operators to be in the classes \( \mathcal{S}^*, \mathcal{C}(\alpha) \) and \( \mathcal{UCV} \), where \( \mathcal{C}(\alpha) \) and \( \mathcal{UCV} \) denote the subclasses of \( \mathcal{A} \) consisting of functions which are, respectively, close-to-convex of order \( \alpha(0 \leq \alpha < 1) \) in \( \mathcal{U} \) and uniformly convex functions.

In the present paper, we obtain some subordination results of the above integral operators \( F_n(z) \) and \( F_{\alpha_1, \ldots, \alpha_n}(z) \).

In order to derive our main results, we have to recall here the following results:

**Lemma 1** ([14]) If \( f \in \mathcal{A} \) satisfies

\[
\text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} < \frac{\beta - 1}{2\delta(\beta + 1)} \quad (z \in \mathcal{U})
\]

for some \( \beta > 1 \) and \( \delta > 0 \), then

\[
(f'(z))^\delta < \frac{\beta(1-z)}{\beta - z} \quad (z \in \mathcal{U}).
\]
Lemma 2 ([14]) If \( f \in A \) satisfies
\[
\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > \frac{1 - \beta}{2\delta(\beta + 1)} \quad (z \in U)
\]
for some \( \beta > 1 \) and \( \delta > 0 \), then
\[
\left( \frac{1}{f'(z)} \right)^\delta \prec \frac{\beta(1 - z)}{\beta - z} \quad (z \in U).
\]

2 Subordination results

We begin by proving the following theorem.

**Theorem 1** Let \( \alpha_i > 0 \) be real numbers for all \( i = 1, \ldots, n \). If each \( f_i \in A \) \( \{i = 1, \ldots, n\} \) satisfies
\[
\Re \left( \frac{zf'_i(z)}{f_i(z)} \right) < 1 + \frac{\beta - 1}{2\delta(\beta + 1)} \sum_{i=1}^n \alpha_i \quad (z \in U)
\]
for some \( \beta > 1 \) and \( \delta > 0 \), then
\[
\left( \prod_{i=1}^n \left( \frac{f_i(z)}{z} \right)^\alpha_i \right)^\delta \prec \frac{\beta(1 - z)}{\beta - z} \quad (z \in U).
\]

**Proof.** It follows from (1) that
\[
F'_n(z) = \left( \frac{f_1(z)}{z} \right)^{\alpha_1} \ldots \left( \frac{f_n(z)}{z} \right)^{\alpha_n}.
\]
Thus we have
\[
F''_n(z) = \left[ \alpha_1 \left( \frac{f'_1(z)}{f_1(z)} - \frac{1}{z} \right) + \ldots + \alpha_n \left( \frac{f'_n(z)}{f_n(z)} - \frac{1}{z} \right) \right] F'_n(z)
\]
or, equivalently,
\[
\frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^n \alpha_i \left( \frac{zf'_i(z)}{f_i(z)} - 1 \right).
\]
Taking the real part of both terms of (10), we have
\[
\Re \left( \frac{zF''(z)}{F'_n(z)} \right) = \sum_{i=1}^{n} \alpha_i \Re \left( \frac{zf'_i(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i \\
= \alpha_1 \Re \left( \frac{zf'_1(z)}{f_1(z)} \right) + \alpha_2 \Re \left( \frac{zf'_2(z)}{f_2(z)} \right) + \cdots \\
+ \alpha_n \Re \left( \frac{zf'_n(z)}{f_n(z)} \right) - \alpha_1 - \alpha_2 - \cdots - \alpha_n.
\]

(11)

Making use of the hypothesis (7), we obtain
\[
\Re \left( \frac{zF''(z)}{F'_n(z)} \right) < \alpha_1 \left( 1 + \frac{\beta - 1}{2\delta(\beta + 1)[\alpha_1 + \cdots + \alpha_n]} \right) + \alpha_2 \left( 1 + \frac{\beta - 1}{2\delta(\beta + 1)[\alpha_1 + \cdots + \alpha_n]} \right) \\
+ \cdots + \alpha_n \left( 1 + \frac{\beta - 1}{2\delta(\beta + 1)[\alpha_1 + \cdots + \alpha_n]} \right) - \alpha_1 - \alpha_2 - \cdots - \alpha_n \\
< \frac{\beta - 1}{2\delta(\beta + 1)}.
\]

Applying Lemma 1, we have
\[
(F'_n(z))^\delta < \frac{\beta(1 - z)}{\beta - z} \quad (z \in \mathcal{U})
\]
or, equivalently,
\[
\left( \prod_{i=1}^{n} \left( \frac{f_i(z)}{z} \right)^{\alpha_i} \right)^\delta < \frac{\beta(1 - z)}{\beta - z} \quad (z \in \mathcal{U}).
\]

This completes the proof.

