A curious synopsis on the Goldbach conjecture, the friendly numbers, the perfect numbers, the Mersenne composite numbers and the Sophie Germain primes

Ikorong Anouk Gilbert Nemron

Abstract

The notion of a friendly number (or amicable number), (see [2] or [10] or [18] or [19]) is based on the idea that a human friend is a kind of alter ego. Indeed, Pythagoras wrote (see [18] or [19]): 

A friend is the other I, such as are 220 and 284. These numbers have a special mathematical property: each is equal to the sum of the other’s proper divisors (divisors other than the number itself).

The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110, and they sum to 284; the proper divisors of 284 are 1, 2, 4, 71, and 142, and they sum to 220. So \{220, 284\} is called a pair of friendly numbers [note \{17296, 18416\} is also a pair of friendly numbers (see [18] or [19])]. More precisely, we say that a number \(a'\) is a friendly number or amicable number, if there exists a number \(a'' \neq a'\) such that \(\{a', a''\}\) is a pair of friendly numbers [example. 220, 284, 17296 and 18416 are friendly numbers]. It is trivial to see that a friendly number is a composite number (we
recall that *a composite number* is a non prime number. Primes are well known (see [1] or [2] or [3] or [4]), and original characterizations of primes via divisibility is given in [15] and [16] and [17], and the *friendly* numbers problem states that there are infinitely many friendly numbers. Pythagoras saw *perfection* in any integer that equaled the sum of all the other integers that divided evenly into it (see [2] or [10] or [17] or [18] or [19]). The first perfect number is 6. It’s evenly divisible by 1, 2, and 3, and it’s also the sum of 1, 2, and 3, [note 28 and 496 and 33550336 are also perfect numbers (see [18] or [19])]. It is immediate that a *perfect* number is a composite number, and the *perfect* numbers problem states that there are infinitely many perfect numbers. It is trivial to see that an integer is perfect if and only if this integer is *self-amicable* (perfect numbers are characterized via divisibility in [15] and [17]). A *Mersenne composite* is a non prime number of the form \( M_m = 2^m - 1 \), where \( m \) is prime (Mersenne composites were characterized via divisibility in [15] and [17]). It is known (see [2] or [3] or [4] or [10] or [18] or [19]) that \( M_{11} \) and \( M_{67} \) are Mersenne composites. A prime \( h \) is called a *Sophie Germain prime* (see [10]), if both \( h \) and \( 2h + 1 \) are prime; the first few Sophie Germain primes are 2, 3, 5, 11, 23, 29, 41, ..., and it is easy to check that 233 is a Sophie Germain prime. Sophie Germain primes are known for some integers > 233 (For original characterizations of Sophie Germain primes via divisibility, see [15] and [17]). The Sophie Germain primes problem asserts that there are infinitely many Sophie Germain primes. That being said, in this paper, we state a simple conjecture (Q.), we generalize the Fermat induction, and we use only the immediate part of the generalized Fermat induction to give a simple and detailed proof that (Q.) is stronger than the Goldbach problem, the friendly numbers problem, the perfect numbers problem, the Mersenne composites problem and the Sophie Germain primes problem; *this helps us to explain why it is natural and not surprising to conjecture that*
the friendly numbers problem, the perfect numbers problem, the Mersenne composites problem, and the Sophie Germain primes problem are simultaneously special cases of the Goldbach problem (we recall (see [1] or [5] or [6] or [7] or [8] or [9] or [11] or [13] or [14]) that the Goldbach problem states that every even integer $e \geq 4$ is of the form $e = p + q$, where $(p, q)$ is a couple of prime(s).

2010 Mathematics Subject Classification: 11AXX, 03Bxx, 05A05.

Key words and phrases: goldbach, goldbachian, cache, Sophie Germain primes, Mersenne composites, friendly numbers, perfect numbers.

