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Uniqueness of solutions in non-Lipschitzian dynamics [1]

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Abstract

Results which ensure local and global uniqueness of solutions, for a class of autonomous dynamical systems with non-Lipschitzian right-hand side, are presented. Examples where the results can be applied are given.

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Key words and phrases: Non-Lipschitzian dynamics, uniqueness of solutions, dynamical systems.

1 Introduction

There are two fundamental issues concerning the solutions of ordinary differential equations. The first one is the existence and the second one is the uniqueness of a solution. For the case of an autonomous dynamical system of order n (i.e. $\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x_0}$ where $\mathbf{x} \in \mathbb{R}^n$ and

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$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$), the existence of a solution for the initial value problem is guaranteed by the continuity of the mapping $\mathbf{f}$. On the other hand, the uniqueness of a solution can not be ensured only by the continuity of $\mathbf{f}$. Extra conditions are needed. The most common of these conditions is the continuous differentiability of $\mathbf{f}$ ($\mathbf{f} \in C^1$) or at least the satisfaction of a Lipschitz condition for $\mathbf{f}$, (see e.g. [4], [5] or any usual text on differential equations).

In non-Lipschitzian dynamics a unique solution can not always be guaranteed. In [7], a dynamical system with non-Lipschitzian right-hand side which admits multiple solutions is presented. The author of [7] proves that non-Lipschitzian dynamics can exhibit a rather complex and unpredictable behavior because of the lack of a unique solution. In the present article we indicate that non-Lipschitzian dynamics can also exhibit a smooth dynamic behavior. We describe a class of dynamical systems with non-Lipschitzian right-hand side which, under certain conditions, admit a unique solution. It is also shown that this class includes the second-order (planar) Hamiltonian systems as a special case. Note that this special case was explicitly treated in [6]. A theorem which ensures local uniqueness and two corollaries for global uniqueness are proved and examples of third-order systems are given. First, let us present a well-known result that we will need later in the proof of Theorem 1.

For the special case of an one-dimensional autonomous dynamical system (i.e. $f : \mathbb{R} \to \mathbb{R}$), only the continuity of f is sufficient to ensure local uniqueness as long as the initial condition x_0 is not an equilibrium point, i.e. $f(x_0) \neq 0$ (see [1], p.36). Consequently, if f is continuous, multiple solutions may arise only when $f(x_0) = 0$. It is interesting to note that a generalization of the aforementioned result has been presented recently; see [3] and the references therein.

Now we are ready to present the structure of the dynamical system. We consider $n-1$ C^1 functions $f_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, \ldots, n-1$. For these functions we define the following $(n-1)\times(n-1)$ Jacobian matrices:

$$J_i(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_{n-1}}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_{i-1}} & \dots & \frac{\partial f_{n-1}}{\partial x_{i-1}} \\ \frac{\partial f_1}{\partial x_{i+1}} & \dots & \frac{\partial f_{n-1}}{\partial x_{i+1}} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \dots & \frac{\partial f_{n-1}}{\partial x_n} \end{pmatrix}, i = 1, \dots, n$$

where the derivatives of f_k, $k = 1, \dots, n-1$ with respect to x_i have been omitted. The dynamical system that we will consider throughout the article has the following form:

$$\dot{x}_j = (-1)^{j+1}|J_j(\mathbf{x})|, j = 1, \dots, n \tag{1}$$

where $|J_j(\mathbf{x})|$ denotes the determinant of $J_j(\mathbf{x})$. The initial condition for (1) is $\mathbf{x}(0) = \mathbf{x_0}$ (i.e. $x_j(0) = x_{j0}, j = 1, \dots, n$). From now on all the vectors are columns and $n > 1$. Note that the continuity of $|J_j(\mathbf{x})|, j = 1, \dots, n$ ensures the existence of a solution for (1). We do not claim that $|J_j(\mathbf{x})|, j = 1, \dots, n$ are continuously differentiable or Lipschitzian, since that claim would result the uniqueness of the solution. An important feature of the system is that along its solution the functions $f_k(\mathbf{x})$ remain constant, i.e.

$$\frac{df_k(\mathbf{x}(t))}{dt} = 0, k = 1, \dots, n-1.$$

We can see this fact easily if we consider the following $n \times n$ matrices:

$$R_k(\mathbf{x}) = (\triangledown f_k, \triangledown f_1, \triangledown f_2, \dots, \triangledown f_{n-1}), k = 1, \dots, n-1.$$

The matrix $R_k(\mathbf{x})$ is singular (two same columns) and so the expansion of its determinant along the first column is:

$$|R_k(\mathbf{x})| = \sum_{j=1}^{n} (-1)^{j+1} \frac{\partial f_k(\mathbf{x})}{\partial x_j} |J_j(\mathbf{x})| = \frac{df_k(\mathbf{x}(t))}{dt} = 0, k = 1, \dots, n-1.$$

Thus $f_k(\mathbf{x}(t)) = f_k(\mathbf{x_0}), k = 1, \ldots, n-1$ are considered to be constant energy functions on the solution of (1). It is apparent now that the dynamical system (1) for $n = 2$ becomes a Hamiltonian system with f_1 representing its energy function.

2 Main Results

In this section we present a theorem and two corollaries, which ensure local and global uniqueness of the solutions associated to (1). Our main theorem follows.

Theorem 1 *If there exists a nonsingular $J_k(\mathbf{x_0})$, then (1) admits a locally unique solution.*

Proof. Without loss of generality consider that $k = n$ (in fact we can change the order of the elements of $\mathbf{x}$ properly to construct a dynamical system similar to (1) that has $|J_n(\mathbf{x_0})| \neq 0$). First, we consider the $n-1$ C^1 functions $f_i(\mathbf{x}) - f_i(\mathbf{x_0})$, $i = 1, \ldots, n-1$. Since the Jacobian matrix $J_n(\mathbf{x_0})$ of $f_i(\mathbf{x})$ (or equivalent of $f_i(\mathbf{x}) - f_i(\mathbf{x_0})$, $i = 1, \ldots, n-1$) at $\mathbf{x_0}$ is nonsingular, we can apply the implicit function theorem (see e.g. [2]) for the functions $f_i(\mathbf{x}) - f_i(\mathbf{x_0}), i = 1, \ldots, n-1$ at $\mathbf{x_0}$. From that theorem we have that there are $n-1$ C^1 unique mappings θ_i defined on a neighborhood of $x_n(0) = x_{n0}$ say $|x_n - x_{n0}| < \epsilon$, such that $x_{i0} = \theta_i(x_{n0}), i = 1, \ldots, n-1$ and $f_i(\theta_1(x_n), \ldots, \theta_{n-1}(x_n), x_n) - f_i(\mathbf{x_0}) = 0, i = 1, \ldots, n-1$. We define the open region $M = \{\mathbf{x} \in \mathbb{R}^n : |x_n - x_{n0}| < \epsilon\}$ of $\mathbf{x_0}$, where the implicit function theorem holds.

Next, consider the functions $f_i(\mathbf{x}) - f_i(\mathbf{x_0}), i = 1, \ldots, n-1$, plus $f_n(\mathbf{x}) = x_n$. The $n \times n$ Jacobian matrix of these n functions at $\mathbf{x_0}$ is also nonsingular, i.e. $|J(\mathbf{x_0})| = |J_n(\mathbf{x_0})| \neq 0$. We know that $f_i(\mathbf{x}) - f_i(\mathbf{x_0})$, $i = 1, \ldots, n-1$ and $f_n(\mathbf{x}) = x_n$ are C^1. Applying the inverse function theorem (see [2]), we have that there is a neighborhood B of $\mathbf{x_0}$ such that for every $\mathbf{x} \in B$ the mapping $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}) - f_1(\mathbf{x_0}), ..., f_{n-1}(\mathbf{x}) -$

$f_{n-1}(\mathbf{x_0}), f_n(\mathbf{x}))^T$ is a diffeomorphism. We define the open region $A = M \cap B$ of $\mathbf{x_0}$, where both theorems hold.

Suppose that we have a point of the solution of (1) $\mathbf{x}(a)$, which belongs to A. For this point we have that:

1. $f_i(\mathbf{x}(a)) - f_i(\mathbf{x_0}) = 0, i = 1, \ldots, n-1$, from the fact that $df_i(\mathbf{x}(t))/dt = 0$ and $f_n(\mathbf{x}(a)) = x_n(a)$.

2. $f_i(\theta_1(x_n(a)), \ldots, \theta_{n-1}(x_n(a)), x_n(a)) - f_i(\mathbf{x_0}) = 0, i = 1, \ldots, n-1$, from the implicit function theorem and $f_n(\mathbf{x}(a)) = x_n(a)$.

Since $\mathbf{F}(\mathbf{x})$ is a diffeomorphism, it is also a bijectivity in A. Since the images are the same, we have that the corresponding points are also the same i.e. $x_i(a) = \theta_i(x_n(a)), i = 1, \ldots, n-1$. Because $\mathbf{x}(a)$ was arbitrary we conclude that for every $\mathbf{x}(t) \in A$, which of course belongs to the solution of (1), it holds that $x_i(t) = \theta_i(x_n(t)), i = 1, \ldots, n-1$.

Until now we have proved that every point of the solution of (1), that belongs to A, must satisfy $x_i(t) = \theta_i(x_n(t)), i = 1, \ldots, n-1$. Now we can construct the following initial value problem in one variable:

$$(2) \quad \dot{x}_n(t) = (-1)^{n+1} |J_n(\theta_1(x_n(t)), \ldots, \theta_{n-1}(x_n(t)), x_n(t))|, x_n(0) = x_{n0}.$$

From the fact that $|J_n(\mathbf{x_0})| \neq 0$ and the continuity of $|J_n(\mathbf{x})|$, we conclude that there is a neighborhood N of x_{n0} such that (2) has a unique solution $x_n(t)$ (recall the result from Section 1). The uniqueness for the rest $x_i(t)$ holds, since every variable of the solution in $A \cap N$ must satisfy $x_i(t) = \theta_i(x_n(t)), i = 1, \ldots, n-1$ and $x_n(t)$ is unique. Thus $\mathbf{x}(t)$ is unique in $A \cap N$ and the proof is complete.

Let us now claim that for every $\mathbf{x} \in \mathbb{R}^n$ there is a nonsingular $J_k(\mathbf{x})$. If this holds, we have that the solution $\mathbf{x}(t)$ is globally unique in $\mathbb{R}^n$ for every $\mathbf{x_0}$. Note that under this assumption, the dynamical system (1) does not have equilibrium points and vice versa. The following corollary states exactly that:

Corollary 1 *Suppose that (1) does not have equilibrium points. Then its solution is globally unique.*

At this point it is necessary to understand that the existence and the uniqueness theorems have a meaning only as long as the solution of (1) has a finite norm. This means that these results may be applied locally or globally in $\mathbb{R}^n$, while their application takes place on an open interval of time (maximal interval), where the solution remains finite. For instance in the case of finite escape time of a solution the maximal interval can be (a, b), where $b < \infty$.

The lack of equilibria in (1) is not the only case where the theorem can be applied globally. Suppose that there is an f_i of the $n-1$ C^1 functions whose value at $\mathbf{x_0}$ is different from the value at any equilibrium of (1). Then $f_i(\mathbf{x}(t)) = c$ guarantees that $\mathbf{x}(t)$, a point on the solution, will remain away from the equilibria of (1). Thus for every $\mathbf{x}(t)$ on the solution we have that there is a $|J_k(\mathbf{x}(t))| \neq 0$ and so the uniqueness theorem is applied globally. We briefly state that as:

Corollary 2 *Suppose that there is an f_i such that $f_i(\mathbf{x_0}) = c$ and $f_i(\mathbf{x}) \neq c$, for every $\mathbf{x}$ which is an equilibrium point of (1). Then the solution of (1) is globally unique.*

We stress that Corollary 2 can be applied only with respect to the initial condition $\mathbf{x_0}$. It is not as general as Corollary 1 because there might be initial points of (1) for which Corollary 2 cannot be applied. Nevertheless, as we will see in the examples its application is sometimes useful and effective to guarantee uniqueness of solution.

3 Examples

A third order dynamical system of the form (1) is considered. The functions $f_1(x, y, z)$ and $f_2(x, y, z)$ are C^1. Let us first assume that

$f_1(x,y,z) = x + z^2$ and $f_2(x,y,z) = x^2 + h(y) + z^2$, where $h(y) = (2/3)y^{3/2} + y$ for $y \geq 0$ and $h(y) = (-2/3)(-y)^{3/2} + y$ for $y < 0$. Note that $h(y)$ is a C^1 function of y while its second derivative does not exist for $y = 0$. Its first derivative is $h^{'}(y) = y^{1/2} + 1$ for $y \geq 0$ and $h^{'}(y) = (-y)^{1/2} + 1$ for $y < 0$, which is continuous. Now we have the system: $\dot{x} = -2zh^{'}(y)$, $\dot{y} = 4xz - 2z$, $\dot{z} = h^{'}(y)$. Since $|J_3(x,y,z)| = h^{'}(y) > 0$ from Corollary 1 we conclude that the system has a unique solution globally, even if we start from points where $y = 0$. We stress that $h^{'}(y)$ does not satisfy a local Lipschitz condition at $y = 0$ and of course it has discontinuous derivatives at $y = 0$. Thus we cannot conclude uniqueness of the solution from known results.

Let us next define $f_1(x,y,z) = k(x) + k(y) + z^2$ and $f_2(x,y,z) = y - x$, where $k(x) = (2/3)x^{3/2}$ for $x \geq 0$ and $k(x) = (-2/3)(-x)^{3/2}$ for $x < 0$. Again $k(x)$ is a C^1 function of x while its second derivative does not exist for $x = 0$. Its first derivative is $k^{'}(x) = x^{1/2}$ for $x \geq 0$ and $k^{'}(x) = (-x)^{1/2}$ for $x < 0$, which is continuous. We have the system: $\dot{x} = -2z$, $\dot{y} = -2z$, $\dot{z} = k^{'}(x) + k^{'}(y)$. The system has an equilibrium at $(0,0,0)$. If we start from a point $\mathbf{x_0} = (x_0, y_0, z_0)$ where $x_0 \neq y_0$ then $f_2(\mathbf{x_0}) = y_0 - x_0 \neq 0$. Since $f_2(0,0,0) = 0$ from Corollary 2 we have that the solution is unique. Note that for this case the Jacobian matrix $|J_3(x(t),y(t),z(t))| = k^{'}(x(t)) + k^{'}(y(t)) > 0$. Again we cannot imply global uniqueness from known results (although local uniqueness for this case is ensured), since $k^{'}(x) + k^{'}(y)$ does not satisfy a global Lipschitz condition and it has discontinuous derivatives at x=0 and y=0.

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Differential sandwich theorems for some analytic functions defined by certain linear operators [1]

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Abstract

In this investigation, we obtain some applications of first order differential subordination and superordination results involving the multiplier transformation and Al-Oboudi operator for certain normalized analytic functions in the open unit disc.

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Key words and phrases: Analytic functions, differential subordination , superordination, sandwich theorems, multiplier transformation, Al-Oboudi operator.

1 Introduction

Let $H(U)$ be the class of analytic functions in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, k]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} ... \ (a \in \mathbb{C}). \tag{1.1}$$

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Also, let A be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + a_2 z^2 + \dots . \tag{1.2}$$

If f, $g \in H(U)$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence, (cf., e.g.,[5], [10]; see also [11]):

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $p, h \in H(U)$ and let $\varphi(r, s, t; z) : C^3 \times U \to C$. If p and $\varphi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z), \tag{1.3}$$

then p is a solution of the differential superordination (1.3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant if $q(z) \prec p(z)$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants of (1.3) is called the best subordinant. Recently Miller and Mocanu [12] obtained conditions on the functions h, q and φ for which the following implication holds :

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [12], Bulboača [4] considered certain classes of first order differential superordinations as well as superordination preserving integral operators [6]. Ali et al. [1], have used the results of Bulboača [4] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U. Also, Tuneski [16] obtained a sufficient condition for starlikeness of f in terms of the quantity $\dfrac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [15] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = 1$ and $q_2(0) = 1$. They [15] also obtained results for functions defined by using Carlson-Shaffer operator and Salagean operator.

