Differential sandwich theorems for some analytic functions defined by certain linear operators

M. K. Aouf, A. O. Mostafa

Abstract

In this investigation, we obtain some applications of first order differential subordination and superordination results involving the multiplier transformation and Al-Oboudi operator for certain normalized analytic functions in the open unit disc.

2010 Mathematics Subject Classification: 30C45.

Key words and phrases: Analytic functions, differential subordination , superordination, sandwich theorems, multiplier transformation, Al-Oboudi operator.

1 Introduction

Let $H(U)$ be the class of analytic functions in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and let $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form:

\[ f(z) = a + a_k z^k + a_{k+1} z^{k+1} \ldots \quad (a \in \mathbb{C}). \]
Also, let $A$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + a_2 z^2 + \ldots.$$  

(1.2)

If $f, g \in H(U)$, we say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g(z)$ is univalent in $U$, then we have the following equivalence, (cf., e.g.,[5], [10]; see also [11]):

$$f(z) \prec g(z) (z \in U) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $p, h \in H(U)$ and let $\varphi(r, s, t; z) : C^3 \times U \to C$. If $p$ and $\varphi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if $p$ satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z),$$

(1.3) then $p$ is a solution of the differential superordination (1.3). Note that if $f$ is subordinate to $g$, then $g$ is superordinate to $f$. An analytic function $q$ is called a subordinant if $q(z) \prec p(z)$ for all $p$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q < \tilde{q}$ for all subordinants of (1.3) is called the best subordinant. Recently Miller and Mocanu [12] obtained conditions on the functions $h, q$ and $\varphi$ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

(1.4) Using the results of Miller and Mocanu [12], Bulboǎca [4] considered certain classes of first order differential superordinations as well as superordination preserving integral operators [6]. Ali et al. [1], have used the results of Bulboǎca [4] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$
where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \). Also, Tuneski [16] obtained a sufficient condition for starlikeness of \( f \) in terms of the quantity \( \frac{f''(z)f(z)}{(f'(z))^2} \). Recently, Shanmugam et al. [15] obtained sufficient conditions for the normalized analytic function \( f \) to satisfy

\[
q_1(z) < \frac{f(z)}{zf'(z)} < q_2(z)
\]

and

\[
q_1(z) < \frac{z^2f'(z)}{f(z)} < q_2(z),
\]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = 1 \) and \( q_2(0) = 1 \). They [15] also obtained results for functions defined by using Carlson-Shaffer operator and Salagean operator.

For any real number \( \sigma \), Cho and Kim [7] defined the multiplier transformations \( I_\delta \) of functions \( f \in A \) by

\[
I_\delta f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\delta)^\sigma}{1+\delta} a_k z^k \quad (\delta > -1).
\]

Obviously, we observe that

\[
I_\delta^\sigma (I_\delta f(z)) = I_\delta^{\sigma+\gamma} f(z) \quad (\sigma, \gamma \text{ real}),
\]

\[
I_\delta^\sigma f(z) = f(z), \quad I_\delta^1 f(z) = zf'(z), \quad I_\delta^2 f(z) = z(f'(z) + zf''(z))
\]

and

\[
z(I_\delta^\sigma f(z))' = (1+\delta)I_\delta^{\sigma+1} f(z) - \delta I_\delta^\sigma f(z).
\]

For all real integer \( \sigma \) and \( \delta = 1 \), the operator \( I_\delta^\sigma \) was studied by Uralegaddi and Somanatha [17], and for \( \sigma = -1 \) the operator \( I_\delta^\sigma \) is the integral operator studied by Owa ana Srivastava [13]. Furthermore, for any negative real number \( \sigma \) and \( \delta = 1 \), the operator \( I_\delta^\sigma \) is the multiplier transformation studied by Jung et al. [9] (see also [3]), and for any non-negative integer \( \sigma \) and \( \delta = 0 \), the operator \( I_\delta^\sigma \) is the differential operator...
defined by Sălăgean [14]. In fact the operator $I_\alpha^\lambda$ is related rather closely to the multiplier transformation studied by Flett [8].