1 Prologue.

Briefly, the immediate part of the generalized Fermat induction is based around the following simple definitions. Let $n$ be an integer $\geq 2$, we say that $c(n)$ is a cache of $n$, if $c(n)$ is an integer of the form $0 \leq c(n) < n$ [Example.1 If $n = 4$, then $c(n)$ is a cache of $n$ if and only if $c(n) \in \{0, 1, 2, 3\}$]. Now, for every couple of integers $(n, c(n))$ such that $n \geq 2$ and $0 \leq c(n) < n$ [observe that $c(n)$ is a cache of $n$], we define $c(n, 2)$ as follows: $c(n, 2) = 1$ if $c(n) \equiv 1 \mod 2$; and $c(n, 2) = 0$ if $c(n) \not\equiv 1 \mod 2$. It is immediate that $c(n, 2)$ exists and is well defined, since $n \geq 2$ [Example.2 If $n = 6$, then $c(n, 2) = 0$ if $c(n) \in \{0, 2, 4\}$ and $c(n, 2) = 1$ if $c(n) \in \{1, 3, 5\}$]. In this paper, induction will be made on $n$ and $c(n, 2)$ [where $n$ is an integer $\geq 2$ and $c(n)$ is a cache of $n$].

2 Introduction and non-standard definitions.

Definitions.1. We say that $e$ is goldbach, if $e$ is an even integer $\geq 4$ and is of the form $e = p + q$, where $(p, q)$ is a couple of prime(s). The
Goldbach problem states that every even integer \( e \geq 4 \) is goldbach. We say that \( e \) is goldbachian, if \( e \) is an even integer \( \geq 4 \), and if every even integer \( v \) with \( 4 \leq v \leq e \) is goldbach [there is no confusion between goldbach and goldbachian, since goldbachian clearly implies goldbach].

Example 3. 12 is goldbachian [indeed, 12 and 10 and 8 and 6 and 4 are goldbach (note 4 = 2 + 2, where 2 is prime; 6 = 3 + 3, where 3 is prime; 8 = 5 + 3 where 3 and 5 are prime; 10 = 7 + 3, where 3 and 7 are prime, and 12 = 5 + 7, where 5 and 7 are prime); so 12 is an even integer \( \geq 4 \), and every even integer \( v \) of the form \( 4 \leq v \leq 12 \) is goldbach; consequently 12 is goldbachian]. Since 12 is goldbachian, then it becomes trivial to deduce that 10 and 8 and 6 and 4 are goldbachian. Example 4. 1000000 is goldbachian [indeed, it known that every even integer \( v \) of the form \( 4 \leq v \leq 1000000 \) is goldbach; so 1000000 is an even integer \( \geq 4 \), and every even integer \( v \) of the form \( 4 \leq v \leq 1000000 \) is goldbach; consequently 1000000 is goldbachian]. It is immediate to see that if \( d \) is goldbachian and if \( d' \) is an even integer of the form \( 4 \leq d' \leq d \), then \( d' \) is also goldbachian. Example 5. Let the Mersenne composite \( M_{11} \) (see Abstract and definitions); then \( 2M_{11} + 2 \) is goldbachian [indeed \( 2M_{11} + 2 \) is an even integer such that \( 4 \leq 2M_{11} + 2 \leq 1000000 \); observing that 1000000 is goldbachian (use Example 4), then it becomes immediate to deduce that \( 2M_{11} + 2 \) is goldbachian (since 1000000 is goldbachian and \( 2M_{11} + 2 \) is an even integer such that \( 4 \leq 2M_{11} + 2 \leq 1000000 \)].

Definitions 2. For every integer \( n \geq 2 \), we define \( G'(n) \), \( g'_n \); \( H(n) \), \( h_n \), \( h_{n,1} \); \( MC(n) \), \( c_n \), \( c_{n,1} \); \( A(n) \), \( a_n \), \( a_{n,1} \); \( D(n) \), \( d_n \), and \( d_{n,1} \) as follows: \( G'(n) = \{ g' ; 1 < g' \leq 2n, \text{ and } g' \text{ is goldbachian} \} \), \( g'_n = \max_{g' \in G'(n)} g' \); \( H(n) = \{ x ; 1 < x < 2n \text{ and } x \text{ is a Sophie Germain prime} \} \), \( h_n = \max_{h \in H(n)} h \), and \( h_{n,1} = 4h_n^{b_n} \) [observing (see Abstract and Definitions) that 233 is a Sophie Germain prime, then it becomes immediate to deduce that for every integer \( n \geq 233 \), 233 \( \in H(n) \)] ; \( MC(n) = \{ x ; 1 <
\[ x < 2n \text{ and } x \text{ is a Mersenne composite} \} \), \( c_n = 2 \max_{c \in \mathcal{MC}(n)} c \), and \( c_{n,1} = 4c_n^{c_n} \) [observing (see Abstract and Definitions) that \( M_{11} \) is a Mersenne composite, then it becomes immediate to deduce that for every integer \( n \geq M_{11}, M_{11} \in \mathcal{MC}(n) \); \( A(n) = \{ x; 1 < x < 2n \text{ and } x \text{ is a friendly number} \} \), \( a_n = 2 \max_{a \in A(n)} a \), and \( a_{n,1} = 4a_n^{a_n} \) [observing (see Abstract and Definitions) that 284 is a friendly number, then it becomes immediate to deduce that for every integer \( n \geq 284, 284 \in A(n) \) ; \( D(n) = \{ x; 1 < x < 2n \text{ and } x \text{ is a perfect number} \} \), \( d_n = 2 \max_{d \in D(n)} d \), and \( d_{n,1} = 4d_n^{d_n} \) [observing (see Abstract and Definitions) that 496 is a perfect number, then it becomes immediate to deduce that for every integer \( n \geq 496, 496 \in D(n) \).