For any real number σ, Cho and Kim [7] defined the multiplier transformations I_δ^σ of functions $f \in A$ by

$$I_\delta^\sigma f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\delta}{1+\delta}\right)^\sigma a_k z^k \quad (\delta > -1). \tag{1.5}$$

Obviously, we observe that

$$I_\delta^\sigma (I_\delta^\gamma f(z)) = I_\delta^{\sigma+\gamma} f(z) \ (\sigma, \gamma \text{ real}),$$

$$I_\delta^0 f(z) = f(z),\ I_0^1 f(z) = zf'(z),\ I_0^2 f(z) = z(f'(z) + zf''(z))$$

and

$$z(I_\delta^\sigma f(z))' = (1+\delta) I_\delta^{\sigma+1} f(z) - \delta I_\delta^\sigma f(z). \tag{1.6}$$

For all real integer σ and $\delta = 1$, the operator I_δ^σ was studied by Uralegaddi and Somanatha [17], and for $\sigma = -1$ the operator I_δ^σ is the integral operator studied by Owa ana Srivastava [13]. Furthermore, for any negative real number σ and $\delta = 1$, the operator I_δ^σ is the multiplier transformation studied by Jung et al. [9] (see also [3]), and for any non-negative integer σ and $\delta = 0$, the operator I_δ^σ is the differential operator defined by Sălăgean [14]. In fact the operator I_δ^σ is related rather closely to the multiplier transformation studied by Flett [8].

For $f(z) \in A$, $\lambda > 0$ real number, Al-Oboudi [2] defined the differential operator $D_\lambda^n f(z)$ ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, ...\}$) as follows:

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= D_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z) \quad , \lambda > 0 , \\ D_\lambda^2 f(z) &= D_\lambda (D_\lambda f(z)), \end{aligned}$$

and

$$D_\lambda^n f(z) = D_\lambda(D_\lambda^{n-1} f(z)) \ (n \in \mathbb{N}; \lambda > 0).$$

When $\lambda = 1$ we get the Sălăgean differential operator [14]. It is easily seen that

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda (k-1)]^n a_k z^k \ (n \in \mathbb{N}_0; \lambda > 0). \tag{1.7}$$

Obviously, we observe that

$$\lambda z (D_\lambda^n f(z))' = D_\lambda^{n+1} f(z) - (1-\lambda) D_\lambda^n f(z) \ (\lambda > 0). \tag{1.8}$$

2 Definitions and preliminaries

In order to prove our results, we shall make use of the following known results.

Definition 1 [12]. *Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\xi \in \partial U : \lim_{z \to \xi} f(z) = \infty\},$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

Lemma 1 [11]. *Let q be univalent in the unit disk U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z). \tag{2.1}$$

Suppose that

(*i*) $\psi(z)$ *is starlike univalent in* U,
(*ii*) $\mathrm{Re}\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ *for* $z \in U$.
If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1, Shanmugam et al. [15] obtained the following lemma.

Lemma 2 [15]**.** *Let q be univalent in U with $q(0) = 1$. Let $\alpha \in \mathbb{C}, \gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, further assume that*

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \max\{0, -\mathrm{Re}(\alpha/\gamma)\}.$$

If p is analytic in U, and

$$\alpha p(z) + \gamma zp'(z) \prec \alpha q(z) + \gamma zq'(z),$$

then $p \prec q$ and q is the best dominant.

Lemma 3 [4]. *Let q be convex univalent in U and ϑ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(*i*) $\mathrm{Re}\{\vartheta'(q(z))/\phi(q(z))\} > 0$ *for* $z \in U$,
(*ii*) $\psi(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.

If $p(z) \in H[q(0),1] \cap Q$*, with* $p(U) \subseteq D$*, and* $\vartheta(p(z)) + zp'(z)\phi(p(z))$ *is univalent in* U *and*

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{2.3}$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 3, Shanmugam et al. [15] obtained the following lemma.

Lemma 4 [15]**.** *Let* q *be convex univalent in* U*,* $q(0) = 1$*. Let* $\alpha, \gamma \in \mathbb{C}$ *and* $\mathrm{Re}\{\alpha/\gamma\} > 0$*. If* $p \in H[q(0),1] \cap Q$*,* $\alpha p(z) + \gamma z p'(z)$ *is univalent in* U *and*

$$\alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z),$$

then $q \prec p$ *and* q *is the best subordinant.*

3 Applications to multiplier transformation and sandwich theorems

Theorem 1 *Let* q *be convex univalent in* U *with* $q(0) = 1$*,* $\gamma \in \mathbb{C}^*$*. Further, assume that*

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -\mathrm{Re}(1/\gamma)\}. \tag{3.1}$$

If $f \in A$*, and*

$$\frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} + \gamma(1+\delta)\left[1 - \frac{I_\delta^\sigma f(z) I_\delta^{\sigma+2} f(z)}{[I_\delta^{\sigma+1} f(z)]^2}\right] \prec q(z) + \gamma z q'(z), \tag{3.2}$$

then

$$\frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} \prec q(z)$$

and q *is the best dominant.*

Proof. Define a function p by

$$p(z) = \frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} \quad (z \in U). \tag{3.3}$$

Then the function p is analytic in U and $p(0) = 1$. Therefore, differentiating (3.3) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

$$\frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} + \gamma(1+\delta)\left[1 - \frac{I_\delta^\sigma f(z) I_\delta^{\sigma+2} f(z)}{[I_\delta^{\sigma+1} f(z)]^2}\right] = p(z) + \gamma z p'(z),$$

that is, that

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z)$$

and therefore, the theorem follows by applying Lemma 2.

Now, by appealing to Lemma 4, it can be easily prove the following theorem.

Theorem 2 *Let q be convex univalent in U. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f \in A$, $\frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} \in H[1,1] \cap Q$,*

$$\frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} + \gamma(1+\delta)\left[1 - \frac{I_\delta^\sigma f(z) I_\delta^{\sigma+2} f(z)}{[I_\delta^{\sigma+1} f(z)]^2}\right],$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} + \gamma(1+\delta)\left[1 - \frac{I_\delta^\sigma f(z) I_\delta^{\sigma+2} f(z)}{[I_\delta^{\sigma+1} f(z)]^2}\right],$$

then

$$q(z) \prec \frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)}$$

and q is the best subordinant.

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.

Theorem 3 *Let $\gamma \in \mathbb{C}$ with* $\operatorname{Re}\gamma > 0$, q_1 *be convex univalent in* U *and* q_2 *be univalent in* U, $q_2(0) = 1$ *and satisfies* (3.1). *If* $f \in A$, $0 \neq \frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} \in H[1,1] \cap Q$,

$$\frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} + \gamma(1+\delta)\left[1 - \frac{I_\delta^\sigma f(z) I_\delta^{\sigma+2} f(z)}{[I_\delta^{\sigma+1} f(z)]^2}\right]$$

is univalent in U, *and*

$$q_1(z) + \gamma z q_1'(z) \prec \frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} + \gamma(1+\delta)\left[1 - \frac{I_\delta^\sigma f(z) I_\delta^{\sigma+2} f(z)}{[I_\delta^{\sigma+1} f(z)]^2}\right] \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{I_\delta^\sigma f(z)}{I_\delta^{\sigma+1} f(z)} \prec q_2(z)$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant.*

Taking $\delta = 0$ in Theorems 1, 2 and 3, we have results improve that obtained by Shanmugam et al. [15, Theorems 5.1, 5.2 and 5.3, respectively].

Taking $\delta = \sigma = 0$ in Theorems 1, 2 and 3, we obtain the results obtained by Shanmugam et al. [15, Theorems 3.1, 3.2 and Corollary 3.3, respectively].

Theorem 4 *Let* q *be convex univalent in* U, $\gamma \in \mathbb{C}$, $\operatorname{Re}\{\gamma\} > 0$. *Further, assume that* (3.1) *holds. If* $f \in A$ *satisfies*

$$[1+\gamma(1+\delta)]\frac{z I_\delta^{\sigma+1} f(z)}{[I_\delta^\sigma f(z)]^2} + \gamma(1+\delta)\left[\frac{z I_\delta^{\sigma+2} f(z)}{[I_\delta^\sigma f(z)]^2}\right] - 2\frac{z[I_\delta^{\sigma+1} f(z)]^2}{[I_\delta^\sigma f(z)]^3} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{z I_\delta^{\sigma+1} f(z)}{[I_\delta^\sigma f(z)]^2} \prec q(z)$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) = \frac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2} \ \ (z \in U).$$

Then, simple computations show that

$$p(z)+\gamma zp'(z)=[1+\gamma(1+\delta)]\frac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2}+\gamma(1+\delta)\left[\frac{zI_\delta^{\sigma+2}f(z)}{[I_\delta^\sigma f(z)]^2}\right]-2\frac{z[I_\delta^{\sigma+1}f(z)]^2}{[I_\delta^\sigma f(z)]^3}.$$

Applying Lemma 2 , the theorem follows.

Theorem 5 *Let q be convex univalent in U. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If* $f \in A,\ \dfrac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2} \in H[1,1] \cap Q,$

$$[1+\gamma(1+\delta)]\frac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2}+\gamma(1+\delta)\left[\frac{zI_\delta^{\sigma+2}f(z)}{[I_\delta^\sigma f(z)]^2}\right]-2\frac{z[I_\delta^{\sigma+1}f(z)]^2}{[I_\delta^\sigma f(z)]^3}$$

is univalent in U, and

$$q(z)+\gamma zq'(z) \prec [1+\gamma(1+\delta)]\frac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2}+\gamma(1+\delta)\left[\frac{zI_\delta^{\sigma+2}f(z)}{[I_\delta^\sigma f(z)]^2}\right]-2\frac{z[I_\delta^{\sigma+1}f(z)]^2}{[I_\delta^\sigma f(z)]^3},$$

then

$$q(z) \prec \frac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2},$$

and q is the best subordinant.

Proof. The proof follows by applying Lemma 4.

Combining Theorem 4 and Theorem 5, we get the following sandwich theorem.

Theorem 6 *Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$, q_1 be convex univalent in U and q_2 be univalent in U, $q_2(0) = 1$ and satisfies* (3.1). *If $f \in A$,* $\dfrac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2} \in H[1,1] \cap Q,$

$$[1+\gamma(1+\delta)]\frac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2}+\gamma(1+\delta)\left[\frac{zI_\delta^{\sigma+2}f(z)}{[I_\delta^\sigma f(z)]^2}\right]-2\frac{z[I_\delta^{\sigma+1}f(z)]^2}{[I_\delta^\sigma f(z)]^3}$$

is univalent in U, *and*

$$q_1(z)+\gamma zq_1'(z) \prec [1+\gamma(1+\delta)]\frac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2}+\gamma(1+\delta)\left[\frac{zI_\delta^{\sigma+2}f(z)}{[I_\delta^\sigma f(z)]^2}\right]-2\frac{z[I_\delta^{\sigma+1}f(z)]^2}{[I_\delta^\sigma f(z)]^3},$$

$$\prec q_2(z)+\gamma zq_2'(z),$$

then

$$q_1(z) \prec \frac{zI_\delta^{\sigma+1}f(z)}{[I_\delta^\sigma f(z)]^2} \prec q_2(z)$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant.*

Taking $\delta = 0$ in Theorems 4, 5 and 6, we get the results obtained by Shanmugam et al. [15, Theorems 5.4, 5.5 and 5.6, respectively].

Taking $\sigma = \delta = 0$ in Theorems 4, 5 and 6, we get the results obtained by Shanmugam et al. [15, Theorems 3.4, 3.5 and Corollary 3.6, respectively].

4 Applications to Al-Oboudi operator and sandwich theorems

Theorem 7 *Let* q *be convex univalent in* U *with* $q(0) = 1$, $\gamma \in \mathbb{C}^*$. *Further, assume that the condition (3.1) holds. If* $f \in A, \lambda > 0$ *and*

$$\frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)}+\frac{\gamma}{\lambda}\left[1-\frac{D_\lambda^n f(z)D_\lambda^{n+2}f(z)}{[D_\lambda^{n+1}f(z)]^2}\right] \prec q(z)+\gamma zq'(z), \tag{4.1}$$

then

$$\frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)} \prec q(z)$$

and q *is the best dominant.*

Proof. Define a function p by

$$p(z) = \frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)} \quad (z \in U). \tag{4.2}$$

Then the function p is analytic in U and $p(0) = 1$. Therefore, differentiating (4.2) logarithmically with respect to z and using the identity (1.8) in the resulting equation, we have

$$\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} + \frac{\gamma}{\lambda}\left[1 - \frac{D_\lambda^n f(z) D_\lambda^{n+2} f(z)}{[D_\lambda^{n+1} f(z)]^2}\right] = p(z) + \gamma z p'(z) \ \ (\lambda > 0),$$

that is, that

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z)$$

and therefore, the theorem follows by applying Lemma 2.

Now, by appealing to Lemma 4, it can be easily prove the following theorem.

Theorem 8 *Let q be convex univalent in U. Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. If $f \in A, \lambda > 0$, $\dfrac{D_\lambda^n f(z)}{D_\lambda^n f(z)} \in H[1,1] \cap Q$,*

$$\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} + \frac{\gamma}{\lambda}\left[1 - \frac{D_\lambda^n f(z) D_\lambda^{n+2} f(z)}{[D_\lambda^{n+1} f(z)]^2}\right]$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} + \frac{\gamma}{\lambda}\left[1 - \frac{D_\lambda^n f(z) D_\lambda^{n+2} f(z)}{[D_\lambda^{n+1} f(z)]^2}\right],$$

then

$$q(z) \prec \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)}$$

and q is the best subordinant.

Combining Theorem 7 and Theorem 8, we get the following sandwich theorem.

Theorem 9 *Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$, q_1 be convex univalent in U and q_2 be univalent in U, $q_2(0) = 1$ and satisfies (3.1). If $f \in A, \lambda > 0$, $\dfrac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \in H[1,1] \cap Q$,*

$$\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} + \frac{\gamma}{\lambda}\left[1 - \frac{D_\lambda^n f(z) D_\lambda^{n+2} f(z)}{[D_\lambda^{n+1} f(z)]^2}\right]$$

is univalent in U*, and*

$$q_1(z)+\gamma zq_1'(z)\prec \frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)}+\frac{\gamma}{\lambda}\left[1-\frac{D_\lambda^n f(z)D_\lambda^{n+2}f(z)}{[D_\lambda^{n+1}f(z)]^2}\right]$$

$$\prec q_2(z)+\gamma zq_2'(z),$$

then

$$q_1(z)\prec \frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)}\prec q_2(z)$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant.*

Remark 1 (i) *Taking* $\lambda=1$ *in Theorems 7, 8 and 9, we have results improve that obtained by Shanmugam et al. [15, Theorems 5.1, 5.2 and 5.3, respectively].*

(ii) Taking $\lambda=1$ and $n=0$ in Theorems 7, 8 and 9, we obtain the results obtained by Shanmugam et al. [15, Theorems 3.1, 3.2 and Corollary 3.3, respectively].