For $f(z) \in A$, $\lambda > 0$ real number, Al-Oboudi [2] defined the differential operator $D_\lambda^n f(z)$ ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, \ldots\}$) as follows:

$$D_\lambda^0 f(z) = f(z)$$
$$D_\lambda^1 f(z) = D_\lambda f(z) = (1 - \lambda) f(z) + \lambda z f'(z), \quad \lambda > 0,$$
$$D_\lambda^2 f(z) = D_\lambda (D_\lambda f(z)),$$

and

$$D_\lambda^n f(z) = D_\lambda (D_\lambda^{n-1} f(z)) \quad (n \in \mathbb{N}; \lambda > 0).$$

When $\lambda = 1$ we get the Sălăgean differential operator [14]. It is easily seen that

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \left[1 + \lambda (k - 1)\right]^n a_k z^k \quad (n \in \mathbb{N}_0; \lambda > 0).$$

(1.7)

Obviously, we observe that

$$\lambda z (D_\lambda^n f(z))' = D_\lambda^{n+1} f(z) - (1 - \lambda) D_\lambda^n f(z) \quad (\lambda > 0).$$

(1.8)

2 Definitions and preliminaries

In order to prove our results, we shall make use of the following known results.

**Definition 1** [12]. Denote by $Q$, the set of all functions $f$ that are analytic and injective on $U \setminus E(f)$, where

$$E(f) = \{\xi \in \partial U : \lim_{z \to \xi} f(z) = \infty\},$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$. 
Lemma 1 [11]. Let $q$ be univalent in the unit disk $U$ and $\theta$ and $\varphi$ be analytic in a domain $D$ containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set

\[(2.1) \quad \psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z).\]

Suppose that

\[\begin{align*}
\text{(i)} & \quad \psi(z) \text{ is starlike univalent in } U, \\
\text{(ii)} & \quad \Re\left\{\frac{zq'(z)}{\varphi(q(z))}\right\} > 0 \text{ for } z \in U.
\end{align*}\]

If $p$ is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

\[(2.2) \quad \theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(q(z)) + zq'(z)\varphi(q(z)),\]

then $p(z) < q(z)$ and $q$ is the best dominant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1, Shanmugam et al. [15] obtained the following lemma.

Lemma 2 [15]. Let $q$ be univalent in $U$ with $q(0) = 1$. Let $\alpha \in \mathbb{C}, \gamma \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, further assume that

\[
\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \max\{0, -\Re(\alpha/\gamma)\}.
\]

If $p$ is analytic in $U$, and

\[\alpha p(z) + \gamma zp'(z) < \alpha q(z) + \gamma zq'(z),\]

then $p < q$ and $q$ is the best dominant.

Lemma 3 [4]. Let $q$ be convex univalent in $U$ and $\vartheta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$. Suppose that

\[\begin{align*}
\text{(i)} & \quad \Re\{\vartheta'(q(z))/\phi(q(z))\} > 0 \text{ for } z \in U, \\
\text{(ii)} & \quad \psi(z) = zq'(z)\phi(q(z)) \text{ is starlike univalent in } U.
\end{align*}\]
If \( p(z) \in H[q(0), 1] \cap Q \), with \( p(U) \subseteq D \), and \( \vartheta(p(z)) + z p'(z) \phi(p(z)) \) is univalent in \( U \) and

\[
(2.3) \quad \vartheta(q(z)) + z q'(z) \phi(q(z)) \prec \vartheta(p(z)) + z p'(z) \phi(p(z)),
\]

then \( q(z) \prec p(z) \) and \( q \) is the best subordinant.

Taking \( \theta(w) = \alpha w \) and \( \varphi(w) = \gamma \) in Lemma 3, Shanmugam et al. [15] obtained the following lemma.