Letting \( n = 1 \), \( \alpha_1 = \alpha \) and \( f_1 = f \) in Theorem 1, we have

\textbf{Corollary 1} Let \( \alpha > 0 \). If \( f \in A \) satisfies
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) < 1 + \frac{\beta - 1}{2\delta \alpha(\beta + 1)} \quad (z \in \mathcal{U})
\]

(12)
for some $\beta > 1$ and $\delta > 0$, then
\[(13) \quad \left( \frac{f(z)}{z} \right)^{\alpha \delta} < \frac{\beta(1-z)}{\beta - z} \quad (z \in U).\]

Next, we prove

**Theorem 2** Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If
\[(14) \quad \sum_{i=1}^{n} \alpha_i > n + \frac{1 - \beta}{2\delta(\beta + 1)}\]
for some $\beta > 1$ and $\delta > 0$, and each $f_i \in \mathcal{A}, \{i = 1, \ldots, n\}$ satisfies
\[(15) \quad \text{Re} \left( \frac{zf_i'(z)}{f_i(z)} \right) < \frac{1}{\alpha_i} \quad (z \in U)\]
then
\[(16) \quad \left( \prod_{i=1}^{n} \left( \frac{f_i(z)}{z} \right)^{\alpha_i} \right)^{\delta} < \frac{\beta(1-z)}{\beta - z} \quad (z \in U).\]

**Proof.** From (11) and using (14), (15), we get
\[
\text{Re} \left( \frac{zF''_n(z)}{F'_n(z)} \right) = \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zf'_i(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i
\]
\[
= \alpha_1 \text{Re} \left( \frac{zf'_1(z)}{f_1(z)} \right) + \alpha_2 \text{Re} \left( \frac{zf'_2(z)}{f_2(z)} \right) + \cdots
\]
\[
+ \alpha_n \text{Re} \left( \frac{zf'_n(z)}{f_n(z)} \right) - \sum_{i=1}^{n} \alpha_i
\]
\[
= \alpha_1 \left( \frac{1}{\alpha_1} \right) + \alpha_2 \left( \frac{1}{\alpha_2} \right) + \cdots + \alpha_n \left( \frac{1}{\alpha_n} \right) - \sum_{i=1}^{n} \alpha_i
\]
\[
= n - \sum_{i=1}^{n} \alpha_i < \frac{\beta - 1}{2\delta(\beta + 1)}
\]

The result of Theorem 2, now follows by applying Lemma 1.

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2, we have
Corollary 2 Let $\beta > 1$ and $\delta > 0$ with

\begin{equation}
0 < 1 + \frac{1 - \beta}{2\delta(\beta + 1)} < \alpha.
\end{equation}

If $f \in \mathcal{A}$ satisfies

\begin{equation}
\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{1}{\alpha} \quad (z \in \mathcal{U})
\end{equation}

then

\begin{equation}
\left(\frac{f(z)}{z}\right)^{\alpha \delta} < \frac{\beta(1 - z)}{\beta - z} \quad (z \in \mathcal{U}).
\end{equation}

Next, applying Lemma 2, we obtain the following two results.

Theorem 3 Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$ satisfies

\begin{equation}
\operatorname{Re}\left(\frac{zf'_i(z)}{f_i(z)}\right) > 1 + \frac{1 - \beta}{2\delta(\beta + 1)} \sum_{i=1}^{n} \alpha_i \quad (z \in \mathcal{U}),
\end{equation}

for some $\beta > 1$ and $\delta > 0$, then

\begin{equation}
\left(\prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right) \right)^{-1} < \frac{\beta(1 - z)}{\beta - z} \quad (z \in \mathcal{U}).
\end{equation}

Proof. Employing the same manner as in the proof of Theorem 1, from (11), (20) and applying Lemma 2, it can be easily established the subordination (21).