Theorem 10 *Let* q *be convex univalent in* U, $\gamma\in\mathbb{C}^*$. *Further, assume that* (3.1) *holds. If* $f\in A, \lambda>0$ *satisfies*

$$(1+\frac{\gamma}{\lambda})\frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2}+\frac{\gamma}{\lambda}\left[\frac{zD_\lambda^{n+2}f(z)}{[D_\lambda^n f(z)]^2}-2\frac{z[D_\lambda^{n+1}f(z)]^2}{[D_\lambda^n f(z)]^3}\right]$$

$$\prec q(z)+\gamma zq'(z),$$

then

$$\frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2}\prec q(z)$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z)=\frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2}\quad (z\in U).$$

Then, simple computations show that

$$p(z)+\gamma zp'(z) = (1+\frac{\gamma}{\lambda})\frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2} + \frac{\gamma}{\lambda}\left[\frac{zD_\lambda^{n+2}f(z)}{[D_\lambda^n f(z)]^2} - 2\frac{z[D_\lambda^{n+1}f(z)]^2}{[D_\lambda^n f(z)]^3}\right].$$

Applying Lemma 2 , the theorem follows.

Theorem 11 *Let q be convex univalent in U. Let $\gamma \in \mathbb{C}$ with* $\mathrm{Re}\,\gamma > 0$. *If $f \in A, \lambda > 0,$ $\dfrac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2} \in H[1,1] \cap Q,$*

$$(1+\frac{\gamma}{\lambda})\frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2} + \frac{\gamma}{\lambda}\left[\frac{zD_\lambda^{n+2}f(z)}{[D_\lambda^n f(z)]^2} - 2\frac{z[D_\lambda^{n+1}f(z)]^2}{[D_\lambda^n f(z)]^3}\right],$$

is univalent in U, and

$$q(z)+\gamma zq'(z) \prec (1+\frac{\gamma}{\lambda})\frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2} + \frac{\gamma}{\lambda}\left[\frac{zD_\lambda^{n+2}f(z)}{[D_\lambda^n f(z)]^2} - 2\frac{z[D_\lambda^{n+1}f(z)]^2}{[D_\lambda^n f(z)]^3}\right]$$

then

$$q(z) \prec \frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2},$$

and q is the best subordinant.

Proof. The proof follows by applying Lemma 4.

Combining Theorem 10 and Theorem 11, we get the following sandwich theorem.

Theorem 12 *Let $\gamma \in \mathbb{C}$ with* $\mathrm{Re}\,\gamma > 0$, *q_1 be convex univalent in U and q_2 be univalent in U, $q_2(0) = 1$ and satisfies* (3.1). *If $f \in A, \lambda > 0,$ $\dfrac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2} \in H[1,1] \cap Q,$*

$$(1+\frac{\gamma}{\lambda})\frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2} + \frac{\gamma}{\lambda}\left[\frac{zD_\lambda^{n+2}f(z)}{[D_\lambda^n f(z)]^2} - 2\frac{z[D_\lambda^{n+1}f(z)]^2}{[D_\lambda^n f(z)]^3}\right]$$

is univalent in U, and

$$q_1(z)+\gamma zq_1'(z) \prec (1+\frac{\gamma}{\lambda})\frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2} + \frac{\gamma}{\lambda}\left[\frac{zD_\lambda^{n+2}f(z)}{[D_\lambda^n f(z)]^2} - 2\frac{z[D_\lambda^{n+1}f(z)]^2}{[D_\lambda^n f(z)]^3}\right],$$

$$\prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{zD_\lambda^{n+1}f(z)}{[D_\lambda^n f(z)]^2} \prec q_2(z)$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant.*

Remark 2 (i) *Taking* $\lambda = 1$ *in Theorems 10, 11 and 12, we obtain the results obtained by Shanmugam et al. [15, Theorems 5.4, 5.5 and 5.6].*

(ii) Taking $\lambda = 1$ and $n = 0$ in Theorems 10, 11 and 12, we obtain the results obtained by Shanmugam et al. [15, Theorems 3.4, 3.5 and Corollary 3.6].

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A strong form of strong θ-continuity [1]

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Abstract

A new class of function, called strongly faintly λ-continuous function, has been defined and studied. Relationships among strongly faintly λ- continuous functions and λ-connected spaces, λ-normal spaces and λ-compact spaces are investigated. Furthermore, the relationships between strongly faintly λ-continuous functions and graphs are also investigated.

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1 Introduction

Recent progress in the study of characterizations and generalizations of continuity, compactness, connectedness, separation axioms etc. has been done by means of several generalized closed sets. The first step of generalizing closed set was done by Levine in 1970 [12]. The notion of generalized

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closed sets has been studied extensively in recent years by many topologist because generalized closed sets are the only natural generalization of closed sets. More importantly, they also suggest several new properties of topological spaces. Maki [14] introduced the notion of Λ-sets in topological spaces. A subset A of a topological space (X, τ) is said to be a Λ-set if it coincides with its kernel (the intersection of all open supersets of A). As a generalization of closed sets, λ-closed sets were introduced and studied by Arenas et al. [1] and presented fundamental results for these sets. Nasef and Noiri [21] introduce three classes of strong forms of faintly continuity namely: strongly faint semicontinuity, strongly faint precontinuity and strongly faint β-continuity. recently Nasef [19] defined strongly forms of faint continuity under the terminologies strongly faint α-continuity and strongly faint γ-continuity. In this paper using λ-open, strongly faint λ-continuity is introduced and studied. Moreover, basic properties and preservation theorems of strongly faintly λ- continuous functions are investigated and relationships between strongly faintly λ-continuous functions and graphs are investigated.

2 Preliminaries

Throughout the paper (X, τ) and (Y, σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$, $int(A)$ and $X \backslash A$ denote the closure of A, the interior of A and the complement of A in X, respectively. A subset A is said to be θ-open [13] if for each $x \in X$ there exists an open set U such that $x \in U \subset cl(U) \subset A$. It follows from [25] that the collection of θ-open sets in a topological space (X, τ) forms a topology τ_θ for X. The complement of a θ-open is said to be θ-closed. A point $x \in X$ is called a θ-cluster point of A if $cl(V) \cap A \neq \emptyset$ for every open set V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $cl_\theta(A)$. Alternatively A

is θ-closed if $cl_\theta(A) = A$.

We recall the following definitions, which are well-known and are useful in the sequel.

Definition 1 *A subset A of a space (X, τ) is called semi-open [11] (resp. α-open [18], preopen [15], β-open [2], γ-open [8]) if $A \subset cl(Int(A))$ (resp. $A \subset int(cl(int(A)))$, $A \subset int(cl((A))$, $A \subset cl(int(cl(A)))$, $A \subset cl(int(A)) \cup int(cl(A))$).*

Let A be subset of X. Then A is said to be λ-closed [1] if $A = B \cap C$, where B is a Λ-set and C is a closed set. The complement of a λ-closed set is called a λ-open set. A point x in a topological space (X, τ) is called a λ-*cluster point* of A [5] if every λ-open set U of X containing x such that $A \cap U \neq \emptyset$. The set of all λ-cluster points is called the λ-*closure* of A and is denoted by $cl_\lambda(A)$ ([1, 5]).
A Point $x \in X$ is said to be a λ-interior point of A if there exists a λ-open set U containing x such that $U \subset A$. The set of all λ-interior points of A is said to be λ-interior of A and is denoted by $int_\lambda(A)$.

Lemma 1 *[1, 5]. Let A , B and A_i $(i \in I)$ be subsets of a topological space (X, τ). The following properties hold:*
(1) If A_i is λ-closed for each $i \in I$, then $\cap_{i\in I} A_i$ is λ-closed.
(2) If A_i is λ-open for each $i \in I$, then $\cup_{i\in I} A_i$ is λ-open.
(3) A is λ-closed if and only if $A = cl_\lambda(A)$.
(4) A is λ-open if and only if $A = int_\lambda(A)$.
(5) $cl_\lambda(A) = \cap\{F \in \lambda C(X, \tau) : A \subset F\}$.
(6) $A \subset cl_\lambda(A)$.
(7) If $A \subset B$, then $cl_\lambda(A) \subset cl_\lambda(B)$.
(8) $cl_\lambda(A)$ is λ-closed.
(9) $cl_\lambda(X \backslash A) = X \backslash int_\lambda(A)$.

We have the following diagram in which the converses of implications need not be true.

$$\begin{array}{ccccccc}
\lambda\text{-}open & \leftarrow & open & \leftarrow & \theta\text{-}open & \rightarrow & \beta\text{-}open \\
 & & \downarrow & \searrow & & & \uparrow \\
 & & preopen & \leftarrow & \alpha\text{-}open & \rightarrow & semiopen
\end{array}$$

Definition 2 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly faintly semicontinuous [21] (resp. strongly faintly precontinuous [20], strongly faintly β-continuous [21], strongly faintly α-continuous [19], strongly faintly γ-continuous [19]) if for each $x \in X$ and each semiopen (resp. preopen, β-open, α-open, γ-open) set V of Y containing $f(x)$, there exists a θ-open set U of X containing x such that $f(U) \subset V$.*

Definition 3 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:*
(i) quasi θ-continuous [10], if $f^{-1}(V)$ is θ-open in X for every θ-open set V of Y.
(ii) λ-irresolute [5], if $f^{-1}(V)$ is λ-open in X for every λ-open set V of Y.
(iii) λ-continuous [1], if $f^{-1}(V)$ is λ-open in X for every open set V of Y.
(iv) strongly θ-continuous [23], if $f^{-1}(V)$ is θ-open in X for every open set V of Y.

3 Strongly faintly λ-continuous functions

Definition 4 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly faintly λ-continuous if for each $x \in X$ and each λ-open set V of Y containing $f(x)$, there exists a θ-open set U of X containing x such that $f(U) \subset V$.*

Theorem 1 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
(i) f is strongly faintly λ-continuous;
(ii) $f : (X, \tau_\theta) \to (Y, \sigma)$ is strongly λ-continuous;
(iii) $f^{-1}(V)$ is θ-open in X for every λ-open set V of Y;
(iv) $f^{-1}(F)$ is θ-closed in X for every λ-closed subset F of Y.

Proof. (i)⇒(iii): Let V be an λ-open set of Y and $x \in f^{-1}(V)$. Since $f(x) \in V$ and f is strongly faintly λ-continuous, there exists a θ-open set U of X containing x such that $f(U) \subset V$. It follows that $x \in U \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is θ-open in X.
(iii)⇒(i): Let $x \in X$ and V be an λ-open set of Y containing $f(x)$. By (iii), $f^{-1}(V)$ is a θ-open set containing x. Take $U = f^{-1}(V)$. Then $f(U) \subset V$. This shows that f is strongly faintly λ-continuous.
(iii)⇒(iv): Let V be any λ-closed set of Y. Since $Y \setminus V$ is an λ-open set, by (iii), it follows that $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is θ-open. This shows that $f^{-1}(V)$ is θ-closed in X.
(iv)⇒(iii): Let V be an λ-open set of Y. Then $Y \setminus V$ is λ-closed in Y. By (iv), $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is θ-closed and thus $f^{-1}(V)$ is θ-open.
(i)⇔(ii): Clear.

The relationships between this new class of functions and other corresponding types of functions are shown in the following diagram.

s.θ-continuous ← *s.faintlyλ-continuous*
↖
↓ *s.faintlyα-continuous*
↗ ↖
s.faintly precontinuous *s.faintly semicontinuous*
↖ ↗
s.faintlyγ-continuous
↑
s.faintlyβ-continuous

However, none of these implications is reversible as shown by the following examples and well-known facts.

Example 1 *(i) In ([21], Examples 3.2) is showed a strong faint semi-continuity which is not a strong faint precontinuity.*

(ii) In ([19], Examples 4.3, (resp. Examples: 4.4 and 4.5)) is showed a strong faint semicontinuity which is not a strong faint γ-continuity resp. a strong faint precontinuity which is not a strong faint γ-continuity and a strong faint γ-continuity which is not a strong faint β-continuity).
(iii) Using Example 3.2 of [21], this is easily observed that a strongly faint α-continuity need not be strongly faint λ-continuity.

A function $f : (X, \tau) \to (Y, \sigma)$ is called strongly λ-continuous if $f^{-1}(V)$ is open in X for every λ-open set V of Y.

Theorem 2 *If a function $f : (X, \tau) \to (Y, \sigma)$ is strongly faintly λ-continuous function, then it is strongly λ-continuous.*

If (X, τ) is a regular space, we have $\tau = \tau_\theta$ and the next theorem follows immediately.

Theorem 3 *Let (X, τ) be a regular space. Then for a function $f : (X, \tau) \to (Y, \sigma)$ the following properties are equivalent:*
(i) f is strongly λ-continuous.
(ii) f is strongly faintly λ-continuous.

Recall that, a topological space (Y, σ) is said to be locally indiscrete if every open subset of (Y, σ) is closed.

Lemma 2 *[4] A space (Y, σ) is locally indiscrete if and only if every λ-open set of (Y, σ) is open.*

A space X is said to be submaximal if each dense subset of X is open in X and extremaly disconnected (ED for short) if the closure of each open set of X is open in X.

Theorem 4 *Let (Y, σ) be a submaximal ED, locally indiscrete space. Then the following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*
(i) f is strongly faintly α-continuous,
(ii) f is strongly faintly γ-continuous,

(iii) f is strongly faintly semicontinuous.
(iv) f is strongly faintly precontinuous.
(v) f is strongly faintly β-continuous.
(vi) f is strongly θ-continuous.
(vii) f is strongly faintly λ-continuous.

Proof. By [21] we have $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$ (since that X is submaximal and DE then $\sigma = \sigma^\alpha = \gamma O(Y,\sigma) = SO(Y,\sigma) = PO(Y,\sigma) = \beta O(Y,\sigma)$).
$(vi) \Leftrightarrow (vii)$: This follows from the fact that if (Y,σ) is locally indiscrete then $\sigma = \lambda O(Y,\sigma)$.

Theorem 5 *If $f : X \to Y$ is strongly faintly λ-continuous and $g : Y \to Z$ is λ-irresolute, then $g \circ f : X \to Z$ is strongly faintly λ-continuous.*

Proof. Let $G \in \lambda O(Z)$. Then $g^{-1}(G) \in \lambda O(Y)$ and hence $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$ is θ-open in X. Therefore $g \circ f$ is strongly faintly λ-continuous.

Theorem 6 *The following statements hold for functions $f : X \to Y$ and $g : Y \to Z$:*
(i) If both f and g are strongly faintly δ-almost continuous, then the composition $g \circ f : X \to Z$ is strongly faintly δ-almost continuous.
(ii) If f strongly faintly λ-continuous and g is a λ-continuous, then $g \circ f$ is strongly λ-continuous.
(iii) If f strongly faintly λ-continuous and g is a λ-continuous, then $g \circ f$ is strongly θ-continuous.
(iv) If f is quasi θ-continuous and g is strongly faintly λ-continuous, then $g \circ f$ is strongly faintly λ-continuous.
(v) If f is strongly θ-continuous and g is strongly faintly λ-continuous, then $g \circ f$ is strongly faintly λ-continuous.

Definition 5 *A θ-frontier of a subset A of (X,τ) is $Fr_\theta(A) = cl_\theta(A) \cap cl_\theta(X \backslash A)$.*

Theorem 7 *Let (X,τ) be a regular space. Then the set of all points $x \in X$ in which a function $f : (X,\tau) \to (Y,\sigma)$ is not strongly faintly λ-continuous at x is identical with the union of the θ-frontier of the inverse images of λ-open subsets of Y containing $f(x)$.*

Proof. Necessity. Suppose that f is not strongly faintly λ-continuous at $x \in X$. Then there exists a λ-open set V of Y containing $f(x)$ such that $f(U)$ is not a subset of V for each $U \in \tau_\theta$ containing x. Hence we have $U \cap (X \backslash f^{-1}(V)) \neq \emptyset$ for each $U \in \tau_\theta$ containing x. Since X is regular, it follows that $x \in cl_\theta(X \backslash f^{-1}(V))$. On the other hand we have that, $x \in f^{-1}(V) \subset cl_\theta(f^{-1}(V))$. This means that $x \in Fr_\theta(f^{-1}(V))$.
Sufficiency. Suppose that $x \in Fr_\theta(f^{-1}(V))$ for some $V \in \lambda O(Y, f(x))$. Now, we assume that f is strongly faintly λ-continuous at $x \in X$. Then there exists $U \in \tau_\theta$ containing x such that $f(U) \subset V$. Therefore, we have $U \subset f^{-1}(V)$ and hence $x \in int_\theta(f^{-1}(V)) \subset X \backslash Fr_\theta(f^{-1}(V))$. This is a contradiction. This means that f is not strongly faintly λ-continuous.