**Lemma 4** [15]. Let \( q \) be convex univalent in \( U \), \( q(0) = 1 \). Let \( \alpha, \gamma \in \mathbb{C} \) and \( \text{Re}\{\alpha/\gamma\} > 0 \). If \( p \in H[q(0), 1] \cap Q \), \( \alpha p(z) + \gamma z p'(z) \) is univalent in \( U \) and

\[
\alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z),
\]

then \( q \prec p \) and \( q \) is the best subordinant.

### 3 Applications to multiplier transformation and sandwich theorems

**Theorem 1** Let \( q \) be convex univalent in \( U \) with \( q(0) = 1 \), \( \gamma \in \mathbb{C}^* \). Further, assume that

\[
(3.1) \quad \text{Re}\left\{1 + \frac{z q''(z)}{q'(z)}\right\} > \max\{0, -\text{Re}(1/\gamma)\}.
\]

If \( f \in A \), and

\[
(3.2) \quad \frac{I^\gamma f(z)}{I^{\gamma+1} f(z)} + \gamma(1 + \delta) \left[1 - \frac{I^\gamma f(z)I^\gamma f(z)}{[I^{\gamma+1} f(z)]^2}\right] \prec q(z) + \gamma z q'(z),
\]

then

\[
\frac{I^\gamma f(z)}{I^{\gamma+1} f(z)} \prec q(z)
\]

and \( q \) is the best dominant.
Proof. Define a function $p$ by

$$(3.3) \quad p(z) = \frac{I_\delta^\gamma f(z)}{I_\delta^{\gamma+1} f(z)} \quad (z \in U).$$

Then the function $p$ is analytic in $U$ and $p(0) = 1$. Therefore, differentiating (3.3) logarithmically with respect to $z$ and using the identity (1.6) in the resulting equation, we have

$$\frac{I_\delta^\gamma f(z)}{I_\delta^{\gamma+1} f(z)} + \gamma (1 + \delta) \left[ 1 - \frac{I_\delta^\gamma f(z) I_\delta^{\gamma+2} f(z)}{[I_\delta^{\gamma+1} f(z)]^2} \right] = p(z) + \gamma z p'(z),$$

that is, that

$$p(z) + \gamma z p'(z) \preceq q(z) + \gamma z q'(z)$$

and therefore, the theorem follows by applying Lemma 2.

Now, by appealing to Lemma 4, it can be easily prove the following theorem.

Theorem 2 Let $q$ be convex univalent in $U$. Let $\gamma \in \mathbb{C}$ with $\Re \gamma > 0$. If $f \in A$, $\frac{I_\delta^\gamma f(z)}{I_\delta^{\gamma+1} f(z)} \in H[1, 1] \cap Q$,

$$\frac{I_\delta^\gamma f(z)}{I_\delta^{\gamma+1} f(z)} + \gamma (1 + \delta) \left[ 1 - \frac{I_\delta^\gamma f(z) I_\delta^{\gamma+2} f(z)}{[I_\delta^{\gamma+1} f(z)]^2} \right],$$

is univalent in $U$, and

$$q(z) + \gamma z q'(z) \preceq \frac{I_\delta^\gamma f(z)}{I_\delta^{\gamma+1} f(z)} + \gamma (1 + \delta) \left[ 1 - \frac{I_\delta^\gamma f(z) I_\delta^{\gamma+2} f(z)}{[I_\delta^{\gamma+1} f(z)]^2} \right],$$

then

$$q(z) \preceq \frac{I_\delta^\gamma f(z)}{I_\delta^{\gamma+1} f(z)}$$

and $q$ is the best subordinant.