Using the previous definitions, let us define:

Definition 3 (Fundamental.1). For every integer \( n \geq 2 \), we put
\[ Z(n.1) = \{ h_{n,1} \} \bigcup \{ c_{n,1} \} \bigcup \{ a_{n,1} \} \bigcup \{ d_{n,1} \} , \]
where \( h_{n,1} \) and \( c_{n,1} \) and \( a_{n,1} \) and \( d_{n,1} \) are introduced in Definitions.2.

From Definition 3 and Definitions.2, then the following two assertions are immediate.

Assertion 1. Let \( n \) be an integer \( \geq 2 \). Then:
(1.0) \( g'_{n+1} \leq 2n + 2 \).
(1.1) \( g'_{n+1} < 2n + 2 \) if and only if \( g'_{n+1} = g'_n \).
(1.2) \( g'_{n+1} = 2n + 2 \) if and only if \( 2n + 2 \) is goldbachian.
(1.3) \( 2n + 2 \) is goldbachian if and only if \( 2n \) is goldbachian and \( 2n + 2 \) is goldbach.

Assertion 2. Let \( n \) be an integer \( \geq M_{11} \); consider \( z_{n,1} \in Z(n.1) \), and look at \( z_n \) [Example.6. If \( z_{n,1} = c_{n,1} \), then \( z_n = c_n \) and we are playing with the Mersenne composites; if \( z_{n,1} = a_{n,1} \), then \( z_n = a_n \) and we are playing with the friendly numbers; if \( z_{n,1} = d_{n,1} \), then \( z_n = d_n \) and we
are playing with the perfect numbers; and if $z_{n,1} = h_{n,1}$, then $z_n = h_n$ and we are playing with the Sophie Germain primes]. Then $232 < z_n < z_{n,1}$, and $z_{n,1} > 233^{233} > M_{11}$ and $z_{n-1,1} \leq z_{n,1}$.

Now, using the previous definitions, let $(Q.)$ be the following statement:

(Q.): For every integer $r \geq M_{11}$, one and only one of the following two properties $w(Q.r)$ and $x(Q.r)$ is satisfied.

$w(Q.r)$: $2r + 2$ is not goldbach.

$x(Q.r)$: For every $z_{r,1} \in Z(r,1)$, we have $z_{r,1} > g'_{r+1}$.

We will see that if for every integer $r \geq M_{11}$, property $x(Q.r)$ of statement (Q.) is satisfied, then the Sophie Germain primes problem, the Mersenne composites problem, the friendly numbers problem, and the perfect numbers problem are simultaneously special cases of the Goldbach conjecture. It is easy to see that property $x(Q.r)$ of statement (Q.) is satisfied for large values of $r \geq M_{11}$. In this paper, using only the immediate part of the generalized Fermat induction, we prove a Theorem which immediately implies the following result (E.):

(E.): Suppose that statement (Q.) is true. Then the Sophie Germain primes problem, the Merseenne composites problem, the friendly numbers problem, the perfect numbers problem and the Goldbach problem are simultaneously true.