Definition 6 *(i) A space (X,τ) is said to be λ-connected [3] (resp. θ-connected) if X cannot be written as the union of two nonempty disjoint λ-open (resp. θ-open) sets.*
(ii) A subset K of a (X,τ) space is said to be, λ-compact [3] (resp. θ-compact [10]) relative to (X,τ), if for every cover of K by λ-open (resp. θ-open) has a finite subcover. A topological space (X,τ) is λ-compact (resp. θ-compact) if the set X is λ-compact (resp. θ-compact) relative to (X,τ).

Theorem 8 *If $f : (X,\tau) \to (Y,\sigma)$ is a strongly faintly λ-continuous surjection function and (X,τ) is a θ-connected space, then Y is a λ-connected space.*

Proof. Assume that (Y,σ) is not λ-connected. Then there exist nonempty λ-open sets V_1 and V_2 of Y such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since

f is surjective, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty subsets of X. Since f is strongly faintly λ-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are θ-open sets of X. Therefore (X, τ) is not θ-connected. This is a contradiction and hence (Y, σ) is λ-connected.

Theorem 9 *If $f : (X, \tau) \to (Y, \sigma)$ is a strongly faintly λ-continuous, then $f(K)$ is λ-compact relative to (Y, σ) for each subset K which is θ-compact relative to (X, τ).*

Proof. Let $\{V_i : i \in I\}$ be any cover of cover of $f(K)$ by λ-open sets. For each $x \in K$, there exists $i_x \in I$, such that $f(x) \in V_{i_x}$. Since f is strongly faintly λ-continuous, there exists $U_x \in \tau_\theta$ containing x such that $f(U_x) \subset V_{i_x}$. The family $\{U_x) : x \in K\}$ is a cover of K by θ sets of (X, τ). Since K is θ-compact relative to (X, τ), there exists a finite subset K_0 of K such that $K \subset \bigcup\{U_x : x \in K_0\}$. Therefore, we obtain $f(K) \subset \bigcup\{f(U_x) : x \in K_0\} \subset \bigcup\{V_{i_x} : x \in K_0\}$. Therefore, $f(K)$ is λ-compact relative to (Y, σ).

Theorem 10 *The surjective strongly faintly λ-continuous image of a θ-compact space is λ-compact.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a strongly faintly λ-continuous function from a θ-compact space X onto a space Y. Let $\{G_\alpha$: $\alpha \in I\}$ be any λ-open cover of Y. Since f is strongly faintly λ-continuous, $\{f^{-1}(G_\alpha)$: $\alpha \in I\}$ is a θ-open cover of X. Since X is θ-compact, there exists a finite subcover $\{f^{-1}(G_i) : i = 1, 2, ... n\}$ of X. Then it follows that $\{G_i$: $i = 1, 2, ... n\}$ is a finite subfamily which cover Y. Hence Y is λ-compact.

4 Separation Axioms

Recall, that a topological space (X, τ) is said to be:
(i) λ-T_1 [5] (resp. θ-T_1) if for each pair of distinct points x and y of X, there exists λ-open (resp. θ-open) sets U and V containing x and y,

respectively such that $y \notin U$ and $x \notin V$.
(ii) λ-T_2 [5] (resp. θ-T_2 [24]) if for each pair of distinct points x and y in X, there exists disjoint λ-eopen (resp. θ-open) sets U and V in X such that $x \in U$ and $y \in V$.

Remark 1 *(i) [6] Hausdorff* $\Leftrightarrow \theta$-T_1.
(ii) [9] $T_0 \Leftrightarrow \lambda$-$T_1 \Leftrightarrow \lambda$-$T_2$.

Theorem 11 *If* $f : (X, \tau) \to (Y, \sigma)$ *is strongly faintly* λ*-continuous injection and* Y *is a* T_0 *space, then* X *is a* θ-T_1 *(or Hausdorff) space.*

Proof. Suppose that Y is T_0. For any distinct points x and y in X, then by Remark 1(ii) there exist $V, W \in \lambda O(Y)$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is strongly faintly λ-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are θ-open subsets of (X, τ) such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is θ-T_1 (equivalently Hausdorff by Remark 1(i)).

Theorem 12 *If* $f : (X, \tau) \to (Y, \sigma)$ *is strongly faintly* λ*-continuous injection and* Y *is a* T_0 *space, then* X *is a* θ-T_2 *space.*

Proof. Suppose that Y is T_0. For any pair of distinct points x and y in X, there exist disjoint λ-open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$ (Remark 1(ii)). Since f is strongly faintly λ-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are θ-open in X containing x and y, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ because $U \cap V = \emptyset$. This shows that X is θ-T_2.

Theorem 13 *If* $f, g : X \to Y$ *are strongly faintly* λ*-continuous functions and* Y *is* T_0*, then* $E = \{x \in X\colon f(x) = g(x)\}$ *is closed in* X.

Proof. Suppose that $x \notin E$. Then $f(x) \neq g(x)$. Since Y is T_0 and by Remark 1(ii), there exist $V \in \lambda O(Y, f(x))$ and $W \in \lambda O(Y, g(x))$ such that $V \cap W = \emptyset$. Since f and g are strongly faintly λ-continuous, there exist a θ-open U set of X containing x and a θ-open G set of X containing x such that $f(U) \subset V$ and $g(G) \subset W$. Set $D = U \cap G$. then $D \cap E = \emptyset$ with D a subset θ-open hence open such that $x \in D$. Then $x \notin cl(E)$ and thus E is closed in X.

Definition 7 *A space (X, τ) is said to be: (i) θ-regular (resp. λ-regular) if for each θ-closed (resp. λ-closed) set F and each point $x \notin F$, there exist disjoint θ-open (resp. λ-open) sets U and V such that $F \subset U$ and $x \in V$.*
(ii) θ-normal (resp. λ-normal) if for any pair of disjoint θ-closed (resp. λ-closed) subsets F_1 and F_2 of X, there exist disjoint θ-open (resp. λ-open) sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Recall that, a function $f : (X, \tau) \to (Y, \sigma)$ is called $\theta\lambda$-open if $f(V) \in \lambda O(Y)$ for each $V \in \tau_\theta$.

Theorem 14 *If f is strongly faintly λ-continuous $\theta\lambda$-open injective function from a θ-regular space (X, τ) onto a space (Y, σ), then (Y, σ) is λ-regular.*

Proof. Let F be an λ-closed subset of Y and $y \notin F$. Take $y = f(x)$. Since f is strongly faintly λ-continuous, $f^{-1}(F)$ is θ-closed in X such that $f^{-1}(y) = x \notin f^{-1}(F)$. Take $G = f^{-1}(F)$. We have $x \notin G$. Since X is θ-regular, then there exist disjoint θ-open sets U and V in X such that $G \subset U$ and $x \in V$. We obtain that $F = f(G) \subset f(U)$ and $y = f(x) \in f(U)$ such that $f(U)$ and $f(V)$ are disjoint λ-open sets. This shows that Y is λ-regular.

Theorem 15 *If f is strongly faintly λ-continuous $\theta\lambda$-open injective function from a θ-normal space (X, τ) onto a space (Y, σ), then Y is λ-normal.*

Proof. Let F_1 and F_2 be disjoint λ-closed subsets of Y. Since f is strongly faintly λ-continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are θ-closed sets.

Take $U = f^{-1}(F_1)$ and $V = f^{-1}(F_2)$. We have $U \cap V = \emptyset$. Since X is θ-normal, there exist disjoint θ-open sets A and B such that $U \subset A$ and $V \subset B$. We obtain that $F_1 = f(U) \subset f(A)$ and $F_2 = f(V) \subset f(B)$ such that $f(A)$ and $f(B)$ are disjoint λ-open sets. Thus, Y is λ-normal.

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 8 *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be (θ, λ)-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist a θ-open U set of X containing x and λ-open V set of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 3 *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is (θ, λ)-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a θ-open U set of X containing x and λ-open V set of Y containing y such that $f(U) \cap V = \emptyset$.*

Proof. It is an immediate consequence of Definition 8.

Theorem 16 *If $f : (X, \tau) \to (Y, \sigma)$ is strongly faintly λ-continuous function and (Y, σ) is T_0, then $G(f)$ is (θ, λ)-closed.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$, then $f(x) \neq y$. Since Y is T_0, then by Remark 1(ii) there exist λ-open sets V and W in Y such that $f(x) \in V$, $y \in W$ and $V \cap W = \emptyset$. Since f is strongly faintly λ-continuous, $f^{-1}(V)$ is θ-open in X containing x . Take $U = f^{-1}(V)$. We have $f(U) \subset V$. Therefore, we obtain $f(U) \cap W = \emptyset$. This shows that $G(f)$ is (θ, λ)-closed.

Theorem 17 *Let $f : (X, \tau) \to (Y, \sigma)$ has (θ, λ)-closed graph $G(f)$. If f is a strongly faintly λ-continuous injection, then (X, τ) is θ-T_2.*

Proof. Let x and y be any two distinct points of X. Then since f is injective, we have $f(x) \neq f(y)$. Then, we have $(x, f(y)) \in (X \times Y)\backslash G(f)$. By Lemma 3, there exist a θ-open U set of X and λ-open V set of Y such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \emptyset$. Hence $U \cap f^{-1}(V) = \emptyset$ and $y \notin U$. Since f is strongly faintly λ-continuous, there exists a θ-open W set of X containing y such that $f(W) \subset V$. Therefore, we have $f(U) \cap f(W) = \emptyset$. Since f is injective, we obtain $U \cap W = \emptyset$. This implies that (X, τ) is θ-T_2.

Remark 2 *We recall that the space X is called a λ-space [1] if the set of all λ-open subsets form a topology on X. Clearly a space X is a λ- space if and only if the intersection of two λ-open sets is λ-open. A concrete example of a λ-space is a $T_{\frac{1}{2}}$-space, where a space X is called $T_{\frac{1}{2}}$ if every singleton is open or closed [7].*

Theorem 18 *Let (Y, σ) be a λ-space. If $f : (X, \tau) \to (Y, \sigma)$ has the (θ, λ)-closed graph, then $f(K)$ is λ-closed in (Y, σ) for each subset K which is θ-compact relative to X.*

Proof. Suppose that $y \notin f(K)$. Then $(x, y) \notin G(f)$ for each $x \in K$. Since $G(f)$ is (θ, λ)-closed, there exist a θ-open U_x set of X containing x and λ-open V_x set of Y containing y such that $f(U_x) \cap V_x = \emptyset$. by Lemma 4.7. The family $\{U_x : x \in K\}$ is a cover of K by θ-open sets. Since K is θ-compact relative to (X, τ), there exists a finite subset K_0 of K such that $K \subset \bigcup\{U_x : x \in K_0\}$. Set $V = \bigcap\{V_x : x \in K_0\}$. Then V is a λ-open set in Y containing y. Therefore, we have $f(K) \cap V \subset [\bigcup_{x \in K_0} f(U_x)] \cap V \subset \bigcup_{x \in K_0}[f(U_x) \cap V] = \emptyset$. It follows that $y \notin cl_\lambda(f(K))$. Therefore, $f(K)$ is λ-closed in (Y, σ).

Corollary 1 *Let (Y, σ) be a λ-space. If $f : (X, \tau) \to (Y, \sigma)$ is strongly faintly λ-continuous and (Y, σ) is T_0, then $f(K)$ is λ-closed in (Y, σ) for each subset K which is θ-compact relative to (X, τ).*

Proof. The proof follows from Theorems 16 and 18.

5 Conclusion.

Maps have always been of tremendous importance in all branches of mathematics and the whole science. On the other hand, topology plays a significant role in quantium physics, high energy physics and superstring theory [16, 17]. Thus we have obtained a new class of mappings called strongly faintly λ-continuous which may have possible application in quantium physics, high energy physics and superstring theory.

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Subordination results of certain analytic functions [1]

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Abstract

In this paper, we obtain some subordination results for two integral operators defined in the open unit disk.

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1 Introduction and definitions

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. For the functions f and g in $\mathcal{A}$, we say that f is subordinate to g in $\mathcal{U}$, and write $f \prec g$, if there exists a Schwarz function w in $\mathcal{U}$ with $|w(z)| < 1$ and $w(0) = 0$ such that $f(z) = g(w(z))$ in $\mathcal{U}$ (see [9]).

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Breaz and Breaz [4] and Breaz *et al.* [8] introduced and studied the integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1}$$

and

$$F_{\alpha_1,\dots,\alpha_n}(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \dots \left(f_n'(t)\right)^{\alpha_n} dt \tag{2}$$

where $f_i \in \mathcal{A}$ and $\alpha_i > 0$, for all $i = 1, \dots, n\,, n \in \mathbb{N}$ (see also [1, 2, 3, 5, 7, 12, 13]).

Breaz and Güney [6] considered the above integral operators and they obtained their properties on the classes $\mathcal{S}^*_\alpha(b)$ and $\mathcal{C}_\alpha(b)$ of starlike and convex functions of complex order b and type α introduced and studied by Frasin [10].

Recently, Frasin [11] obtained some sufficient conditions for the above integral operators to be in the classes $\mathcal{S}^*$, $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$, where $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, close -to-convex of order $\alpha (0 \le \alpha < 1)$ in $\mathcal{U}$ and uniformly convex functions.

In the present paper, we obtain some subordination results of the above integral operators $F_n(z)$ and $F_{\alpha_1,\dots,\alpha_n}(z)$.

In order to derive our main results, we have to recall here the following results:

Lemma 1 *([14]) If $f \in \mathcal{A}$ satisfies*

$$Re\left\{\frac{zf''(z)}{f'(z)}\right\} < \frac{\beta - 1}{2\delta(\beta + 1)} \qquad (z \in \mathcal{U}) \tag{3}$$

for some $\beta > 1$ and $\delta > 0$, then

$$\left(f'(z)\right)^\delta \prec \frac{\beta(1 - z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{4}$$

Lemma 2 *([14]) If $f \in \mathcal{A}$ satisfies*

$$Re\left\{\frac{zf''(z)}{f'(z)}\right\} > \frac{1-\beta}{2\delta(\beta+1)} \qquad (z \in \mathcal{U}) \tag{5}$$

for some $\beta > 1$ and $\delta > 0$, then

$$\left(\frac{1}{f'(z)}\right)^{\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{6}$$

2 Subordination results

We begin by proving the following theorem.

Theorem 1 *Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If each $f_i \in \mathcal{A}$ $\{i = 1, \ldots, n\}$ satisfies*

$$Re\left(\frac{zf_i'(z)}{f_i(z)}\right) < 1 + \frac{\beta - 1}{2\delta(\beta+1)\sum\limits_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}) \tag{7}$$

for some $\beta > 1$ and $\delta > 0$, then

$$\left(\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\right)^{\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{8}$$

Proof. It follows from (1) that

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z}\right)^{\alpha_n}.$$

Thus we have

$$F_n''(z) = \left[\alpha_1\left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \ldots + \alpha_n\left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right)\right] F_n'(z) \tag{9}$$

or, equivalently,

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{10}$$

Taking the real part of both terms of (10), we have

$$\begin{aligned}\mathrm{Re}\left(\frac{zF_n''(z)}{F_n'(z)}\right) &= \sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)-\sum_{i=1}^{n}\alpha_i\\ &= \alpha_1\mathrm{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right)+\alpha_2\mathrm{Re}\left(\frac{zf_2'(z)}{f_2(z)}\right)+\cdots\\ &\quad +\alpha_n\mathrm{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right)-\alpha_1-\alpha_2-\cdots-\alpha_n. \end{aligned} \tag{11}$$

Making use of the hypothesis (7), we obtain

$$\begin{aligned}&\mathrm{Re}\left(\frac{zF_n''(z)}{F_n'(z)}\right)\\ <\ &\alpha_1\left(1+\frac{\beta-1}{2\delta(\beta+1)[\alpha_1+\cdots+\alpha_n]}\right)+\alpha_2\left(1+\frac{\beta-1}{2\delta(\beta+1)[\alpha_1+\cdots+\alpha_n]}\right)\\ &+\cdots+\alpha_n\left(1+\frac{\beta-1}{2\delta(\beta+1)[\alpha_1+\cdots+\alpha_n]}\right)-\alpha_1-\alpha_2-\cdots-\alpha_n\\ <\ &\frac{\beta-1}{2\delta(\beta+1)}.\end{aligned}$$

Applying Lemma 1, we have

$$(F_n'(z))^{\delta}\prec\frac{\beta(1-z)}{\beta-z}\qquad(z\in\mathcal{U})$$

or, equivalently,

$$\left(\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\right)^{\delta}\prec\frac{\beta(1-z)}{\beta-z}\qquad(z\in\mathcal{U}).$$

This completes the proof.