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.
Theorem 3 Let $\gamma \in \mathbb{C}$ with $\text{Re}\, \gamma > 0$, $q_1$ be convex univalent in $U$ and $q_2$ be univalent in $U$, $q_2(0) = 1$ and satisfies (3.1). If $f \in A$, $0 \neq \frac{I_\delta^\gamma f(z)}{I_\delta^{\sigma+1} f(z)} \in H[1,1] \cap Q$, 
\[
\frac{I_\delta^\gamma f(z)}{I_\delta^{\sigma+1} f(z)} + \gamma(1 + \delta) \left[ 1 - \frac{I_\delta^\gamma f(z)I_\delta^{\sigma+2} f(z)}{[I_\delta^{\sigma+1} f(z)]^2} \right]
\]
is univalent in $U$, and 
\[
q_1(z) + \gamma z q_1'(z) \prec \frac{I_\delta^\gamma f(z)}{I_\delta^{\sigma+1} f(z)} + \gamma(1 + \delta) \left[ 1 - \frac{I_\delta^\gamma f(z)I_\delta^{\sigma+2} f(z)}{[I_\delta^{\sigma+1} f(z)]^2} \right]
\]
\[
\prec q_2(z) + \gamma z q_2'(z),
\]
then 
\[
q_1(z) \prec \frac{I_\delta^\gamma f(z)}{I_\delta^{\sigma+1} f(z)} \prec q_2(z)
\]
and $q_1$ and $q_2$ are, respectively, the best subordinant and the best dominant.

Taking $\delta = 0$ in Theorems 1, 2 and 3, we have results improve that obtained by Shanmugam et al. [15, Theorems 5.1, 5.2 and 5.3, respectively].

Taking $\delta = \sigma = 0$ in Theorems 1, 2 and 3, we obtain the results obtained by Shanmugam et al. [15, Theorems 3.1, 3.2 and Corollary 3.3, respectively].

Theorem 4 Let $q$ be convex univalent in $U$, $\gamma \in \mathbb{C}$, $\text{Re}\{\gamma\} > 0$. Further, assume that (3.1) holds. If $f \in A$ satisfies 
\[
[1+\gamma(1+\delta)]\frac{z I_\delta^{\sigma+1} f(z)}{[I_\delta^\gamma f(z)]^2} + \gamma(1+\delta) \left[ \frac{z I_\delta^{\sigma+2} f(z)}{[I_\delta^\gamma f(z)]^2} \right] - 2 \frac{z [I_\delta^{\sigma+1} f(z)]^2}{[I_\delta^\gamma f(z)]^3} \prec q(z) + \gamma z q'(z),
\]
then 
\[
\frac{z I_\delta^{\sigma+1} f(z)}{[I_\delta^\gamma f(z)]^2} \prec q(z)
\]
and $q$ is the best dominant.
Differential sandwich theorems for some analytic... 19

Proof. Define the function $p(z)$ by

$$p(z) = \frac{zI_0(z)^{\gamma+1}f(z)}{[I_0(z)^{\gamma}f(z)]^2} \ (z \in U).$$

Then, simple computations show that

$$p(z)+\gamma z p'(z) = [1+\gamma(1+\delta)] \frac{zI_0(z)^{\gamma+1}f(z)}{[I_0(z)^{\gamma}f(z)]^2} + \gamma(1+\delta) \left[ \frac{zI_0(z)^{\gamma+2}f(z)}{[I_0(z)^{\gamma}f(z)]^2} \right] - 2z \frac{[I_0(z)^{\gamma+1}f(z)]^2}{[I_0(z)^{\gamma}f(z)]^3}.$$ 

Applying Lemma 2, the theorem follows.

Theorem 5 Let $q$ be convex univalent in $U$. Let $\gamma \in \mathbb{C}$ with $\text{Re} \gamma > 0$. If $f \in A$, $\frac{zI_0(z)^{\gamma+1}f(z)}{[I_0(z)^{\gamma}f(z)]^2} \in H[1,1] \cap Q$, $\left[ 1+\gamma(1+\delta) \right] \frac{zI_0(z)^{\gamma+1}f(z)}{[I_0(z)^{\gamma}f(z)]^2} + \gamma(1+\delta) \left[ \frac{zI_0(z)^{\gamma+2}f(z)}{[I_0(z)^{\gamma}f(z)]^2} \right] - 2z \frac{[I_0(z)^{\gamma+1}f(z)]^2}{[I_0(z)^{\gamma}f(z)]^3}$ is univalent in $U$, and