Result (E.) helps us to explain why it is natural and not surprising to conjecture that the friendly numbers problem, the perfect numbers problem, the Mersenne composites problem, and the Sophie Germain primes problem are simultaneously special cases of the Goldbach problem.
3 The proof of Theorem which implies result (E.)

The following Theorem immediately implies result (E.) mentioned above.

**Theorem 1.** Let \((n, c(n))\) be a couple of integers such that \(n \geq M_{11}\) and \(c(n)\) is a cache of \(n\). Now suppose that statement (Q.) is true. We have the following.

(0.) If \(c(n) \equiv 0 \mod 2\), then \(2n + 2 - c(n)\) is goldbachian.

(1.) If \(c(n) \equiv 1 \mod 2\), then for every \(z_{n,1} \in \mathcal{Z}(n.1)\), we have \(z_{n,1} > 1 + g'_{n+1} - c(n)\).

To prove Theorem 1, we use:

**Lemma 1.** Let \((n, c(n))\) be a couple of integers, where \(c(n)\) is a cache of \(n\). Suppose that \(n = M_{11}\). Then Theorem 1 is contented.

**Proof.** We have to distinguish two cases (namely case where \(c(n)\) is even and case where \(c(n)\) is odd).

**Case 0.** \(c(n)\) is even. Clearly \(c(n) \equiv 0 \mod 2\) and we have to show that property (0.) of Theorem 1 is satisfied by the couple \((n, c(n))\). Recall \(n = M_{11}\) and so \(2n + 2 = 2M_{11} + 2\); observe that \(2M_{11} + 2\) is goldbachian (by Example 5); in particular \(2n + 2 - c(n)\) is goldbachian [use the definition of goldbachian (see Definitions 1) and note (by the previous) that \(2n + 2\) is goldbachian (since \(n = M_{11}\) and \(c(n) \in \{0, 2, \ldots, M_{11} - 1\}\)]. So property (0.) of Theorem 1 is satisfied by the couple \((n, c(n))\), and Theorem 1 is contented. Case 0 follows.

**Case 1.** \(c(n)\) is odd. Clearly \(c(n) \equiv 1 \mod 2\) and we have to show that property (1.) of Theorem 1 is satisfied by the couple \((n, c(n))\). Since \(n = M_{11}\) and since \(2M_{11} + 2\) is goldbachian (by Example 5), clearly \(g'_{n+1} = g'_{M_{11} + 1} = 2M_{11} + 2, 233 \in \mathcal{H}(n), h_n \geq 233, h_{n,1} > 233^{233} > 2M_{11} + 2; M_{11} \in \mathcal{MC}(n), c_n \geq M_{11}, c_{n,1} > M_{11}^M_{11} > 2M_{11} + 2; 284 \in \mathcal{A}(n), a_n \geq 284, a_{n,1} > 284^{284} > 2M_{11} + 2; 496 \in \mathcal{D}(n), d_n \geq 496, and d_{n,1} > 496^{496} > 2M_{11} + 2;\) clearly \(\mathcal{Z}(n.1) = \{h_{n,1}, c_{n,1}, a_{n,1}, d_{n,1}\}\), and using the
previous inequalities and the fact that \( g'_{n+1} = g'_{M_{11}+1} = 2M_{11} + 2 \), it becomes immediate to see that for every \( z_{n,1} \in \mathcal{Z}(n.1) \), we have \( z_{n,1} > g'_{n+1} \); in particular for every \( z_{n,1} \in \mathcal{Z}(n.1) \), we have \( z_{n,1} > 1 + g'_{n+1} - c(n) \). So property (1.) of Theorem 1 is satisfied by the couple \( (n, c(n)) \), and Theorem 1 is contented. Case 1 follows, and Lemma 1 immediately follows.

Using Lemma 1 and the meaning of Theorem 1 and the fact \( c(n,2) \in \{0,1\} \), then it becomes easy to see:

**Remark 1.** If Theorem 1 is false, then there exists \( (n, c(n)) \) such that \( (n, c(n)) \) is a counter-example with \( n \) minimum and \( c(n,2) \) maximum.

**Consequence 1.** (Application of Remark 1 and Lemma 1). Suppose that Theorem 1 is false, and let \( (n, c(n)) \) be a counter-example with \( n \) minimum and \( c(n,2) \) maximum. Then \( n \geq M_{11} + 1 \).