Letting $n=1$, $\alpha_1=\alpha$ and $f_1=f$ in Theorem 1, we have

Corollary 1 *Let* $\alpha>0$. *If* $f\in\mathcal{A}$ *satisfies*

$$Re\left(\frac{zf'(z)}{f(z)}\right)<1+\frac{\beta-1}{2\delta\alpha(\beta+1)}\qquad(z\in\mathcal{U}) \tag{12}$$

for some $\beta > 1$ *and* $\delta > 0$, *then*

$$\left(\frac{f(z)}{z}\right)^{\alpha\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{13}$$

Next, we prove

Theorem 2 *Let* $\alpha_i > 0$ *be real numbers for all* $i = 1, \ldots, n$. *If*

$$\sum_{i=1}^{n} \alpha_i > n + \frac{1-\beta}{2\delta(\beta+1)} \tag{14}$$

for some $\beta > 1$ *and* $\delta > 0$, *and each* $f_i \in \mathcal{A}$, $\{i = 1, \ldots, n\}$ *satisfies*

$$Re\left(\frac{zf_i'(z)}{f_i(z)}\right) < \frac{1}{\alpha_i} \qquad (z \in \mathcal{U}) \tag{15}$$

then

$$\left(\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\right)^{\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{16}$$

Proof. From (11) and using (14), (15) , we get

$$\begin{aligned}
\mathrm{Re}\left(\frac{zF_n''(z)}{F_n'(z)}\right) &= \sum_{i=1}^{n} \alpha_i \mathrm{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) - \sum_{i=1}^{n} \alpha_i \\
&= \alpha_1 \mathrm{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right) + \alpha_2 \mathrm{Re}\left(\frac{zf_2'(z)}{f_2(z)}\right) + \cdots \\
&\quad + \alpha_n \mathrm{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) - \sum_{i=1}^{n} \alpha_i \\
&= \alpha_1\left(\frac{1}{\alpha_1}\right) + \alpha_2\left(\frac{1}{\alpha_2}\right) + \cdots + \alpha_n\left(\frac{1}{\alpha_n}\right) - \sum_{i=1}^{n} \alpha_i \\
&= n - \sum_{i=1}^{n} \alpha_i < \frac{\beta - 1}{2\delta(\beta+1)}
\end{aligned}$$

The result of Theorem 2, now follows by applying Lemma 1.

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2, we have

Corollary 2 *Let* $\beta > 1$ *and* $\delta > 0$ *with*

$$0 < 1 + \frac{1-\beta}{2\delta(\beta+1)} < \alpha. \tag{17}$$

If $f \in \mathcal{A}$ *satisfies*

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{1}{\alpha} \qquad (z \in \mathcal{U}) \tag{18}$$

then

$$\left(\frac{f(z)}{z}\right)^{\alpha\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{19}$$

Next, applying Lemma 2, we obtain the following two results.

Theorem 3 *Let* $\alpha_i > 0$ *be real numbers for all* $i = 1, \dots, n$. *If* $f_i \in \mathcal{A}$ *for all* $i = 1, \dots, n$ *satisfies*

$$Re\left(\frac{zf_i'(z)}{f_i(z)}\right) > 1 + \frac{1-\beta}{2\delta(\beta+1)\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}), \tag{20}$$

for some $\beta > 1$ *and* $\delta > 0$, *then*

$$\left(\left[\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\right]^{-1}\right)^{\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{21}$$

Proof. Employing the same manner as in the proof of Theorem 1, from (11), (20) and applying Lemma 2, it can be easily established the subordination (21).

Theorem 4 *Let* $\alpha_i > 0$ *be real numbers for all* $i = 1, \dots, n$. *If*

$$\sum_{i=1}^{n}\alpha_i < n + \frac{\beta - 1}{2\delta(\beta+1)} \tag{22}$$

for some $\beta > 1$ *and* $\delta > 0$, *and each* $f_i \in \mathcal{A}, \{i = 1, \dots, n\}$ *satisfies*

$$Re\left(\frac{zf_i'(z)}{f_i(z)}\right) > \frac{1}{\alpha_i} \qquad (z \in \mathcal{U}) \tag{23}$$

then

$$\left(\left[\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\alpha_i}\right]^{-1}\right)^{\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{24}$$

Proof. Employing the same manner as in the proof of Theorem 2, from (22), (23) and applying Lemma 2, it can be easily established the subordination (24).

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3, we have

Corollary 3 *Let* $\alpha > 0$. *If* $f \in \mathcal{A}$ *satisfies*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 1 + \frac{1-\beta}{2\alpha\delta(\beta+1)} \qquad (z \in \mathcal{U}), \tag{25}$$

for some $\beta > 1$ *and* $\delta > 0$, *then*

$$\left(\frac{z}{f(z)}\right)^{\alpha\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{26}$$

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 4, we have

Corollary 4 *Let* $\beta > 1$ *and* $\delta > 0$ *with*

$$0 < \alpha < 1 + \frac{\beta - 1}{2\delta(\beta+1)}. \tag{27}$$

If $f \in \mathcal{A}$ *satisfies*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{\alpha} \qquad (z \in \mathcal{U}) \tag{28}$$

then

$$\left(\frac{z}{f(z)}\right)^{\alpha\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{29}$$

Now we prove

Theorem 5 *Let $\alpha_i > 0$ be real numbers for all $i = 1, \dots, n$. If $f_i \in \mathcal{A}$ for all $i = 1, \dots, n$ satisfies*

$$\text{Re}\left(\frac{zf_i''(z)}{f_i'(z)}\right) < \frac{\beta - 1}{2\delta(\beta+1)\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}) \tag{30}$$

for some $\beta > 1$ and $\delta > 0$, then

$$\left(\prod_{i=1}^{n}\left(f_i'(z)\right)^{\alpha_i}\right)^{\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{31}$$

Proof. From (2) we easily get

$$\frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}\right). \tag{32}$$

Thus we have

$$\begin{aligned}\text{Re}\left(\frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)}\right) &= \sum_{i=1}^{n}\alpha_i\text{Re}\left(\frac{zf_i''(z)}{f_i'(z)}\right)\\ &= \alpha_1\text{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right) + \alpha_2\text{Re}\left(\frac{zf_2'(z)}{f_2(z)}\right) + \cdots\\ &\quad +\alpha_n\text{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right).\end{aligned}$$

From (30), it follows that

$$\begin{aligned}&\text{Re}\left(\frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)}\right) < \alpha_1\left(\frac{\beta-1}{2\delta(\beta+1)[\alpha_1+\cdots+\alpha_n]}\right)\\ &+\alpha_2\left(\frac{\beta-1}{2\delta(\beta+1)[\alpha_1+\cdots+\alpha_n]}\right) + \cdots + \alpha_n\left(\frac{\beta-1}{2\delta(\beta+1)[\alpha_1+\cdots+\alpha_n]}\right)\\ &< \frac{\beta-1}{2\delta(\beta+1)}\end{aligned}$$

for all $z \in \mathcal{U}$. Therefore, from Lemma 1, we obtain (31).

Applying Lemma 2, the proof of the next theorem below is much akin to that of Theorem 5 and so we omit for details involved.

Theorem 6 *Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$ satisfies*

$$Re\left(\frac{zf_i''(z)}{f_i'(z)}\right) > \frac{1-\beta}{2\delta(\beta+1)\sum\limits_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}) \tag{33}$$

for some $\beta > 1$ and $\delta > 0$, then

$$\left(\left[\prod_{i=1}^{n}\left(f_i'(z)\right)^{\alpha_i}\right]^{-1}\right)^{\delta} \prec \frac{\beta(1-z)}{\beta - z} \qquad (z \in \mathcal{U}). \tag{34}$$

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A note on a diophantine equation [1]

József Sándor

Abstract

We prove, without using Catalan's equation that, the only solution in positive integers of the equation $5^a - 2^b = 1$ is $a = 1$, $b = 2$. This shows a completely elementary method of solution of an equation from [1].

2010 Mathematics Subject Classification: 11D61.
Key words and phrases: diophantine equations; congruences.

Introduction

1. In paper [1] it is shown that all solutions to the equation

$$2^x + 5^y = z^2 \tag{1}$$

in nonnegative integers are provided by $x = 3$, $y = 0$, $z = 3$ and $x = 2$, $y = 1$, $z = 3$.

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When $x \geq 1$, $y \geq 1$, in the proof offered in [1] we are led to the following equation

$$5^y - 2^{k+1} = 1. \tag{2}$$

Then, the author uses the strong conjecture of Catalan, and proved recently by P. Mihăilescu ([2]), that the only solution to the equation

$$a^b - c^d = 1 \tag{3}$$

in positive integers≥ 2 is offered by $a = 3$, $b = 2$, $c = 2$, $d = 3$. Since $k+1 \geq 2$, clearly, we cannot have $y \geq 2$. When $y = 1$, however, we get $k = 1$.

In what follows, we shall prove elementarily this fact (i.e., without using the theory of equation (3)).

2. The proof First we prove that $k+1$ is even. If $k+1 = b$ is odd, then $2^b + 1$ is divisible by $2+1 = 3$, which is impossible, as 3 doesn't divide 5^y. Put $b = 2B$. If $B = 1$, then we are done, as then $y = 1$, etc. Let $B > 1$. Then we get the equation

$$5^y - 4^B = 1. \tag{4}$$

If $y = 2A$ is even, then $5^{2A} - 1 = 4^B$, and as $5^{2A} - 1$ is divisible by $5^2 - 1 = 24$, which is divisible by 3, we get a contradiction, as 4^B cannot be divisible by 3. Thus y is odd; put $y = 2A+1$. If $A = 0$, then $y = 1$; so we may suppose $A \geq 1$. Then (4) implies $5 \cdot 5^{2A} = 4^B + 1$.

As $5 = 8 - 3$, $5^{2A} = 25^A = (8 \cdot 3 + 1)^A \equiv 1(\text{mod } 8)$, we get that

$$5 \cdot 5^{2A} \equiv -3(\text{ mod } 8).$$

On the other hand, if $B \geq 2$, clearly $4^B + 1 \equiv 1(\text{mod } 8)$. As

$$-3(\text{ mod } 8) \not\equiv 1(\text{ mod } 8),$$

the contradiction follows.

When $B = 1$ we get $5^{2A} = 1$, which is impossible, since we have assumed $A \geq 1$.

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Centralizers of the maximal tori of the classical group $SO(2n, \mathbb{R})$ [1]

Vadoud Najjari

Abstract

The main endeavor in this paper is to characterize the maximal tori of the classical group $SO(2n, \mathbb{R})$. Then we calculate the centralizers of maximal tori in this group, which may be of interest in its own right.

2010 Mathematics Subject Classification: 11F23, 11F99 , 53D22.

Key words and phrases: Centralizer, Maximal torus.

1 Introduction

Brocker and Tom Dieck [2], Lerman and Tolman [4] have characterized the finite subgroups of centralizers of tori in the real symplectic group $Sp(2n, \mathbb{R})$ and then Schmah [7], characterized centralizers of all tori in $Sp(2n, \mathbb{R})$. In this paper we are able to calculate the centralizers of maximal tori of the classical group $SO(2n, \mathbb{R})$, which may be of interest in

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its own right. We encourage readers to see the related useful manuscripts [1, 5, 8, 9].

We review maximal tori in a Lie group and also several useful Lemma and Theorem (see also [3, 6]). Let G be a Lie group. A torus T is maximal if $T \subset U \subset G$ and U is a torus then $T = U$.

Theorem 1 *Any torus is contained in a maximal torus.*

Proof. Consider an increasing sequence of tori:

$$T \subset T_1 \subset T_2 \subset ... \subset G \tag{1}$$

then we must show that this sequence is finite. Take Lie algebra of this chain to get,

$$\mathrm{T}(T) \subset \mathrm{T}(T_1) \subset \mathrm{T}(T_2) \subset ... \subset \mathrm{T}(G). \tag{2}$$

Since this is an increasing sequence of finite dimensional vector spaces, therefor, it must be a finite sequence.

Lemma 1 *If G is connected, then any element of G is conjugate to an element in T.*

Proof. See [3].

Lemma 2 *In a connected group, any two maximal tori are conjugate.*

Proof. See [3].

2 Spacial Case of The Centralizers

In this section we investigate maximal tori of the classical groups $SO(2n, \mathbb{R})$ and $SO(2n + 1, \mathbb{R})$.

Proposition 1 *The maximal torus of $SO(2n+1,\mathbb{R})$ is,*

$$\mathbf{T} = \begin{pmatrix} \begin{pmatrix} cos\theta_1 & -sin\theta_1 \\ sin\theta_1 & cos\theta_1 \end{pmatrix} & & 0 & \\ & \ddots & & \\ 0 & & \begin{pmatrix} cos\theta_n & -sin\theta_n \\ sin\theta_n & cos\theta_n \end{pmatrix} & \\ & & & 1 \end{pmatrix}. \tag{3}$$

Proof. It is clear that $\mathbf{T}$ is a torus and so it only remains to see that it is maximal. Let $A \in SO(2n+1,\mathbb{R})$ which which commutes with $\mathbf{T}$ and let V_i be the subspace of $\mathbb{R}^{2n+1}$ spanned by e_{2i-1} and e_{2i}. Then $\mathbf{T}$ leaves each V_i invariant and there are subgroups $\mathbf{T}_i$ of $\mathbf{T}$ which leave V_i fixed. Notice that the whole of $\mathbf{T}$ leaves e_{2i+1} fixed. Now, we calculate with $t \in \mathbf{T}$:

$$tAe_{2n+1} = Ate_{2n+1} = Ae_{2n+1} \tag{4}$$

since λe_{2n+1} is the only vector fixed by the whole of $\mathbf{T}$ we have $Ae_{2n+1} = \lambda e_{2n+1}$. Since $A \in SO(2n+1,\mathbb{R})$ its only real eigenvalues are ± 1 and so $Ae_{2n+1} = \pm e_{2n+1}$. Now, let $t_i \in \mathbf{T}_i$ and $v_i \in V_i$. Then,

$$t_iAv_i = At_iv_i = Av_i. \tag{5}$$

Thus Av_i is fixed by t_i and $Av_i \neq \lambda e_{2n+1}$. Hence $Av_i \in V_i$ that each space V_i is invariant under A. Since $A \in SO(2n+1,\mathbb{R})$ this means that $A \in \mathbf{T}$.

Proposition 2 *The maximal torus of $SO(2n,\mathbb{R})$ is,*

$$\begin{pmatrix} \begin{pmatrix} cos\theta_1 & -sin\theta_1 \\ sin\theta_1 & cos\theta_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} cos\theta_n & -sin\theta_n \\ sin\theta_n & cos\theta_n \end{pmatrix} \end{pmatrix} \tag{6}$$

where $\theta_1, \theta_2, ..., \theta_n \in \mathbb{R}$

Proof. Consider the injection $SO(2n, \mathbb{R}) \to SO(2n+1, \mathbb{R})$ given by,

$$A \to \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{7}$$

then T is the inverse image of the maximal torus of $SO(2n+1, \mathbb{R})$ and so it is a maximal torus.