$$q(z)+\gamma z q'(z) < \left[ 1+\gamma(1+\delta) \right] \frac{zI_0(z)^{\gamma+1}f(z)}{[I_0(z)^{\gamma}f(z)]^2} + \gamma(1+\delta) \left[ \frac{zI_0(z)^{\gamma+2}f(z)}{[I_0(z)^{\gamma}f(z)]^2} \right] - 2z \frac{[I_0(z)^{\gamma+1}f(z)]^2}{[I_0(z)^{\gamma}f(z)]^3},$$

then

$$q(z) < \frac{zI_0(z)^{\gamma+1}f(z)}{[I_0(z)^{\gamma}f(z)]^2},$$

and $q$ is the best subordinant.

Proof. The proof follows by applying Lemma 4.

Combining Theorem 4 and Theorem 5, we get the following sandwich theorem.

Theorem 6 Let $\gamma \in \mathbb{C}$ with $\text{Re} \gamma > 0$, $q_1$ be convex univalent in $U$ and $q_2$ be univalent in $U$, $q_2(0) = 1$ and satisfies (3.1). If $f \in A$, $\frac{zI_0(z)^{\gamma+1}f(z)}{[I_0(z)^{\gamma}f(z)]^2} \in H[1,1] \cap Q$, $\left[ 1+\gamma(1+\delta) \right] \frac{zI_0(z)^{\gamma+1}f(z)}{[I_0(z)^{\gamma}f(z)]^2} + \gamma(1+\delta) \left[ \frac{zI_0(z)^{\gamma+2}f(z)}{[I_0(z)^{\gamma}f(z)]^2} \right] - 2z \frac{[I_0(z)^{\gamma+1}f(z)]^2}{[I_0(z)^{\gamma}f(z)]^3}$
is univalent in $U$, and
\begin{align*}
q_1(z) + \gamma z q_1'(z) & \prec \left[1 + \gamma (1 + \delta)\right] \frac{z I_{\delta}^{n+1} f(z)}{[I_{\delta}^n f(z)]^2} + \gamma (1 + \delta) \left[\frac{z I_{\delta}^{n+2} f(z)}{[I_{\delta}^n f(z)]^2} - 2 \frac{z [I_{\delta}^{n+1} f(z)]^2}{[I_{\delta}^n f(z)]^3}\right] \\
& \prec q_2(z) + \gamma z q_2'(z),
\end{align*}
then
\begin{align*}
q_1(z) & \prec \frac{z I_{\delta}^{n+1} f(z)}{[I_{\delta}^n f(z)]^2} \prec q_2(z)
\end{align*}
and $q_1$ and $q_2$ are, respectively, the best subordinant and the best dominant.

Taking $\delta = 0$ in Theorems 4, 5 and 6, we get the results obtained by Shanmugam et al. [15, Theorems 5.4, 5.5 and 5.6, respectively].

Taking $\sigma = \delta = 0$ in Theorems 4, 5 and 6, we get the results obtained by Shanmugam et al. [15, Theorems 3.4, 3.5 and Corollary 3.6, respectively].

4 Applications to Al-Oboudi operator and sandwich theorems

**Theorem 7** Let $q$ be convex univalent in $U$ with $q(0) = 1$, $\gamma \in \mathbb{C}^*$. Further, assume that the condition (3.1) holds. If $f \in A, \lambda > 0$ and
\begin{equation}
\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} + \gamma \left[1 - \frac{D_\lambda^n f(z) D_\lambda^{n+2} f(z)}{[D_\lambda^{n+1} f(z)]^2}\right] \prec q(z) + \gamma z q'(z),
\end{equation}
then
\begin{align*}
\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} & \prec q(z)
\end{align*}
and $q$ is the best dominant.

**Proof.** Define a function $p$ by
\begin{equation}
p(z) = \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \quad (z \in U).
\end{equation}
Then the function \( p \) is analytic in \( U \) and \( p(0) = 1 \). Therefore, differentiating (4.2) logarithmically with respect to \( z \) and using the identity (1.8) in the resulting equation, we have

\[
\frac{D^n f(z)}{D^{n+1} f(z)} + \frac{\gamma}{\lambda} \left[ 1 - \frac{D^n f(z)D^{n+2} f(z)}{[D^{n+1} f(z)]^2} \right] = p(z) + \gamma z p'(z) \quad (\lambda > 0),
\]

that is, that

\[
p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z)
\]

and therefore, the theorem follows by applying Lemma 2.