**Proof.** Clearly \( n \geq M_{11} + 1 \) [use Lemma 1].

**Remark 2.** Suppose that Theorem 1 is false, and let \( (n, c(n)) \) be a counter-example with \( n \) minimum and \( c(n,2) \) maximum. We have the following two simple properties (2.0) and (2.1).

(2.0) [The using of the minimality of \( n \)]. Put \( u = n - 1 \), then, for every \( z_{u,1} \in \mathcal{Z}(u.1) \), we have \( z_{u,1} > g'_{u+1} \).

Indeed, let \( u = n - 1 \) and let \( c(u) = j \), where \( j \in \{0,1\} \); now consider the couple \( (u, c(u)) \) [ note that \( u < n, u \geq M_{11} \) (use Consequence 1), \( c(u) \) is a cache of \( u \), and the couple \( (u, c(u)) \) clearly exists ]. Then, by the minimality of \( n \), the couple \( (u, c(u)) \) is not a counter-example to Theorem 1. Clearly \( c(u) \equiv j \mod 2 \) [ because \( c(u) = j \), where \( j \in \{0,1\} \)], and therefore property (j.) of Theorem 1 is satisfied by the couple \( (u, c(u)) \) [[ Example 7. If \( j = 0 \) (i.e. if \( c(u) = j = 0 \)), then property (0.) of Theorem 1 is satisfied by the couple \( (u, c(u)) \); so \( 2u + 2 \) is goldbachian.

Example 8. If \( j = 1 \) (i.e. if \( c(u) = j = 1 \)), then property (1.) of Theorem 1 is satisfied by the couple \( (u, c(u)) \); so, for every \( z_{u,1} \in \mathcal{Z}(u.1) \), we...
have $z_{u,1} > g'_{u+1}$.]

(2.1) [The using of the maximality of $c(n,2)$: the immediate part of the generalized Fermat induction]. If $c(n) \equiv 0 \text{mod}[2]$, then for every $z_{n,1} \in \mathcal{Z}(n.1)$, we have $z_{n,1} > g'_{n+1}$.

Indeed, if $c(n) \equiv 0 \text{mod}[2]$; clearly $c(n,2) = 0$. Now let the couple $(n, y(n))$ such that $y(n) = 1$. Clearly $y(n)$ is a cache of $n$ such that $y(n,2) = 1$ [note that $n \geq M_{11} + 1$ (use Consequence 1)]. Clearly $y(n,2) > c(n,2)$, where $y(n)$ and $c(n)$ are two caches of $n$ [since $c(n,2) = 0$ and $y(n,2) = 1$, by the previous]; then, by the maximality of $c(n,2)$, the couple $(n, y(n))$ is not a counter-example to Theorem 1 [because $(n, c(n))$ is a counter-example to Theorem 1 such that $n$ is minimum and $c(n,2)$ is maximum, and the couple $(n, y(n))$ is of the form $y(n,2) > c(n,2)$, where $y(n)$ and $c(n)$ are two caches]. Note that $y(n) \equiv 1 \text{mod}[2]$ [since $y(n) = 1$, by the definition of $y(n)$], and therefore, property (1.) of Theorem 1 is satisfied by the couple $(n, y(n))$; so, for every $z_{n,1} \in \mathcal{Z}(n.1)$, we have $z_{n,1} > 1 + g'_{n+1} - y(n)$, and clearly, for every $z_{n,1} \in \mathcal{Z}(n.1)$, we have $z_{n,1} > g'_{n+1}$ [because $y(n) = 1$].

Consequence 2. (Application of Remark 2). Suppose that Theorem 1 is false, and let $(n, c(n))$ be a counter-example with $n$ minimum and $c(n,2)$ maximum. Then we have the following four properties.

(2.0) $2n$ is goldbachian [i.e. $g'_{n} = 2n$].

(2.1) For every $z_{n-1.1} \in \mathcal{Z}(n-1.1)$, we have $z_{n-1.1} > g'_{n}$.

(2.2) For every $z_{n,1} \in \mathcal{Z}(n.1)$, we have $z_{n,1} > g'_{n}$.

(2.3) If $c(n) \equiv 0 \text{mod}[2]$, then $2n + 2$ is goldbach.