As it is shown by Schmah [7], all tori in $Sp(2n, \mathbb{R})$ are conjugate to one contained in the following diagonal representation of T^n:

$$\mathbb{T}^n := \begin{pmatrix} \begin{pmatrix} cos\theta_1 & sin\theta_1 \\ -sin\theta_1 & cos\theta_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} cos\theta_n & sin\theta_n \\ -sin\theta_n & cos\theta_n \end{pmatrix} \end{pmatrix} \tag{8}$$

where $\theta_1, \theta_2, ..., \theta_n \in \mathbb{R}$, then $\mathbb{T}^n$ and the maximal tori of the $SO(2n, \mathbb{R})$ are the same. Thus the centralizers of them are equal. As Schmah [7], we consider the standard $\mathbb{T}^n$ to be subgroups of $\mathbb{C}^n$ in the usual way,

$$T^n = \{(z_1, z_2, ..., z_n) \in \mathbb{C}^n : |z_i| = 1, \forall i\}. \tag{9}$$

We denote by exp the map from $\mathbb{R}^n$ to T^n given by $t \to exp^{2\pi it}$ (component-wise exponentiation). Let the map diag:$\mathbb{C}^n \longrightarrow Mat(n, \mathbb{C})$ by

$$\text{diag}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix}. \tag{10}$$

We will identify $Mat(n, \mathbb{C})$ with its representation in $Mat(2n, \mathbb{R})$ induced by,

$$a + ib \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{11}$$

All homomorphisms from T^k to T^n are of the form $exp(t) \longmapsto exp(Mt)$ for some $n \times k$ matrix M with integer entries. For all such matrices M, we define

$$\begin{aligned} \varphi_M : T^k &\longrightarrow \mathbb{T}^n \\ exp(t) &\longmapsto \text{diag}(exp(Mt)) \end{aligned} \tag{12}$$

and let $\Phi_M = Im(\varphi_M)$. All tori in the $\mathbb{T}^n$ are of this form. Note that Φ_M is a K-torus if and only if M has rank k (though φ_M need not be faithful). If all of the PM-blocks are nonzero, then

$$Z(\Phi_M) = W(p_1, q_1) \times_l ... \times_l W(p_r, q_r) \tag{13}$$

By the definition of PM-block form (See [7]), the only block that can be zero is the last one, M_r. Suppose that M_r is zero, then $\Phi_{M_r} = \{I_{2n}\}$. So $Z(\Phi_{M_r}) = SO(2n_r, \mathbb{R})$, and hence,

$$Z(\Phi_M) = W(p_1, q_1) \times_l ... \times_l W(p_{r-1}, q_{r-1}) \times_l SO(2n_r, \mathbb{R}) \tag{14}$$

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About some integral inequalities using Riemann-Liouville integrals [1]

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Abstract

In this paper, the Riemann-Liouville fractional integral is used to establish some integral inequalities.

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Key words and phrases: Integral inequalities, Riemann-Liouville fractional integral.

1 Introduction

In recent years, inequalities are playing a very significant role in al fields of mathematics, and present a very active and attractive field of research. As example, let us cite the field of integration which is dominated by inequalities involving functions and their integrals [6, 9]. One of the famous integral inequalities is (see [1]):

(1) $$T(f, g) \geq 0,$$

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where

(2)
$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx$$

and f and g are two integrable functions which are synchronous on $[a,b]$ $\Big($i.e. $(f(x)-f(y))(g(x)-g(y)) \geq 0$, for any $x,y \in [a,b]\Big)$.
The functional (2) has received a great deal of attention from mathematicians and a number of inequalities have appeared in the literature, see [3, 5, 7, 8].
The purpose of this paper is :
$1-$ to give another (short) proof to Theorems $(3.1, 3.2)$ [2] which is easy than the previous proof of these theorems and
$2-$ to establish some new integral inequalities involving functions and their fractional integrals.

2 Basic Definitions

In the following, we will give some basic definitions and properties. For more details, one can consult [4, 10].

Definition 1 *A real valued function $f : [0,\infty[\to \mathbb{R}$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1 \in C([0,\infty[)$.*

Definition 2 *A function $f : [0,\infty[\to \mathbb{R}$ is said to be in the space $C_\mu^n, n \in \mathbb{R}$, if $f^{(n)} \in C_\mu$.*

Definition 3 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as*

(3)
$$J^\alpha f(x) = \tfrac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt; \quad \alpha > 0, x > 0,$$
$$J^0 f(x) = f(x),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^{\alpha}J^{\beta}f(x) = J^{\alpha+\beta}f(x), \alpha \geq 0, \beta \geq 0, \tag{4}$$

and the commutative property

$$J^{\alpha}J^{\beta}f(x) = J^{\beta}J^{\alpha}f(x). \tag{5}$$

For the expression (3), when $f(x) = x^{\mu}$ we get another expression that will be used later:

$$J^{\alpha}x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}x^{\alpha+\mu}, \alpha > 0; \quad \mu > -1, x > 0. \tag{6}$$

3 Main Results

Our first result is the following theorem for which we will propose another " short" proof.

Theorem 1 *[2] Let f and g be two synchronous functions on $[0,\infty[$. Then for all $t > 0, \alpha > 0$, we have:*

$$J^{\alpha}(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}}J^{\alpha}f(t)J^{\alpha}g(t). \tag{7}$$

Proof. Since the functions f and g are synchronous on $[0,\infty[$, then for all $\tau \geq 0, \rho \geq 0$, we have

$$\Big(f(\tau) - f(\rho)\Big)\Big(g(\tau) - g(\rho)\Big) \geq 0. \tag{8}$$

Therefore

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{9}$$

Multiplying both sides of (9) by $\frac{\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}}{\Gamma^2(\alpha)}$, $\tau\in(0,t)$, $\rho\in(0,t)$, we obtain:

$$\frac{\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}}{\Gamma^2(\alpha)}f(\tau)g(\tau)+\frac{\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}}{\Gamma^2(\alpha)}f(\rho)g(\rho)$$

(10)

$$\geq\frac{\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}}{\Gamma^2(\alpha)}f(\tau)g(\rho)+\frac{\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}}{\Gamma^2(\alpha)}f(\rho)g(\tau).$$

Integrating (10) over $(0,t)^2$, we obtain:

$$\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}f(\tau)g(\tau)d\tau d\rho+$$

$$\tfrac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}f(\rho)g(\rho)d\tau d\rho\geq$$

(11)

$$\tfrac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}f(\tau)g(\rho)d\tau d\rho+$$

$$\tfrac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\Big((t-\tau)(t-\rho)\Big)^{\alpha-1}f(\rho)g(\tau)d\tau d\rho.$$

Then we can write

$$J^\alpha(fg)(t)\int_0^t\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}d\rho+\frac{J^\alpha(1)}{\Gamma(\alpha)}\int_0^t f(\rho)g(\rho)(t-\rho)^{\alpha-1}d\rho\geq$$

(12)

$$\tfrac{J^\alpha f(t)}{\Gamma(\alpha)}\int_0^t(t-\rho)^{\alpha-1}g(\rho)d\rho+\tfrac{J^\alpha g(t)}{\Gamma(\alpha)}\int_0^t(t-\rho)^{\alpha-1}f(\rho)d\rho.$$

Hence

$$J^\alpha(fg)(t)\geq\frac{1}{J^\alpha(1)}J^\alpha f(t)J^\alpha g(t). \tag{13}$$

Theorem 3.1 is thus proved.

The second result for which we give another proof is:

Theorem 2 *[2] Let f and g be two synchronous functions on $[0,\infty[$. Then for all $t>0$, $\alpha>0$, $\beta>0$, we have:*

(14)
$$\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(fg)(t)+\frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(fg)(t)\geq J^{\alpha}f(t)J^{\beta}g(t)+J^{\beta}f(t)J^{\alpha}g(t).$$

Proof. To prove this theorem, we multiply both sides of (9) by $\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$; $\tau,\rho\in(0,t)$.
We have

$$\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}f(\tau)g(\tau)+\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}f(\rho)g(\rho)\geq \tag{15}$$

$$\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}f(\tau)g(\rho)+\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}f(\rho)g(\tau).$$

By integration of (15) over $(0,t)^2$, we obtain the inequality (14).
The third result is the following theorem:

Theorem 3 *Let $p\geq 1, q\geq 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, if $|f|^p(t)$ and $|g|^q(t); t>0$ are two functions in $C_\mu, (\mu\geq -1)$, then for $\alpha>0$, the fractional integral inequality*

$$J^{\alpha}|fg|(t)\leq\Big(J^{\alpha}|f|^{p}(t)\Big)^{1/p}\Big(J^{\alpha}|g|^{q}(t)\Big)^{1/q} \tag{16}$$

is valid.

Proof. To prove Theorem 3.3 we consider the functions:
$F(\tau):=(t-\tau)^{(\alpha-1)/p}f(\tau),\ G(\tau):=(t-\tau)^{(\alpha-1)/q}g(\tau);\ \tau\in(0,t)$,
then we apply Holder inequality. We obtain:

(17)
$$\int_0^t|fg(\tau)(t-\tau)^{\alpha-1}|d\tau\leq\Big(\int_0^t|f|^p(\tau)(t-\tau)^{(\alpha-1)}d\tau\Big)^{1/p}\Big(\int_0^t|g|^q(\tau)(-\tau)^{(\alpha-1)}d\tau\Big)^{1/q}.$$

Now, multiplying both sides of (17) by $\Gamma^{-1}(\alpha)$ and using the fact that $\frac{1}{p}+\frac{1}{q}=1$, we obtain the inequality (16).

Corollary 1 *Let f be a positive and continuous function on $[0,\infty[$, then we have*

$$J^\alpha(f^n)(t) \geq \Big(\frac{1}{J^\alpha(1)}\Big)^{(n-1)}\Big(J^\alpha f(t)\Big)^n; \;\; n \geq 2. \tag{18}$$

Proof. We prove (18) by induction.
Our second corollary is the following

Corollary 2 (i) *Let f be a continuous function on $[0,\infty[$. Then for all $t>0$, $\alpha>0$, $\beta>0$, the inequality:*

$$J^\alpha(1)J^\beta(f^2)(t) + J^\beta(1)J^\alpha(f^2)(t) \geq 2J^\alpha f(t)J^\beta f(t) \tag{19}$$

is valid.
(ii) *Let f be a positive and differentiable function on $[0,\infty[$ such that $f'>0$. Then for all $t>0$, $\alpha>0$, $\beta>0$, we have:*

$$\begin{gathered} J^\alpha(1)J^\beta(f^3)(t) + J^\beta(1)J^\alpha(f^3)(t) \geq \\ (J^\beta(1))^{-1}J^\alpha f(t)(J^\beta f(t))^2 + (J^\alpha(1))^{-1}J^\beta f(t)(J^\alpha f(t))^2. \end{gathered} \tag{20}$$

Proof. (i) Let f be a continuous function on $[0,\infty[$. We have

$$\Big(f(\tau)-f(\rho)\Big)^2 \geq 0. \tag{21}$$

We multiply both sides of (21) by $\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$; $\tau\in(0,t)$, $\rho\in(0,t)$, then we integrate the resulting inequality over $(0,t)^2$, we obtain (19).

(ii) To prove (20), we remark that the conditions $f>0, f'>0$ imply that the functions f and $g := f^2$ are synchronous on $[0,\infty[$, hence by application of Theorem 3.2, we obtain the desired inequality.

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Relationship between a variant of essential spectrum and the Kato essential spectrum [1]

Mohammed Benharrat, Bekkai Messirdi

Abstract

J. P. Labrousse [8] studied and characterized in the case of Hilbert spaces, a relation between the Kato essential spectrum (essential quasi-Fredholm spectrum) and another essential spectrum defined by $\sigma_{ec}(T) = \{\lambda \in \mathbb{C}; R(\lambda I - T) \text{ is not closed}\}$ (see [3]). In this paper, we investigate this relation in the case of Banach spaces.

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1 Introduction

Let X be complex Banach space and let T be closed, densely defined linear operator on X. We denote by $\mathcal{D}(T) \subset X$ its domain, $R(T)$ its

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range, and $N(T)$ its null space. We denote by $\mathcal{C}(X)$ the set of all closed, densely defined linear operators. Let I denote the identity operator in X. T is called a Kato type operator if we can write $T = T_1 \oplus T_0$ where T_0 is a nilpotent operator and T_1 is a semi-regular one. In 1958, Kato proved that a closed semi-Fredholm operator is of Kato type, J.P Labrousse [7] studied and characterized a new class of operators named quasi-Fredholm operators, in the case of Hilbert spaces, and he prove that this class coincide with the set of Kato type operators. But in the case of Banach spaces, the Kato type operator is also quasi-Fredholm, the inverse is not true. The study of such a class of operators give a new definition of essential spectrum called the Kato essential spectrum which is the set of all complex λ such that $\lambda I - T$ is not of Kato type.

The main question motivated by J.P Labrousse [8], in the Hilbert spaces, is the relationship between the Kato essential spectrum and essential spectrum containing all the complex numbers λ such that $R(\lambda I - T)$ is not closed, noted $\sigma_{ec}(T)$ (see [3]), he proved that the symmetric difference between them is at most countable. In this paper, we continue the investigation of this relation in the case of Banach spaces, and we prove the same results given by J.P Labrousse. The poof of this property is now made difficult, because we lose in our case the Hilbertian structure, it requires technical methods of gap theory.

Our paper is organized as follows:

In sction 2, we give some preliminary results in which our investigation will be done.

In section 3, we extended in the Theorem 3 to the Banach space, the result concerning the essential spectrum of the operator T proved by J.P Labrousse [8] in the case of Hilbert spaces.

In section 4, we present the relationship between the Kato essential spectrum and another essential spectrum defined above.

Finally, in section 5 we apply the results obtained in section 4 to study the symmetric difference between the two essential spectrum for two classes of operators, first class is the class of operators which satisfy a polynomial growth condition, and the second is the class of symmetrizable operators, class of operators defined by P. D. Lax in [10]. We shows that the Kato essential spectrum of symmetrizable operators is contained in $\sigma_{ec}(T)$.

2 Preliminary Results

Let T be a closed, densely defined linear operator acting in complex Banach space X. The next result exhibits some useful connections between the null spaces and the ranges of the iterates T^n of a linear operator T on a vector space X.

Lemma 1 *For a linear operator T on a vector space X the following statements are equivalent:*

1. $N(T) \subseteq R(T^m)$, *for all* $m \geq 0$.
2. $N(T^n) \subseteq R(T)$, *for all* $n \geq 0$.
3. $N(T^n) \subseteq R(T^m)$, *for all* $n, m \geq 0$.
4. $N(T^n) = T^m(N(T^{n+m}))$, *for all* $n, m \geq 0$.

Definition 1 (see [13]) *Let $T \in \mathcal{C}(X)$, T is said to be semi-regular if $R(T)$ is closed and T verifies one of the equivalent conditions of lemma 1.*

Definition 2 *An operator $T \in \mathcal{C}(X)$, is said to be of Kato type of order d, if there exists $d \in \mathbb{N}$ and a pair of closed subspaces (M, N) of X such that :*

1. $X = M \oplus N$.
2. $T(M \cap D(T)) \subset M$ *and* $T_{/M}$ *is semi-regular.*

3. $N \subset D(T)$, $T(N) \subset N$ and $(T_{/N})^d = 0$ (i.e $T_{/N}$ is nilpotent).

An operator T is said to be of Kato type if T is a Kato type of order d, for some $d \in \mathbb{N}$.

Clearly, every semi-regular operator is Kato type with $M = X$ and $N = \{0\}$ and a nilpotent operator has a decomposition with $M = \{0\}$ and $N = X$.

