Strong convergence of a modified implicit iteration process for the finite family of $\psi$–uniformly pseudocontractive mappings in Banach spaces

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Abstract

The purpose of this paper is to establish the strong convergence of a implicit iteration process with errors to a common fixed point for a finite family of $\psi$–uniformly pseudocontractive and $\psi$–uniformly accretive mappings in real Banach spaces. The results presented in this paper extend and improve the corresponding results of Refs. [3, 7, 12]. The remark at the end is important.

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1 Introduction

Form now onward, we assume that $E$ is a real Banach space and $K$ be a nonempty convex subset of $E$. Let $J$ denote the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 \text{ and } \|f^*\| = \|x\|\},$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by $j$.

Let $\Psi := \{\psi : [0, \infty) \rightarrow [0, \infty) \text{ is a strictly increasing mapping such that } \psi(0) = 0\}$.

**Definition 1** A mapping $T : K \rightarrow K$ is called $\psi$–uniformly pseudocontractive if there exist mapping $\psi \in \Psi$ and $j(x - y) \in J(x - y)$ such that

$$(1.1) \quad \langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \psi(\|x - y\|), \quad \forall x, y \in K.$$ 

**Definition 2** A mapping $S : D(S) \subset E \rightarrow E$ is called $\psi$–uniformly accretive if there exist mapping $\psi \in \Psi$ and $j(x - y) \in J(x - y)$ such that

$$(1.2) \quad \langle Sx - Sy, j(x - y) \rangle \geq \psi(\|x - y\|), \quad \forall x, y \in E.$$ 

**Remark 1** 1. Taking $\psi(a) := \psi(a)a$, $\forall a \in [0, \infty)$, $(\psi \in \Psi)$, we get the usual definitions of $\psi$– pseudocontractive and $\psi$– accretive mappings.

2. Taking $\psi(a) := \gamma a^2; \gamma \in (0, 1), \forall a \in [0, \infty)$, $(\psi \in \Psi)$, we get the usual definitions of strongly pseudocontractive and strongly accretive mappings.

3. $T$ is $\psi$–uniformly pseudocontractive iff $S = I - T$ is $\psi$–uniformly accretive.

It is known that $T$ is strongly pseudocontractive if and only if $(I - T)$ is strongly accretive.
In 2001, Xu and Ori [12] introduced the following implicit iteration process for a finite family of nonexpansive mappings \( \{T_i : i \in I\} \) (here \( I = \{1, 2, ..., N\} \)), with \( \{\alpha_n\} \) a real sequence in \((0, 1)\), and an initial point \( x_0 \in K \):

\[
\begin{align*}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1}, \\
&\vdots
\end{align*}
\]

which can written in the following compact form:

\[
(1.3) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad \forall n \geq 1,
\]

where \( T_n = T_{n(\text{mod} N)} \) (here the \( \text{mod} N \) function takes values in \( I \)). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters \( \{\alpha_n\} \) are sufficient to guarantee the strong convergence of the sequence \( \{x_n\} \).

In [7], Osilike proved the following theorem.

**Theorem 1** Let \( E \) be a real Banach space and \( K \) be a nonempty closed convex subset of \( E \). Let \( \{T_i : i \in I\} \) be \( N \) strictly pseudocontractive self-mappings of \( K \) with \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \). Let \( \{\alpha_n\}_{n=1}^{\infty} \) be a real sequence satisfying the conditions:

- \( (i) \) \( 0 < \alpha_n < 1 \),
- \( (ii) \) \( \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty \),
- \( (iii) \) \( \sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty \).
From arbitrary \( x_0 \in K \), define the sequence \( \{x_n\} \) by the implicit iteration process (1.3). Then \( \{x_n\} \) converges strongly to a common fixed point of the mappings \( \{T_i : i \in I\} \) if and only if \( \lim_{n \to \infty} d(x_n, F) = 0 \).

**Definition 3** A normed space \( E \) is said to satisfy Opial’s condition if for any sequence \( \{x_n\} \) in \( E \), \( x_n \rightharpoonup x \) implies that \( \limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \) for all \( y \in E \) with \( y \neq x \).

In [3], Chen et al proved the following theorem.

**Theorem 2** Let \( K \) be a nonempty closed convex subset of a \( q \)-uniformly smooth and \( p \)-uniformly convex Banach space \( E \) that has the Opial property. Let \( s \) be any element in \( (0, 1) \) and let \( \{T_i\}_{i=1}^N \) be a finite family of strictly pseudocontractive self-maps of \( K \) such that \( \{T_i\}_{i=1}^N \), have at least one common fixed point. For any point \( x_0 \in K \) and any sequence \( \{\alpha_n\}_{n=1}^\infty \) in \( (0, s) \), define the sequence \( \{x_n\} \) by the implicit iteration process (1.3). Then \( \{x_n\} \) converges weakly to a common fixed point of \( \{T_i\}_{i=1}^N \).