Proof. Property (2.0) is easy [indeed consider the couple $(u, c(u))$ such that $u = n - 1$ and $c(u) = 0$, and apply Example.7 of property (2.0) of Remark 2 ]; property (2.1) is also easy [indeed consider the couple $(u, c(u))$ such that $u = n - 1$ and $c(u) = 1$, and apply Example.8 of property (2.0) of Remark 2 ]; and property (2.2) is an immediate consequence of property (2.1) via Assertion 2 [indeed, note that $z_{n-1.1} \leq z_{n,1}$,
by using Assertion 2]. Now to prove Consequence 2 it suffices to show property (2.3). **Fact:** $2n + 2$ is goldbach. Indeed, observing [ by using property (2.1) of Remark 2] that for every $z_{n,1} \in \mathcal{Z}(n,1)$, we have $z_{n,1} > g_{n+1}'$, clearly property $x(Q,n)$ of statement $\lfloor Q \rfloor$ is satisfied, and recalling that statement $\lfloor Q \rfloor$ is true, then we immediately deduce that property $w(Q,n)$ of statement $\lfloor Q \rfloor$ is not satisfied; therefore $2n + 2$ is goldbach.

**Proof of Theorem 1.** We reason by reduction to absurd. Suppose that Theorem 1 is false and let $(n, c(n))$ be a counter-example with $n$ **minimum** and $c(n,2)$ **maximum** [such a couple exists, by Remark 1]. Then we observe the following.

**Observation 1.0.** $c(n) \not\equiv 0 \mod[2]$.

Otherwise,

$$c(n) \equiv 0 \mod[2] \tag{1},$$

and clearly

$$2n + 2 - c(n) \text{ is not goldbachian} \tag{2}$$

[indeed note $c(n) \equiv 0 \mod[2]$ [by congruence (1)], and in particular, property (0.) of Theorem 1 is not satisfied by the couple $(n, c(n))$; so $2n + 2 - c(n) \text{ is not goldbachian}$. (2) immediately implies that

$$2n + 2 \text{ is not goldbachian} \tag{3}$$

[ indeed, recalling that $c(n)$ is a cache of $n$ such that $c(n) \equiv 0 \mod[2]$ [by congruence (1)], clearly $c(n) \geq 0$ and $2n + 2 - c(n) \geq 4$ [note that $n \geq M_{11} + 1$, by Consequence 1]; now using the previous and the definition of goldbachian via (2), then we immediately deduce that $2n + 2$ is not goldbachian]. Now we have the following two simple **Facts**.

**Fact 1.0.0.** $g'_{n+1} = g'_n$. Indeed, observing [via (3)] that $2n + 2$ is
not goldbachian, clearly $g_{n+1}' < 2n + 2$ [use the definition of $g_{n+1}'$ via the definition of $g_n'$, and observe (by the previous) that $2n + 2$ is not goldbachian] and property (1.1) of Assertion 1 implies that $g_{n+1}' = g_n'$.

Fact 1.0.1. $2n + 2$ is not goldbach. Otherwise, observing [via property (2.0) of Consequence 2] that $2n$ is goldbachian, then, using the previous, it immediately follows that $2n + 2$ is goldbach and $2n$ is goldbachian; consequently $2n + 2$ is goldbachian [use the fact that $2n + 2$ is goldbach and $2n$ is goldbachian and apply property (1.3) of Assertion 1], and this contradicts (3). The Fact 1.0.1 follows.

These two simple Facts made, observing [by Fact 1.0.1] that $2n + 2$ is not goldbach, clearly property $w(Q_n)$ of statement $(Q_n)$ is satisfied, and recalling that statement $(Q_n)$ is true, then we immediately deduce that property $x(Q_n)$ of statement $(Q_n)$ is not satisfied; therefore

$$\text{there exists } z_{n,1} \in \mathcal{Z}(n,1) \text{ such that } z_{n,1} \leq g_{n+1}' \quad (4).$$

Now using Fact 1.0.0, then (4) immediately implies that there exists $z_{n,1} \in \mathcal{Z}(n,1)$ such that $z_{n,1} \leq g_n'$, and this contradicts property (2.2) of Consequence 2. Observation 1.0 follows.