An operator is said to be *essentialy semi-regular* if it admits a decomposition (M, N) such that N is finite-dimensional vector space. Note that if T is essentially semi-regular then $T_{/N}$ is nilpotent and T is of Kato type.

Theorem 1 ([1]) *Let $T \in \mathcal{C}(X)$, and assume that T is of Kato type of order d Then:*

1. *$M \cap N(T) = R(T^n) \cap N(T) = R(T^d) \cap N(T)$ for every $n \in \mathbb{N}$, $n \geq d$.*
2. *$R(T) + N(T^n) = T(M) \oplus N$ for every natural $n \geq d$.*
   Moreover $R(T) + N(T^n)$ is closed in X.

Note that by results of J.P Labrousse [7], in the case of Hilbert spaces, the set of quasi-Fredholm operators (operators verifying conditions (1) and (2) of Theorem 1) coincides with the set of Kato type operators. But in the case of Banach spaces the Kato type operator is also quasi-Fredholm, according to the remark following Theorem 3.2.2 in [7] the converse is true when $R(T^d) \cap N(T)$ and $R(T) + N(T^d)$ are complemented in the Banach space X.

Theorem 2 ([1]) *Suppose that $T \in \mathcal{C}(X)$, is of Kato type. Then there exists an open disc $\mathbb{D}(0, \epsilon)$ for which $\lambda I - T$ is semi-regular for all $\lambda \in \mathbb{D}(0, \epsilon) \setminus \{0\}$.*

Definition 3 *Let $T \in \mathcal{C}(X)$, the Kato essential spectrum is defined by*

$$\sigma_{ek}(T) = \{\lambda \in \mathbb{C}; \lambda I - T \text{ is not of Kato type}\}.$$

Note that the set of all Kato type operators is open by Theorem 2 (see, for example, [6], [1]), consequently the Kato essential spectrum is a closed set of the spectrum $\sigma(T)$ of T.

The reduced minimum modulus of a non-zero operator T is defined by

$$\gamma(T) = \inf_{x \notin N(T)} \frac{\|Tx\|}{\operatorname{dist}(x, N(T))}.$$

Where $\operatorname{dist}(x, N(T)) = \inf_{y \in N(T)} \|x - y\|$. If $T = 0$ then we take $\gamma(T) = \infty$. Note that (see [6]):

$$\gamma(T) > 0 \Leftrightarrow R(T) \text{ is closed.}$$

Let M, N be two closed linear subspaces of the Banach space X and set

$$\delta(M, N) = \sup\{\operatorname{dist}(x, N) : x \in M, \|x\| = 1\},$$

in the case that $M \neq \{0\}$, otherwise we define $\delta(\{0\}, N) = 0$ for any subspace N.

The gap between M and N is defined by

$$\widehat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}.$$

$\widehat{\delta}$ is a metric on the set $\mathcal{F}(X)$ of all linear closed subspaces of X, and the convergence $M_n \longrightarrow M$ in $\mathcal{F}(X)$ is obviously defined by $\widehat{\delta}(M_n, M) \longrightarrow 0$ as $n \longrightarrow \infty$ in $\mathbb{R}$. Moreover, $(\mathcal{F}(X), \widehat{\delta})$ is a complete metric space (see [6]).

Proposition 1 ([1]) *For every operator $T \in \mathcal{C}(X)$, and arbitrary $\lambda, \mu \in \mathbb{C}$, we have:*

1. $\gamma(\lambda I - T)\delta(N(\mu I - T), N(\lambda I - T)) \leq |\mu - \lambda|$.

2. $\min\{\gamma(\mu I - T), \gamma(\lambda I - T)\}\widehat{\delta}(N(\mu I - T), N(\lambda I - T)) \leq |\mu - \lambda|$.

Proof. 1. The statement is trivial for $\lambda = \mu$. Suppose that $\lambda \neq \mu$ and consider an element $0 \neq x \in N(\mu I - T)$. Then $x \notin N(\lambda I - T)$ and hence

$$\gamma(\lambda I - T)\text{dist}(x, N(\lambda I - T)) \leq \|(\lambda I - T)x\| \\ = \|(\lambda I - T)x - (\mu I - T)x\| = |\mu - \lambda| \,.$$

From this estimate we obtain, if $B = \{x \in N(\mu I - T), \|x\| \leq 1\}$, that

$$\gamma(\lambda I - T) \sup_{x \in B} \text{dist}(x, N(\lambda I - T)) \leq |\mu - \lambda|$$

and therefore we deduce 1.
2. Clearly, the inequality follows from 1. by interchanging λ and μ.

Proposition 2 ([1]) *Let $M, N \in \mathcal{F}(X)$. For every $x \in X$ and $0 < \epsilon < 1$, there exists $x_0 \in X$ such that $(x - x_0) \in M$ and*

$$dist(x_0, N) \geq \left((1 - \epsilon) \frac{1 - \delta(M, N)}{1 + \delta(M, N)} \right) \|x_0\| \,. \qquad (1)$$

Proof. If $x \in M$ it suffices to take $x_0 = 0$. Assume therefore that $x \notin M$. Let $\widehat{X} = X/M$ denote the quotient space and put $\widehat{x} = x + M$ the equivalence class of x. Evidently, $\|\widehat{x}\| = \inf_{z \in \widehat{x}} \|z\| > 0$. We claim that there exists an element $x_0 \in X$ such that

$$\|\widehat{x}\| = \text{dist}(x_0, M) \geq (1 - \epsilon) \|x_0\| \,.$$

Indeed, when it is not so, then

$$\|\widehat{x}\| = \|z\| < (1 - \epsilon) \|z\| \text{ for every } z \in \widehat{x},$$

and therefore

$$\|\widehat{x}\| \leq \inf_{z \in \widehat{x}} \|z\| = (1 - \epsilon) \|\widehat{x}\| \,.$$

This is impossible since $\|\widehat{x}\| > 0$.
Let $\mu = \text{dist}(x_0, N) = \inf_{u \in N} \|x_0 - u\|$. We know that there exists $y \in N$

such that $\|x_0 - y\| \leq \mu + \epsilon \|x_0\|$. From that we obtain $\|y\| \leq (1+\epsilon)\|x_0\| + \mu$. On the other hand, we have $\operatorname{dist}(y, M) \leq \delta(N, M)\|y\|$ and hence

$$\begin{aligned}(1-\epsilon)\|x_0\| &\leq \operatorname{dist}(x_0, M) \\ &\leq \|x_0 - y\| + \operatorname{dist}(y, M) \leq \mu + \epsilon\|x_0\| + \delta(N, M)\|y\| \\ &\leq \mu + \epsilon\|x_0\| + \delta(N, M)[(1+\epsilon)\|x_0\| + \mu].\end{aligned}$$

From this we obtain that

$$\mu \geq \left[\frac{1-\epsilon-\delta(N, M)}{1+\delta(N, M)} - \epsilon\right]\|x_0\|.$$

Since $\epsilon > 0$ is arbitrary, this implies the inequality (1).

Proposition 3 ([6]) *Let X be a Banach space and suppose that $T \in \mathcal{C}(X)$ has a closed range. Let Y be a (not necessarily closed) subspace of X. if $Y + N(T)$ is closed then $T(Y)$ is closed.*

Proof Let us denote by $\widehat{x}$ the equivalence class $x + N(T)$ in the quotient space $X/N(T)$ and by $\widehat{T} : X/N(T) \to X$ the canonical injection defined by $\widehat{T}(\widehat{x}) = Tx$, where $x \in \widehat{x}$. Since $T(X)$ is closed $\widehat{T}$ has a bounded inverse $\widehat{T}^{-1} : R(T) \to X/N(T)$. Let $\widehat{Y} = \widehat{y} : y \in Y$. Clearly $T(Y) = \widehat{T}(\widehat{Y})$ is the inverse image of $\widehat{Y}$ under the continuous map $\widehat{T}^{-1}$, so $T(Y)$ is closed if $\widehat{Y}$ is closed . It remains to show that $\widehat{Y}$ is closed if $Y + N(T)$ is closed. Suppose that the sequence $(\widehat{x_n})$ of $\widehat{Y}$ converges to $\widehat{x} \in X/N(T)$. This implies that there exists a sequence (x_n) with $x_n \in \widehat{x_n}$ such that $\operatorname{dist}(x_n - x, N(T))$ converges to zero, and so there exists a sequence $(z_n) \subset N(T)$ such that $x_n - x - z_n \to 0$. Then the sequence $(x_n - z_n) \subset Y + N(T)$ converges to x and since by assumption $Y + N(T)$ is closed, we have $x \in Y + N(T)$. This implies $\widehat{x} \in \widehat{Y}$; thus $\widehat{Y}$ is closed .

3 Main Results

In this section we present some results concerning the essential spectra of the operator T. We begin with the following preparatory result which is crucial for our purpose. For α a nonzero positive real number, we introduce the following set

$$\mathcal{R}(\alpha) = \{\lambda \in \mathbb{C}\ ;\ \gamma(\lambda I - T) \geq \alpha\}$$

Theorem 3 *Let $(\lambda_n)_n \subset \mathcal{R}(\alpha)$ non stationary sequence and $\lambda_n \longrightarrow \lambda_0$ in $\mathbb{C}$, then*

1. $\widehat{\delta}(N(\lambda_n I - T), N(\lambda_0 I - T)) \leq \frac{1}{\alpha}\,|\lambda_n - \lambda_0|$.

2. $\lambda_0 \in \mathcal{R}(\alpha)$.

3. *$\lambda_0 I - T$ is semi-regular.*

Proof. 1. For $n, m \in \mathbb{N}$ by Proposition 1 part 2., we have

$$\widehat{\delta}(N(\lambda_n I - T), N(\lambda_m I - T)) \leq \frac{1}{\min\{\gamma(\lambda_n I - T), \gamma(\lambda_m I - T)\}}\,|\lambda_n - \lambda_m|\,.$$

Since by assumption $\gamma(\lambda_n I - T) \geq \alpha$, for all $n \in \mathbb{N}$ we have $\min\{\gamma(\lambda_n I - T), \gamma(\lambda_m I - T)\} \geq \alpha$ and

$$\widehat{\delta}(N(\lambda_n I - T), N(\lambda_m I - T)) \leq \frac{1}{\alpha}\,|\lambda_n - \lambda_m| \tag{2}$$

The sequence $(N(\lambda_n I - T))_n)$ is a Cauchy sequence in the complete metric space $\mathcal{F}(X)$, thus it converges. Let $F = \lim_{n\to\infty} N(\lambda_n I - T)$. Let $x \in N(\lambda_0 I - T)$, by Proposition 1 part 1., we have

$$\delta(N(\lambda_0 I - T), N(\lambda_n I - T)) \leq \frac{1}{\alpha}\,|\lambda_n - \lambda_0|\,.$$

From this estimate we deduce that $x \in F$ and $N(\lambda_0 I - T) \subset F$.

Conversely, let $x \in F$, by Proposition 2 for $0 < \epsilon < 1$, $N = F$ and $M = N(\lambda_n I - T)$ there exists $x_n \in X$ such that $(x - x_n) \in N(\lambda_n I - T)$ and

$$\mathrm{dist}(x_n, F) \geq \left((1-\epsilon)\frac{1 - \delta(N(\lambda_n I - T), F)}{1 + \delta(N(\lambda_n I - T), F)}\right)\|x_n\|\,.$$

From this we obtain

$$\widehat{\delta}(F, N(\lambda_n I - T)) \geq \operatorname{dist}(x_n, F) \geq \left((1-\epsilon)\frac{1-\delta(N(\lambda_n I - T), F)}{1+\delta(N(\lambda_n I - T), F)}\right)\|x_n\|,$$

which yields, $x_n \to 0$ as $n \to \infty$. On the other hand

$$\begin{aligned}(\lambda_0 I - T)x_n &= (\lambda_0 I - T)x - (\lambda_0 I - T)(x - x_n)\\ &= (\lambda_0 I - T)x + (\lambda_n - \lambda_0)(x - x_n),\end{aligned}$$

and hence $(\lambda_n - \lambda_0)(x - x_n) \to 0$. We obtain that

$$(\lambda_0 I - T)x_n \to (\lambda_0 I - T)x.$$

Hence $N(\lambda_0 I - T)$ is closed, and $(\lambda_0 I - T)x = 0$, consequently $F \subset N(\lambda_0 I - T)$. To end the proof of (1) we take $n \to \infty$ in (2).

2. Suppose that $\lambda_0 \notin \mathcal{R}(\alpha)$, then there exists $x \in \mathcal{D}(T)$ and $0 < \epsilon < 1$ such that $\|x\| = 1$, $x \notin N(\lambda_0 I - T)$ and

$$\|(\lambda_0 I - T)x\| < (1-\epsilon)\alpha \|x\|.$$

We can find $x_n \notin N(\lambda_n I - T)$ such that $(x - x_n) \in N(\lambda_n I - T)$ and take $n \in \mathbb{N}$ such that

$$|\lambda_n - \lambda_0| < \frac{\epsilon}{2}\alpha.$$

Then

$$\|(\lambda_n I - T)x\| \leq \|(\lambda_0 I - T)x\| + |\lambda_n - \lambda_0| \|x\|,$$

and therefore

$$\|(\lambda_n I - T)x\| \leq (1 - \frac{\epsilon}{2})\alpha \|x\|. \tag{3}$$

On other hand, we have $(x - x_n) \in N(\lambda_n I - T)$ and hence

$$\begin{aligned}\|x - x_n\| &\leq \sup\{\operatorname{dist}(y, N(\lambda_n I - T)); y \in N(\lambda_0 I - T), \|y\| = 1\}\\ &\leq \delta(N(\lambda_0 I - T), N(\lambda_n I - T))\|x\|\\ &\leq \widehat{\delta}(N(\lambda_0 I - T), N(\lambda_n I - T))\|x\|\\ &\leq \frac{1}{\alpha}|\lambda_n - \lambda_0| \|x\| \leq \frac{\epsilon}{2}\|x\|.\end{aligned}$$

From the last inequality it follows that

$$\|x_n\| \geq (1 - \frac{\epsilon}{2}) \|x\|,$$

and

$$\|x\| \leq \frac{1}{(1 - \frac{\epsilon}{2})} \|x_n\|. \tag{4}$$

From (3) and (4) we obtain

$$\|(\lambda_n I - T)x_n\| < \alpha \|x_n\|,$$

what contradicts the fact that $(\lambda_n)_n \subset \mathcal{R}(\alpha)$.

3. It is clear that $N(\lambda_n I - T) \subset R((\lambda_0 I - T)^k)$ for every $k \in \mathbb{N}$. For every $x \in N(\lambda_0 I - T)$, $k \in \mathbb{N}$, and $\lambda_n \neq \lambda_0$ we then have

$$\begin{aligned} \operatorname{dist}(x, R((\lambda_0 I - T)^k)) &\leq \operatorname{dist}(x, N(\lambda_n I - T)) \|x\| \\ &\leq \delta(N(\lambda_0 I - T), N(\lambda_n I - T)) \|x\| \\ &\leq \widehat{\delta}(N(\lambda_0 I - T), N(\lambda_n I - T)) \|x\|. \end{aligned}$$

This implies that $x \in \overline{R((\lambda_0 I - T)^k)}$ for every $k \in \mathbb{N}$. Hence $N(\lambda_0 I - T) \subset \overline{R((\lambda_0 I - T)^k)}$ for every $k \in \mathbb{N}$. To establish 3. it suffices to prove that $R((\lambda_0 I - T)^k)$ is closed for $k \in \mathbb{N}$. We proceed by induction. The case $k = 1$ is obvious from 2. Assume that $R((\lambda_0 I - T)^k)$ is closed. Then $N(\lambda_0 I - T) \subset \overline{R((\lambda_0 I - T)^k)} = R((\lambda_0 I - T)^k)$ and hence $N(\lambda_0 I - T) + R((\lambda_0 I - T)^k)$ is closed. By proposition 3 we then conclude that $T(R((\lambda_0 I - T)^k)) = R((\lambda_0 I - T)^{k+1})$ is closed.