Inspired and motivated by the above said facts, we suggest the following implicit iteration process with errors and define the sequence \( \{x_n\} \) as follows

\[
(1.4) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n + u_n, \quad \forall n \geq 1,
\]

where \( T_n = T_n(\text{mod}\, N) \), \( \{\alpha_n\} \) is a sequence in \( (0, 1) \) and \( \{u_n\} \) is a summable sequence in \( K \).

Clearly, this iteration process contains the process (1.3) as its special case.

The purpose of this paper is to study the strong convergence of the implicit iteration process (1.4) to a common fixed point for a finite family of \( \psi \)-uniformly pseudocontractive and \( \psi \)-uniformly accretive mappings in real Banach spaces. The results presented in this paper extend and improve the corresponding results of Refs. [3, 7, 12].
2 Main Results

The following lemma is now well known.

**Lemma 3** Let $J : E \rightarrow 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have

$$||x + y||^2 \leq ||x||^2 + 2 \langle y, j(x + y) \rangle, \forall (x + y) \in J(x + y).$$

**Lemma 4** [5] Let $\{\theta_n\}$ be a sequence of nonnegative real numbers, $\{\lambda_n\}$ be a real sequence satisfying

$$0 \leq \lambda_n \leq 1, \sum_{n=0}^{\infty} \lambda_n = \infty$$

and let $\psi \in \Psi$. If there exists a positive integer $n_0$ such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \psi(\theta_{n+1}) + \sigma_n,$$

for all $n \geq n_0$, with $\sigma_n \geq 0, \forall n \in \mathbb{N}$, and $\sigma_n = 0(\lambda_n)$, then $\lim_{n \rightarrow \infty} \theta_n = 0$.

**Theorem 5** Let $\{T_1, T_2, ..., T_N\} : K \rightarrow K$ be $N, \psi-$uniformly pseudo-contractive mappings with $\{T_n x_n\}$ bounded and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.4) satisfying $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$, $\lim_{n \rightarrow \infty} (1 - \alpha_n) = 0$ and $\|u_n\| = 0(1 - \alpha_n)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.

**Proof.** Since each $T_i$ is $\psi-$uniformly pseudocontractive, we have from (1.1)

$$(2.1) \quad \langle T_i x - T_i y, j(x - y) \rangle \leq ||x - y||^2 - \psi(||x - y||), i = 1, 2, \cdots, N.$$ 

We know that the mappings $\{T_1, T_2, ..., T_N\}$ have a common fixed point in $K$, say $w$, then the fixed point set $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is nonempty.
We will show that $w$ is the unique fixed point of $F$. Suppose there exists $q \in F(T_1)$ such that $w \neq q$ i.e., $\|w - q\| > 0$. Then

\[(AR) \quad \psi(\|w - q\|) > 0.\]

Since $\psi$ is strictly increasing with $\psi(0) = 0$. Then, from the definition of $\psi-$uniformly pseudocontractive mapping,

\[\|w - q\|^2 = \langle w - q, J(w - q) \rangle = \langle T_1w - T_1q, J(w - q) \rangle \leq \|w - q\|^2 - \psi(\|w - q\|),\]

implies

\[\psi(\|w - q\|) \leq 0,\]

contradicting $(AR)$, which implies the uniqueness. Hence $F(T_1) = \{w\}$.

Similarly we can prove that $F(T_i) = \{w\}; i = 2, 3, ..., N$. Thus $F = \{w\}$.

We set

\[M_1 = \|x_0 - w\| + \sup_{n \geq 0} \|T_nx_n - w\|,\]
\[M_2 = 1 + \sup_{n \geq 0} \|u_n\|.\]

Obviously $M_1, M_2 < \infty$. Let $M_3 = M_1 + M_2$.

It is clear that $\|x_0 - w\| \leq M_1 < M_3$. Let $\|x_{n-1} - w\| \leq M_1 < M_3$.

Next we will prove that $\|x_n - w\| \leq M_3$.

Consider

\[\|x_n - w\| = \|\alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n + u_n - w\| \]
\[= \|\alpha_n (x_{n-1} - w) + (1 - \alpha_n) (T_n x_n - w) + u_n\| \]
\[\leq \alpha_n \|x_{n-1} - w\| + (1 - \alpha_n) \|T_n x_n - w\| + \|u_n\| \]
\[\leq \alpha_n M_1 + (1 - \alpha_n) M_1 + M_2 \]
\[= M_1 + M_2 \]
\[= M_3.\]
So, from the above discussion, we conclude that the sequence \( \{x_n - w\} \) is bounded. Let \( M_4 = \sup_{n \geq 0} \|x_n - w\| \).

Denote \( M = M_3 + M_4 \). Obviously \( M < \infty \).