Observation 1.0 implies that

$$c(n) \equiv 1 \mod[2] \quad (5),$$

and clearly

$$\text{there exists } z_{n,1} \in \mathcal{Z}(n,1) \text{ such that } z_{n,1} \leq g_{n+1}' \quad (6)$$

[ indeed note $c(n) \equiv 1 \mod[2]$ (by congruence (5)), and in particular, property (1.) of Theorem 1 is not satisfied by the couple $(n, c(n))$; so there exists $z_{n,1} \in \mathcal{Z}(n,1)$ such that $z_{n,1} \leq 1 + g_{n+1}' - c(n)$, and consequently, there exists $z_{n,1} \in \mathcal{Z}(n,1)$ such that $z_{n,1} \leq g_{n+1}'$, because $c(n) \geq 1$ (since $c(n) \equiv 1 \mod[2]$ [by congruence (5)], and $c(n)$ is a cache of $n$)]. (6) clearly says that property $x(Q_n)$ of statement $(Q_n)$ is not
satisfied, and recalling that statement \((Q.)\) is true, then we immediately deduce that property \(w(Q.n)\) of statement \((Q.)\) is satisfied; therefore

\[
2n + 2 \text{ is not goldbach} \quad (7).
\]

(7) immediately implies that \(g'_{n+1} < 2n + 2\); now using property (1.1) of Assertion 1 and the previous inequality, we immediately deduce that

\[
g'_{n+1} = g'_n \quad (8).
\]

Now using equality (8), then (6) clearly says that there exists \(z_{n,1} \in \mathbb{Z}(n.1)\) such that \(z_{n,1} \leq g'_n\), and this contradicts property (2.2) of Consequence 2. Theorem 1 follows.

**Remark 3.** Note that to prove Theorem 1, we consider a couple \((n, c(n))\) such that \((n, c(n))\) is a counter-example with \(n\) minimum and \(c(n, 2)\) maximum. In properties (2.0),(2.1), and (2.2) of Consequence 2 (via property (2.0) of Remark 2), the minimality of \(n\) is used; and in property (2.3) of Consequence 2 (via property (2.1) of Remark 2), the maximality of \(c(n, 2)\) is used. Consequence 2 helps us to give a simple and detailed proof of Theorem 1.

**Corollary 1.** Suppose that statement \((Q.)\) is true. Then we have the following four properties.

1.0) For every integer \(n \geq 1\), \(2n + 2\) is goldbachian [i.e. \(g'_{n+1} = 2n + 2\)].

1.1) The Goldbach conjecture is true.

1.2) For every integer \(n \geq M_{11}\), and for every \(z_{n,1} \in \mathbb{Z}(n.1)\), we have \(z_{n,1} > 2n + 2\).

1.3) The Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

**Proof.** (1.0). It is immediate if \(n \in \{1, 2, ..., M_{11}\}\) (since \(2M_{11} + 2\) is goldbachian, by using Example.5 given in Definitions.1). If \(n \geq M_{11} + 1\),
consider the couple \((n, c(n))\) with \(c(n) = 0\). The couple \((n, c(n))\) is of the form \(0 \leq c(n) < n\), where \(n \geq M_{11}\), \(c(n) \equiv 0 \mod{2}\), and \(c(n)\) is a cache of \(n\). Then property (0.) of Theorem 1 is satisfied by the couple \((n, c(n))\). So \(2n + 2\) is goldbachian [because \(c(n) = 0\)], and consequently \(g'_{n+1} = 2n + 2\).

(1.1). Indeed, the Goldbach conjecture immediately follows, by using property (1.0).

(1.2). Let the couple \((n, c(n))\) be such that \(c(n) = 1\). The couple \((n, c(n))\) is of the form \(0 \leq c(n) < n\), where \(n \geq M_{11}\), \(c(n) \equiv 1 \mod{2}\), and \(c(n)\) is a cache of \(n\). Then property (1.) of Theorem 1 is satisfied by the couple \((n, c(n))\). So, for every \(z_{n,1} \in \mathcal{Z}(n,1)\), we have \(z_{n,1} > g'_{n+1}\) [because \(c(n) = 1\)]; now observing [by property (1.0)] that \(g'_{n+1} = 2n + 2\), then, we immediately deduce that for every \(z_{n,1} \in \mathcal{Z}(n,1)\), we have \(z_{n,1} > 2n + 2\).