Now, by appealing to Lemma 4, it can be easily prove the following theorem.

**Theorem 8** Let \( q \) be convex univalent in \( U \). Let \( \gamma \in \mathbb{C} \) with \( \text{Re} \gamma > 0 \). If \( f \in A, \lambda > 0, \frac{D^n f(z)}{D^{n+1} f(z)} \in H[1, 1] \cap Q, \)

\[
\frac{D^n f(z)}{D^{n+1} f(z)} + \frac{\gamma}{\lambda} \left[ 1 - \frac{D^n f(z)D^{n+2} f(z)}{[D^{n+1} f(z)]^2} \right]
\]

is univalent in \( U \), and

\[
q(z) + \gamma z q'(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)} + \frac{\gamma}{\lambda} \left[ 1 - \frac{D^n f(z)D^{n+2} f(z)}{[D^{n+1} f(z)]^2} \right],
\]

then

\[
q(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)}
\]

and \( q \) is the best subordinant.

Combining Theorem 7 and Theorem 8, we get the following sandwich theorem.

**Theorem 9** Let \( \gamma \in \mathbb{C} \) with \( \text{Re} \gamma > 0 \), \( q_1 \) be convex univalent in \( U \) and \( q_2 \) be univalent in \( U \), \( q_2(0) = 1 \) and satisfies (3.1). If \( f \in A, \lambda > 0, \frac{D^n f(z)}{D^{n+1} f(z)} \in H[1, 1] \cap Q, \)

\[
\frac{D^n f(z)}{D^{n+1} f(z)} + \frac{\gamma}{\lambda} \left[ 1 - \frac{D^n f(z)D^{n+2} f(z)}{[D^{n+1} f(z)]^2} \right]
\]
is univalent in $U$, and

$$q_1(z) + \gamma z q_1'(z) \prec \frac{D_{\lambda}^n f(z)}{D_{\lambda}^{n+1} f(z)} + \gamma \left[ 1 - \frac{D_{\lambda}^n f(z) D_{\lambda}^{n+2} f(z)}{[D_{\lambda}^{n+1} f(z)]^2} \right]$$

$$\prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{D_{\lambda}^n f(z)}{D_{\lambda}^{n+1} f(z)} \prec q_2(z)$$

and $q_1$ and $q_2$ are, respectively, the best subordinant and the best dominant.

**Remark 1**

(i) Taking $\lambda = 1$ in Theorems 7, 8 and 9, we have results improve that obtained by Shanmugam et al. [15, Theorems 5.1, 5.2 and 5.3, respectively].

(ii) Taking $\lambda = 1$ and $n = 0$ in Theorems 7, 8 and 9, we obtain the results obtained by Shanmugam et al. [15, Theorems 3.1, 3.2 and Corollary 3.3, respectively].

**Theorem 10** Let $q$ be convex univalent in $U$, $\gamma \in \mathbb{C}^*$. Further, assume that (3.1) holds. If $f \in A, \lambda > 0$ satisfies

$$(1 + \frac{\gamma}{\lambda}) z D_{\lambda}^{n+1} f(z) + \frac{\gamma}{\lambda} \left[ \frac{z D_{\lambda}^{n+2} f(z)}{[D_{\lambda}^n f(z)]^2} - 2 \frac{z [D_{\lambda}^{n+1} f(z)]^2}{[D_{\lambda}^n f(z)]^3} \right]$$

$$\prec q(z) + \gamma z q'(z),$$

then

$$\frac{z D_{\lambda}^{n+1} f(z)}{[D_{\lambda}^n f(z)]^2} \prec q(z)$$

and $q$ is the best dominant.