The real function \( f : [0, \infty) \to [0, \infty) \), defined by \( f(t) = t^2 \) is increasing and convex. For all \( \lambda \in [0, 1] \) and \( t_1, t_2 > 0 \) we have

\[
(1 - \lambda)t_1 + \lambda t_2^2 \leq (1 - \lambda)t_1^2 + \lambda t_2^2. \tag{2.2}
\]

Consider

\[
\|x_n - w\|^2 = \|\alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n + u_n - w\|^2 \\
= \|\alpha_n (x_{n-1} - w) + (1 - \alpha_n) (T_n x_n - w) + u_n\|^2 \\
\leq [\alpha_n \|x_{n-1} - w\| + (1 - \alpha_n) \|T_n x_n - w\| + \|u_n\|]^2 \\
\leq \alpha_n \|x_{n-1} - w\|^2 + (1 - \alpha_n) \|T_n x_n - w\|^2 + \|u_n\|^2 + 2M \|u_n\| \\
\leq \|x_{n-1} - w\|^2 + M^2 (1 - \alpha_n) + \|u_n\|^2 + 2M \|u_n\|. \tag{2.3}
\]

From lemma 1 and (1.4), we have

\[
\|x_n - w\|^2 = \|\alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n + u_n - w\|^2 \\
= \|\alpha_n (x_{n-1} - w) + (1 - \alpha_n) (T_n x_n - w) + u_n\|^2 \\
\leq \alpha_n^2 \|x_{n-1} - w\|^2 + 2 (1 - \alpha_n) (T_n x_n - w, j(x_n - w)) \\
\quad + 2 \langle u_n, f(x_n - w) \rangle \\
\leq \alpha_n^2 \|x_{n-1} - w\|^2 + 2 (1 - \alpha_n) \|x_n - w\|^2 \\
\quad - 2 (1 - \alpha_n) \psi(\|x_n - w\|) + 2M \|u_n\|. \tag{2.4}
\]
Substituting (2.3) in (2.4), and with the help of \( \|u_n\| = 0(1 - \alpha_n) \) (implies \( \|u_n\| = (1 - \alpha_n) t_n; \ t_n \to 0 \) as \( n \to \infty \)) we get

\[
\|x_n - w\|^2 \leq [\alpha_n^2 + 2(1 - \alpha_n)] \|x_{n-1} - w\|^2 - 2(1 - \alpha_n) \psi(\|x_n - w\|)
+ 2M^2 (1 - \alpha_n)^2 + 2(1 - \alpha) \|u_n\|^2 + 4M(1 - \alpha) \|u_n\|
+ 2M \|u_n\|
= [1 + (1 - \alpha_n)^2] \|x_{n-1} - w\|^2 - 2(1 - \alpha_n) \psi(\|x_n - w\|)
+ 2M^2 (1 - \alpha_n)^2 + 2(1 - \alpha) \|u_n\|^2 + 2M[1 + 2(1 - \alpha)] \|u_n\|
\leq \|x_{n-1} - w\|^2 - 2(1 - \alpha_n) \psi(\|x_n - w\|) + 3M^2 (1 - \alpha_n)^2
+ 2(1 - \alpha) \|u_n\|^2 + 6M \|u_n\|
\leq \|x_{n-1} - w\|^2 - 2(1 - \alpha_n) \psi(\|x_n - w\|)
+ (1 - \alpha_n)[3M^2 (1 - \alpha_n) + 2(1 + 3M) t_n].
\]

(2.5)

Denote

\[
\theta_n = \|x_{n-1} - w\|,
\lambda_n = 2(1 - \alpha_n),
\sigma_n = (1 - \alpha_n)[3M^2 (1 - \alpha_n) + 2(1 + 3M) t_n].
\]

Condition \( \lim_{n \to \infty} (1 - \alpha_n) = 0 \) assures the existence of a rank \( n_0 \in \mathbb{N} \) such that \( \lambda_n = 2(1 - \alpha_n) \leq 1 \), for all \( n \geq n_0 \). Now with the help of \( \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty \), \( \lim_{n \to \infty} (1 - \alpha_n) = 0 \) and lemma 2, we obtain from (2.5) that

\[
\lim_{n \to \infty} \|x_n - w\| = 0,
\]

completing the proof.

**Corollary 6** Let \( \{T_1, T_2, \ldots, T_N\} : K \to K \) be \( N \), \( \psi \)–uniformly pseudo-contractive mappings with \( \{T_n x_n\} \) bounded and \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \). From arbitrary \( x_0 \in K \), define the sequence \( \{x_n\} \) by the implicit iteration process (1.3) satisfying \( \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty \) and \( \lim_{n \to \infty} (1 - \alpha_n) = 0 \). Then \( \{x_n\} \) converges strongly to a common fixed point of \( \{T_1, T_2, \ldots, T_N\} \).
Remark 2 Theorem 3 extend and improve the theorems 1-2 in the following directions:

1) The strictly pseudocontractive mappings are replaced by the more general $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings;

2) Theorem 3 holds in real Banach space whereas the results of theorem 2 are in $q$-uniformly smooth and $p$-uniformly convex Banach space;

3) We do not need the assumption $\lim_{n \to \infty} d(x_n, F)$ as in theorem 1;

4) Weak convergence in theorem 2 is replaced by the strong convergence in theorem 3;

5) One can easily see that if we take $\alpha_n = 1 - \frac{1}{\sqrt{n}}$, then $\sum (1 - \alpha_n) = \infty$, but $\sum (1 - \alpha_n)^2 = \infty$. Hence the conclusion of theorem 1 is not true in all cases.

References


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