(1.3). Indeed, the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite, by using property (1.2) and the definition of \(\mathcal{Z}(n,1)\) (see Definition 3 for the meaning of \(\mathcal{Z}(n,1)\)).

Using property (1.1) and property (1.3) of Corollary 1, then the following result (E.) becomes immediate.

**Result (E.).** Suppose that statement (Q.) is true. Then, the Goldbach conjecture is true, and moreover, the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

**Conjecture 1.** Statement (Q.) is true.

4 Epilogue.

To conjecture that the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture is not surprising. Indeed, let (Q'.) be the following statement:
(Q'): For every integer \( r \geq M_{11} \), at most one of the following two properties \( w(Q'.r) \) and \( x(Q'.r) \) is true.

- \( w(Q'.r) \): \( 2r + 2 \) is not goldbach.
- \( x(P'.r) \): For every \( z_{r,1} \in \mathbb{Z}(r.1) \), we have \( z_{r,1} > g'_{r+1} \).

Note that statement (Q'), somewhere, resembles to statement (Q). More precisely, statement (Q) implies statement (Q'). [Proof. In particular, the Goldbach conjecture is true [use property (1.1) of Corollary 1]; consequently, statement (Q') is true [use definition of statement (Q') and the previous].]

Conjecture.2. Statement (Q) and statement (Q') are equivalent.

Conjecture.2 implies that the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture.

Proof. Suppose that conjecture.2 is true. If the Goldbach conjecture is true, clearly statement (Q') is true; observing that statement (Q') and statement (Q) are equivalent, then (Q) is true, and result (E) implies that the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

Conjecture.3. Suppose that statement (Q') is true. Then the Goldbach conjecture is true, and moreover, the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

Conjecture.3 immediately implies that the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture.

Proof. Suppose that conjecture.3 is true. If the Goldbach conjecture is true, clearly statement (Q') is true, and in particular the Sophie Ger-
main primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

**Conjecture.4.** For every integer \( r \geq M_{11} \), property \( x(Q',r) \) of statement \( (Q') \) is true [note that property \( x(Q',r) \) of statement \( (Q') \) is exactly property \( x(Q',r) \) of statement \( (Q) \); moreover, it is immediate to see that property \( x(Q',r) \) of statement \( (Q') \) is satisfied for large values of \( r \geq M_{11} \)].

Conjecture.4 also implies that the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture.

**Proof.** Suppose that conjecture.4 is true. If the Goldbach conjecture is true, clearly, \( g'_{n+1} = 2n + 2 \), and so for every \( z_{n,1} \in \mathbb{Z}(n,1) \), we have

\[
z_{n,1} > g'_{n+1} > 2n \quad (9)
\]

Observing that (9) is true for every integer \( n \geq M_{11} \), then in particular, it results that the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

Now, using conjecture.2 and conjecture.3 and conjecture.4, it becomes natural and not surprising to conjecture the following:

**Conjecture.5.** The Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture. More precisely, the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are protected by the umbrella of Goldbach.

**Conjecture.6.** Let \((n,b(n))\) be a couple of integers such that \( n \geq M_{11} \) and \( 0 \leq b(n) < n \). We have the following.
(0.) If \( b(n) \equiv 0 \text{mod}[4] \); then \( 2n + 2 - b(n) \) is goldbachian.
(1.) If \( b(n) \equiv 1 \text{mod}[4] \); then \( h_{n,1} > 1 + g'_{n+1} - b(n) \) and \( c_{n,1} > 1 + g'_{n+1} - b(n) \).
(2.) If \( b(n) \equiv 2 \text{mod}[4] \); then \( a_{n,1} > 2 + g'_{n+1} - b(n) \).
(3.) If \( b(n) \equiv 3 \text{mod}[4] \); then \( d_{n,1} > 3 + g'_{n+1} - b(n) \).

It is easy to see that conjecture.6 simultaneously implies that: not only the Goldbach conjecture is true, but the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite, and to attack this conjecture, we must consider the generalized Fermat induction.

References

A curious synopsis on the Goldbach conjecture ...
Voll (2); 2012, 68 – 80.

**Ikorong Anouk Gilbert Nemron**
Université’ Pierre et Marie Curie (Paris VI) France Centre de Calcul, D’Enseignement et de Recherche e-mail: ikorong@ccr.jussieu.fr