**Proof.** Define the function $p(z)$ by

$$p(z) = \frac{z D_{\lambda}^{n+1} f(z)}{[D_{\lambda}^n f(z)]^2} (z \in U).$$
Then, simple computations show that
\[ p(z) + \gamma z p'(z) = (1 + \gamma) \frac{zD_n^{n+1}f(z)}{D_n f(z)} + \frac{\gamma}{\lambda} \left[ \frac{zD_n^{n+2}f(z)}{D_n f(z)} - 2z\left[\frac{D_n^{n+1}f(z)}{D_n f(z)}\right]^2 \right]. \]

Applying Lemma 2, the theorem follows.

**Theorem 11** Let \( q \) be convex univalent in \( U \). Let \( \gamma \in \mathbb{C} \) with \( \text{Re}\gamma > 0 \).
If \( f \in A, \lambda > 0 \), \( \frac{zD_n^{n+1}f(z)}{D_n f(z)} \in H[1, 1] \cap Q \),
\[ (1 + \gamma) \frac{zD_n^{n+1}f(z)}{D_n f(z)} + \frac{\gamma}{\lambda} \left[ \frac{zD_n^{n+2}f(z)}{D_n f(z)} - 2z\left[\frac{D_n^{n+1}f(z)}{D_n f(z)}\right]^2 \right], \]
is univalent in \( U \), and
\[ q(z) + \gamma q'(z) \prec (1 + \gamma) \frac{zD_n^{n+1}f(z)}{D_n f(z)} + \frac{\gamma}{\lambda} \left[ \frac{zD_n^{n+2}f(z)}{D_n f(z)} - 2z\left[\frac{D_n^{n+1}f(z)}{D_n f(z)}\right]^2 \right] \]
then
\[ q(z) \prec \frac{zD_n^{n+1}f(z)}{D_n f(z)} \]
and \( q \) is the best subordinant.

**Proof.** The proof follows by applying Lemma 4.

Combining Theorem 10 and Theorem 11, we get the following sandwich theorem.

**Theorem 12** Let \( \gamma \in \mathbb{C} \) with \( \text{Re}\gamma > 0 \), \( q_1 \) be convex univalent in \( U \) and \( q_2 \) be univalent in \( U \), \( q_2(0) = 1 \) and satisfies (3.1). If \( f \in A, \lambda > 0 \), \( \frac{zD_n^{n+1}f(z)}{D_n f(z)} \in H[1, 1] \cap Q \),
\[ (1 + \gamma) \frac{zD_n^{n+1}f(z)}{D_n f(z)} + \frac{\gamma}{\lambda} \left[ \frac{zD_n^{n+2}f(z)}{D_n f(z)} - 2z\left[\frac{D_n^{n+1}f(z)}{D_n f(z)}\right]^2 \right], \]
is univalent in \( U \), and
\[ q_1(z) + \gamma q_1'(z) \prec (1 + \gamma) \frac{zD_n^{n+1}f(z)}{D_n f(z)} + \frac{\gamma}{\lambda} \left[ \frac{zD_n^{n+2}f(z)}{D_n f(z)} - 2z\left[\frac{D_n^{n+1}f(z)}{D_n f(z)}\right]^2 \right], \]
\[ q_1(z) < \frac{zD_{\lambda}^{n+1}f(z)}{[D_{\lambda}^{n}f(z)]^2} < q_2(z) \]

and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

**Remark 2**

(i) Taking \( \lambda = 1 \) in Theorems 10, 11 and 12, we obtain the results obtained by Shanmugam et al. [15, Theorems 5.4, 5.5 and 5.6].

(ii) Taking \( \lambda = 1 \) and \( n = 0 \) in Theorems 10, 11 and 12, we obtain the results obtained by Shanmugam et al. [15, Theorems 3.4, 3.5 and Corollary 3.6].

**References**


**M. K. Aouf and A. O. Mostafa**

Mansoura University  
Faculty of Science  
Department of Mathematics  
Mansoura 35516, Egypt  
e-mail: mkaouf127@yahoo.com, adelaeg254@yahoo.com