Theorem 4 Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If

\begin{equation}
\sum_{i=1}^{n} \alpha_i < n + \frac{\beta - 1}{2\delta(\beta + 1)}
\end{equation}
for some $\beta > 1$ and $\delta > 0$, and each $f_i \in A, \{i = 1, \ldots, n\}$ satisfies

\begin{equation}
Re \left( \frac{zf_i'(z)}{f_i(z)} \right) > \frac{1}{\alpha_i} \quad (z \in \mathcal{U})
\end{equation}

then

\begin{equation}
\left( \prod_{i=1}^{n} \left( \frac{f_i(z)}{z} \right)^{\alpha_i} \right)^{-1} \beta(1-z) \beta - z \quad (z \in \mathcal{U}).
\end{equation}

**Proof.** Employing the same manner as in the proof of Theorem 2, from (22), (23) and applying Lemma 2, it can be easily established the subordination (24).

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3, we have

**Corollary 3** Let $\alpha > 0$. If $f \in A$ satisfies

\begin{equation}
Re \left( \frac{zf'(z)}{f(z)} \right) > 1 + \frac{1 - \beta}{2\alpha(\beta + 1)} \quad (z \in \mathcal{U}),
\end{equation}

for some $\beta > 1$ and $\delta > 0$, then

\begin{equation}
\left( \frac{z}{f(z)} \right)^{\alpha\delta} \beta(1-z) \beta - z \quad (z \in \mathcal{U}).
\end{equation}

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 4, we have

**Corollary 4** Let $\beta > 1$ and $\delta > 0$ with

\begin{equation}
0 < \alpha < 1 + \frac{\beta - 1}{2\delta(\beta + 1)}.
\end{equation}

If $f \in A$ satisfies

\begin{equation}
Re \left( \frac{zf'(z)}{f(z)} \right) > \frac{1}{\alpha} \quad (z \in \mathcal{U})
\end{equation}

then

\begin{equation}
\left( \frac{z}{f(z)} \right)^{\alpha\delta} \beta(1-z) \beta - z \quad (z \in \mathcal{U}).
\end{equation}
Now we prove

**Theorem 5** Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in A$ for all $i = 1, \ldots, n$ satisfies

\[
\text{Re} \left( \frac{z f''_i(z)}{f'_i(z)} \right) < \frac{\beta - 1}{2\delta(\beta + 1) \sum_{i=1}^{n} \alpha_i} \quad (z \in \mathcal{U})
\]

for some $\beta > 1$ and $\delta > 0$, then

\[
\left( \prod_{i=1}^{n} (f'_i(z))^{\alpha_i} \right)^{\delta} < \frac{\beta(1-z)}{\beta - z} \quad (z \in \mathcal{U}).
\]

**Proof.** From (2) we easily get

\[
\frac{z F''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{z f''_i(z)}{f'_i(z)} \right).
\]

Thus we have

\[
\text{Re} \left( \frac{z F''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} \right) = \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{z f''_i(z)}{f'_i(z)} \right)
= \alpha_1 \text{Re} \left( \frac{z f'_1(z)}{f_1(z)} \right) + \alpha_2 \text{Re} \left( \frac{z f'_2(z)}{f_2(z)} \right) + \cdots
+ \alpha_n \text{Re} \left( \frac{z f'_n(z)}{f_n(z)} \right).
\]

From (30), it follows that

\[
\text{Re} \left( \frac{z F''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} \right) < \alpha_1 \left( \frac{\beta - 1}{2\delta(\beta + 1)[\alpha_1 + \cdots + \alpha_n]} \right)
+ \alpha_2 \left( \frac{\beta - 1}{2\delta(\beta + 1)[\alpha_1 + \cdots + \alpha_n]} \right) + \cdots + \alpha_n \left( \frac{\beta - 1}{2\delta(\beta + 1)[\alpha_1 + \cdots + \alpha_n]} \right)
\leq \frac{\beta - 1}{2\delta(\beta + 1)}
\]
for all $z \in \mathcal{U}$. Therefore, from Lemma 1, we obtain (31).

Applying Lemma 2, the proof of the next theorem below is much akin to that of Theorem 5 and so we omit for details involved.

**Theorem 6** Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$ satisfies

$$
\Re \left( \frac{zf_i''(z)}{f_i'(z)} \right) > \frac{1 - \beta}{2\delta(\beta + 1) \sum_{i=1}^{n} \alpha_i} \quad (z \in \mathcal{U})
$$

for some $\beta > 1$ and $\delta > 0$, then

$$
\left( \prod_{i=1}^{n} (f_i'(z))^\alpha_i \right)^{-1} \delta < \frac{\beta(1 - z)}{\beta - z} \quad (z \in \mathcal{U}).
$$

**References**


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