4 Relationship between Kato essential spectrum and another essential spectrum

The essential spectra were studied by many authors (see [1, 9, 11, 14]). Now, the main question is the relationship between them. Motivated by a problem concerning the essential quasi-Fredholm spectrum posed in [9], J. P. Labrousse [8] characterized in the case of Hilbert spaces, a relation of the essential quasi-Fredholm spectrum (Kato essential spectrum) and another not closed essential spectrum defined in [3] by

$$\sigma_{ec}(T) = \{\lambda \in \mathbb{C}; R(\lambda I - T) \text{ is not closed}\}.$$

Now, we study this relation in the case of Banach spaces. Let X a Banach space and $T \in \mathcal{C}(X)$. We have

Corollary 1 *$\sigma_{ek}(T) \cap \mathcal{R}(\alpha)$ is at most countable.*

Proof. Let λ_0 be a non-isolated point of $\sigma_{ek}(T) \cap \mathcal{R}(\alpha)$. Then there exists $(\lambda_n)_n \subset \sigma_{ek}(T) \cap \mathcal{R}(\alpha)$ such that $\lambda_n \to \lambda_0$, by Theorem 3 $\lambda_0 \notin \sigma_{ek}(T)$. This contradicts that $\sigma_{ek}(T)$ is closed.

Corollary 2 *$\sigma_{ek}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T))$ is at most countable.*

Proof. Note that $\mathbb{C} \setminus \sigma_{ec}(T) = \bigcup_{n=1}^{\infty} \mathcal{R}(\frac{1}{n})$ and

$$\sigma_{ek}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T)) = \bigcup_{n=1}^{\infty} (\sigma_{ek}(T) \cap \mathcal{R}(\frac{1}{n}))$$

This union of sets is at most countable by Corollary 1, so $\sigma_{ek}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T))$ is at most countable.

Corollary 3 *If $\lambda \in \sigma_{ec}(T)$ is non-isolated point then $\lambda \in \sigma_{ek}(T)$.*

Proof. Let $\lambda \in \sigma_{ec}(T)$ be a non-isolated point. Assume that $\lambda I - T$ is of Kato type operator. Then by Theorem 2 there exists an open disc $\mathbb{D}(\lambda, \epsilon)$ such that $\mu I - T$ is semi-regular in $\mathbb{D}(\lambda, \epsilon) \setminus \{\lambda\}$, so $R(\mu I - T)$ is closed for all $\mu \in \mathbb{D}(\lambda, \epsilon) \setminus \{\lambda\}$. This contradicts our assumption that λ non-isolated point.

Corollary 4 *$\sigma_{ek}(T) \Delta \sigma_{ec}(T)$ is at most countable, where Δ is the symmetric difference of the sets $\sigma_{ek}(T)$ and $\sigma_{ec}(T)$.*

Proof. We have

$$\sigma_{ek}(T)\Delta\sigma_{ec}(T) = (\sigma_{ek}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T)) \cup (\sigma_{ec}(T) \cap (\mathbb{C} \setminus \sigma_{ek}(T)))$$

From the Corollary 2 the set $\sigma_{ek}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T))$ is at most countable, and the set $\sigma_{ec}(T) \cap (\mathbb{C} \setminus \sigma_{ek}(T))$ is discrete by Corollary 3.

5 Examples and Applications

Example 1. Let the class $\mathcal{P}_g(X)$ of operators on a Banach space X which satisfy a polynomial growth condition. An operator T satisfies this condition if there exists $K > 0$, and $\delta > 0$ for which

$$\|exp(i\lambda T)\| \leq K(1 + |\lambda|^{\delta}) \text{ for all } \lambda \in \mathbb{R},$$

Examples of operators which satisfy a polynomial growth condition are Hermitian operators on Hilbert spaces, nilpotent and projection operators, algebraic operators with real spectra. It is shown that $\mathcal{P}_g(X)$ coincides with the class of all generalized scalar operators having real spectra. We first note that the polynomial growth condition may be reformulated as follows: $T \in \mathcal{P}_g(X)$ if and only if $\sigma(T) \subseteq \mathbb{R}$ and there is a constant $K > 0$, and $\delta > 0$ such that

$$\left\|(\lambda I - T)^{-1}\right\| \leq K(1 + |Im\lambda|^{-\delta}) \text{ for all } \lambda \in \mathbb{C} \text{ with } Im\lambda \neq 0, \quad (5)$$

We claim that if $T \in \mathcal{P}_g(X)$ and $c = Im\lambda > 0$ then

$$(\lambda I - T)^{-1} = -i \int_0^{\infty} e^{i\lambda t} e^{-itT} dt. \quad (6)$$

Indeed, for every $s > 0$,

$$(\lambda I - T) \int_0^s e^{i(\lambda I - T)t} dt = \int_0^s \frac{d}{dt}(e^{i(\lambda I - T)t}) dt = e^{i(\lambda I - T)s} - I.$$

From the estimate

$$\left\|e^{i(\lambda I - T)s}\right\| = e^{-cs} \left\|e^{-isT}\right\| \leq K e^{-cs}(1 + s\delta),$$

it follows that $\left\|e^{i(\lambda I-T)s}\right\| \to 0$ as $s \to \infty$. From this we obtain that

$$(\lambda I - T)\int_0^\infty e^{i(\lambda I-T)t}dt = I,$$

from which the equality (6) follows.

We recall that for every linear operator T on a vector space X, the ascent of T, is the smallest positive integer $p = p(T)$ such that

$$N(T^p) = N(T^{p+1}).$$

If there is no such integer we set $p(T) = \infty$. The descent of T is the smallest positive integer $q = q(T)$ such that

$$R(T^q) = R(T^{q+1}).$$

If such an integer does not exist, we put $q(T) = \infty$. The finiteness of the ascent and the descent of a linear operator T is related to a certain decomposition of X.

Theorem 4 ([6]) *Suppose that $T \in \mathcal{C}(X)$. If both $p(T)$ and $q(T)$ are finite then $p(T) = q(T) = p$, and we have the decomposition*

$$X = R(T^p) \oplus N(T^p)$$

Conversely, if for a natural number p we have the decomposition $X = R(T^p) \oplus N(T^p)$ then $p(T) = q(T) \leq p$. In this case $T_{/R(T^p)}$ is bijective. Moreover, $\lambda \in \sigma(T)$ is a pole of the resolvent $(\lambda I - T)^{-1}$ if and only if $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$.

The following proposition establish the finiteness of the ascent of a linear operator $T \in \mathcal{P}_g(X)$.

Proposition 4 *Assume that $T \in \mathcal{P}_g(X)$, for every $\lambda \in \sigma(T)$ we have:*

1. $p(\lambda I - T) \leq m < \infty$.

2. $\overline{R((\lambda I-T)^{p})}=\overline{R((\lambda I-T)^{p+k})}$; $k\in\mathbb{N}$. *and* $p=p(\lambda I-T)$.

Proof. Of course, there is no loss of generality if we assume $\lambda=0$. If δ as is the appearing constant in (5), put $m=[\delta]+1$. Then

$$\lim_{t\to 0^{+}}(it)^{m}(itI-T)^{-1}=0.$$

Suppose $p(T)>m$. Let $x\in X$ and $f\in X^{*}$ such that,

$$T^{m+1}x=0,\quad T^{m}x\neq 0,\ \text{and}\ f(T^{m}x)=1.$$

Define a linear continuous functional ϕ on $\mathcal{C}(X)$ by

$$\phi(T)=f(Tx)\ \text{for every}\ T\in\mathcal{C}(X).$$

From (6) we have for all $t>0$

$$(it\lambda I-T)^{-1}=-i\int_{0}^{\infty}e^{-tx}e^{-ixT}dx,$$

and therefore, for all $t>0$

$$\phi((it\lambda I-T)^{-1})=-i\int_{0}^{\infty}e^{-tx}\left(\sum_{n=0}^{\infty}\frac{(-it)^{n}}{n!}\phi(T^{n})\right)dx.$$

Clearly $\phi(T^{n})=f(T^{n}x)=0$ for all $n>m$, so for every $t>0$ we have

$$\begin{aligned}(it)^{m}\phi((itI-T)^{-1}&=-(i)^{m+1}t^{m}\sum_{n=0}^{m}\frac{(-i)^{n}}{n!}\left[\sum_{n=0}^{\infty}(x)^{n}e^{-tx}\right]\\&=-(i)^{m+1}t^{m}\sum_{n=0}^{m}\frac{(-i)^{n}}{n!}\left[\frac{(n+1)!}{t^{n+1}}\right]\phi(T^{n})\\&=-(i)^{2m+1}(m+1)t^{-1}\\&\quad+\{\text{terms with non-negative powers of t}\}.\end{aligned}$$

This shows that $(it)^{m}\phi((itI-T)^{-1}$ does not converge to 0 as $t\to 0^{+}$, which is a contradiction. Hence $p(\lambda I-T)\le m$ for every $\lambda\in\mathbb{C}$.

Now, if $T \in \mathcal{P}_g(X)$ then $T^* \in \mathcal{P}_g(X^*)$, so $p = p(\lambda I - T) < \infty$. This means that

$$N((\lambda I - T^*)^p) = N((\lambda I - T^*)^{p+k}),$$

for every $k \in \mathbb{N}$, and hence

$$\overline{R((\lambda I - T)^p)} = \overline{R((\lambda I - T)^{p+k})};\ k \in \mathbb{N}.$$

Proposition 5 *Let $T \in \mathcal{P}_g(X)$, we have:*

1. *If $\lambda \in \sigma(T) \setminus \sigma_{ec}(T)$, then λ is an isolated point in $\sigma(T)$.*

2. *If $\lambda \in \sigma_{ec}(T)$ and $R((\lambda I - T)^p)$ is closed for some $p \in \mathbb{N}$, then λ is a pole of the resolvent of T.*

Proof. 1. If we assume that $T \in \mathcal{P}_g(X)$ and $R((\lambda I - T))$ is closed for some $\lambda \in \mathbb{C}$ then also $q(\lambda I - T)$ is finite, In fact, if $R(\lambda I - T)$ is closed then $R((\lambda I - T)) + N((\lambda I - T)^p)$ is closed and $R((\lambda I - T)) + N((\lambda I - T)^p) = R((\lambda I - T)) + N((\lambda I - T)^n)$ for all $n \geq p$, thus $q(\lambda I - T) < \infty$. Since $p(\lambda I - T) < \infty$, it follows that λ is an isolated point in $\sigma(T)$.

2. If $R((\lambda I - T)^p)$ is closed, then $R((\lambda I - T)^p) = R((\lambda I - T)^{p+k});\ k \in \mathbb{N}$, so $q(\lambda I - T) < \infty$, it follows that λ is a pole of the resolvent of T.

Corollary 5 *Let $T \in \mathcal{P}_g(X)$, then $\sigma_{ek}(T) \Delta \sigma_{ec}(T)$ is at most countable.*

Proof. From the proposition 5, if $\lambda \notin \sigma_{ec}(T)$, then λ is a pole of the resolvent $(\lambda I - T)^{-1}$. This implies that $\lambda \notin \sigma_{ek}(T)$ and the set $\sigma_{ek}(T) \setminus \sigma_{ec}(T))$ is empty. Now, if $\lambda \in \sigma_{ec}(T)$, we have two cases. First if there exists $p \in \mathbb{N}$ such that $R((\lambda I - T)^p)$ is closed, by the Proposition 5 part 2, λ is a pole of the resolvent and $\lambda \notin \sigma_{ek}(T))$, thus $\sigma_{ec}(T)) \setminus \sigma_{ek}(T)$ is at most countable . Now, if $R((\lambda I - T)^p)$ is not closed for every $p \in \mathbb{N}$, then $R(((\lambda I - T)_{/M})^p)$ is not closed for every T-invariant closed subset M and $p \in \mathbb{N}$, so $\lambda I - T$ is not of Kato type and $\lambda \in \sigma_{ek}(T)$, the set $\sigma_{ec}(T)) \setminus \sigma_{ek}(T)$ is then empty.

Example 2. Let X be a Banach space and H a Hilbert space such that:

1. $X \subset H$, and the embedding mapping ; $X \longrightarrow H$ is continous.
2. X is dense in H.

Definition 4 (P. D. Lax [10]) *A linear bounded operator T in X is said to be symmetrizable over X if T is symmetric with respect to the inner product of H, i.e $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in X$.*

We note that if T is in X and symmetrizable it also bounded in H (with respect to the norm of H). In the sequel, for every a bounded operator T in X, we shall denote by $\alpha(T|X)$ the nullity of $T|X$, defined as $\alpha(T|X) = \dim N(T|X)$, whilst the deficiency $\beta(T|X)$ of $T|X$ is defined by $\beta(T|X) = \text{codim} R(T|X)$, and the number $\text{ind}(T|X) = \alpha(T|X) - \beta(T|X)$ is called the index of $T|X$. Here we denoted $T|X$ the operator considered in X. If the operator T has a index zero, it is called a Weyl operator. We denote $\sigma_{ew}(T)$ the set of all complex λ for with $\lambda I - T$ is not Weyl operator and we call it the Weyl essential spectrum of T. J. Nieto in the next theorem characterized the Weyl essential spectrum of symmetrizable operators.

Theorem 5 ([15]) *Let T be a symmetrizable operator over X. Then $\sigma_{ew}(T|X)$ consists precisely of those $\lambda \in \sigma(T|X)$ which are not isolated eigenvalues of finite multiplicity.*

Note that the theorem above shows that any isolated point of spectrum of symmetrizable operator T over X which is not in $\sigma_{ew}(T|X)$ is a pole of the resolvent and both $\sigma_{ek}(T|X)$ and $\sigma_{ec}(T|X)$ are contained in $\sigma_{ew}(T|X)$. The next result shows that the Kato essential spectrum of symmetrizable operator T is contained in $\sigma_{ec}(T|X)$.

Proposition 6 *Let T be a symmetrizable operator over X. Then*

$$\sigma_{ek}(T|X) \subseteq \sigma_{ec}(T|X).$$

Proof. Suppose that $\lambda \notin \sigma_{ec}(T|X)$, then $R(\lambda I - T|X)$ is closed and, by Theorem 5, it must be $\alpha(\lambda I - T|X) < \infty$, this implies that $\lambda I - T$ is upper semi-Ferdholm operator over X, so is of Kato type, hence $\lambda \notin \sigma_{ek}(T|X)$.

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Some comments on "Halpern iteration for firmly type nonexpansive mappings" [1]

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Abstract

In this note we point out some major bugs appeared in [1].

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1 Introduction and Comments

In [1], Song and Chai established some results for firmly type nonexpansive mappings, however their proofs contained some major bugs.

The main result presented by Song and Chai [1] is as follows:

Theorem 1 *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm and with the fixed point property for nonexpansive self-mappings. Assume that K is a nonempty closed convex*

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subset of E and $T : K \to K$ is a firmly type nonexpansive mapping with a fixed point. For arbitrary initial value $x_0 \in K$ and fixed anchor $u \in K$, define iteratively a sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n.$$

Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the conditions (C1) $\lim_{n\to\infty} \alpha_n = 0$ and (C2) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$. Then as $n \to \infty$, $\{x_n\}$ converges strongly to some fixed point p of T.

On page 8, the following inequality holds:

$$\text{(SC)} \qquad \|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - k \|x_n - T x_n\|^2 + \alpha_n M.$$

One can easily see that, with $\sum_{n=1}^{+\infty} \alpha_n = +\infty$, (SC) does not imply that the sequence $\{x_n\}$ is bounded. In otherwords, if $k \|x_n - T x_n\|^2 - \alpha_n M \leq 0$ then from (SC), we obtain that the sequence $\{x_n\}$ is not bounded.

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