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A curious synopsis on the Goldbach conjecture, the friendly numbers, the perfect numbers, the Mersenne composite numbers and the Sophie Germain primes [1]

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Abstract

The notion of a *friendly number* (or *amicable number*), (see [2] or [10] or [18] or [19]) is based on the idea that a human friend is a kind of alter ego. Indeed, Pythagoras wrote (see [18] or [19]): *A friend is the other I, such as are* 220 *and* 284. These numbers have a special mathematical property: each is equal to the sum of the other's proper divisors *(divisors other than the number itself)*. The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110, and they sum to 284; the proper divisors of 284 are 1, 2, 4, 71, and 142, and they sum to 220. So $\{220, 284\}$ is called a pair of *friendly* numbers [note $\{17296, 18416\}$ is also a pair of *friendly* numbers (see [18] or [19])]. More precisely, we say that a number a' is a *friendly number* or *amicable number*, if there exists a number $a'' \neq a'$ such that $\{a', a''\}$ is a pair of *friendly* numbers [example. 220, 284, 17296 and 18416 are *friendly* numbers]. It is trivial to see that a *friendly* number is a *composite* number (we

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recall that *a composite number* is a non prime number. Primes are well known (see [1] or [2] or [3] or [4]), and original characterizations of primes via divisibility is given in [15] and [16] and [17] $\Big)$, and the *friendly* numbers problem states that there are infinitely many friendly numbers. Pythagoras saw *perfection* in any integer that equaled the sum of all the other integers that divided evenly into it (see [2] or [10] or [17] or [18] or [19]). The first perfect number is 6. It's evenly divisible by 1, 2, and 3, and it's also the sum of 1, 2, and 3, [note 28 and 496 and 33550336 are also perfect numbers (see [18] or [19])]. It is immediate that a *perfect* number is a *composite* number, and the *perfect* numbers problem states that there are infinitely many perfect numbers. It is trivial to see that an integer is perfect if and only if this integer is *self-amicable* (perfect numbers are characterized via divisibility in [15] and [17]). A *Mersenne composite* is a non prime number of the form $M_m = 2^m - 1$, where m is prime (Mersenne composites were characterized via divisibility in [15] and [17]). It is known (see [2] or [3] or [4] or [10] or [18] or [19]) that M_{11} and M_{67} are Mersenne composites. A prime h is called a *Sophie Germain prime* (see [10]), if both h and $2h + 1$ are prime; the first few Sophie Germain primes are $2, 3, 5, 11, 23, 29, 41, ...$, and it is easy to check that 233 is a Sophie Germain prime. Sophie Germain primes are known for some integers > 233 $\Big($ For original characterizations of Sophie Germain primes via divisibility, see [15] and [17] $\Big)$. The Sophie Germain primes problem asserts that there are infinitely many Sophie Germain primes. That being said, in this paper, we state a simple conjecture **(Q.)**, we generalize the Fermat induction, and we use only the immediate part of the generalized Fermat induction to give a simple and detailed proof that **(Q.)** is stronger than the Goldbach problem, the friendly numbers problem, the perfect numbers problem, the Mersenne composites problem and the Sophie Gernain primes problem; *this helps us to explain why it is natural and not surprising to conjecture that*

the friendly numbers problem, the perfect numbers problem, the Mersenne composites problem, and the Sophie Germain primes problem are simultaneously special cases of the Goldbach problem $\Big($ we recall (see [1] or [5] or [6] or [7] or [8] or [9] or [11] or [13] or [14])$\Big)$ that the Goldbach problem states that every even integer $e \geq 4$ is of the form $e = p + q$, where (p, q) is a couple of prime(s).

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1 Prologue.

Briefly, the immediate part of the generalized Fermat induction is based around the following simple definitions. Let n be an integer ≥ 2, we say that $c(n)$ is a *cache* of n, if $c(n)$ is an integer of the form $0 \leq c(n) < n$ [Example.**1** If $n = 4$, then $c(n)$ is a cache of n if and only if $c(n) \in \{0, 1, 2, 3\}$]. Now, for every couple of integers $(n, c(n))$ such that $n \geq 2$ and $0 \leq c(n) < n$ [observe that $c(n)$ is a cache of n], we define $c(n, 2)$ as follows: $c(n, 2) = 1$ **if** $c(n) \equiv 1 mod[2]$; and $c(n, 2) = 0$ **if** $c(n) \not\equiv 1 mod[2]$. It is immediate that $c(n, 2)$ exists and is well defined, since $n \geq 2$ [Example.**2** If $n = 6$, then $c(n, 2) = 0$ if $c(n) \in \{0, 2, 4\}$ and $c(n, 2) = 1$ if $c(n) \in \{1, 3, 5\}$]. In this paper, induction will be made on n and $c(n, 2)$ [where n is an integer ≥ 2 and $c(n)$ is a cache of n].

2 Introduction and non-standard definitions.

Definitions.1. We say that e is *goldbach*, if e is an even integer ≥ 4 and is of the form $e = p + q$, where (p, q) is a couple of prime(s). The

Goldbach problem states that every even integer $e \geq 4$ is goldbach. We say that e is *goldbachian*, if e is an even integer ≥ 4, and if every even integer v with $4 \leq v \leq e$ is goldbach [there is no confusion between *goldbach* and *goldbachian*, since *goldbachian* clearly implies *goldbach*]. $\Big($ Example.**3.** 12 is *golbachian* [indeed, 12 and 10 and 8 and 6 and 4 are *goldbach* (note $4 = 2 + 2$, where 2 is prime; $6 = 3 + 3$, where 3 is prime; $8 = 5 + 3$ where 3 and 5 are prime; $10 = 7 + 3$, where 3 and 7 are prime, and $12 = 5 + 7$, where 5 and 7 are prime); so 12 is an even integer ≥ 4, and every even integer v of the form $4 \leq v \leq 12$ is *goldbach*; consequently 12 is *goldbachian*]. Since 12 is *goldbachian*, then it becomes trivial to deduce that 10 and 8 and 6 and 4 are *goldbachian*. Example.**4.** 1000000 is *goldbachian* [indeed, it known that every even integer v of the form $4 \leq v \leq 1000000$ is *goldbach*; so 1000000 is an even integer ≥ 4, and every even integer v of the form $4 \leq v \leq 1000000$ is *goldbach*; consequently 1000000 is *goldbachian*]. It is immediate to see that if d is *goldbachian* and if d' is an even integer of the form $4 \leq d' \leq d$, then d' is also *goldbachian*. Example.**5.** Let the Mersenne composite M_{11} (see **Abstract and definitions**); then $2M_{11} + 2$ is *goldbachian* [indeed $2M_{11} + 2$ is an even integer such that $4 \leq 2M_{11} + 2 \leq 1000000$; observing that 1000000 is *goldbachian* (use Example.**4**), then it becomes immediate to deduce that $2M_{11} + 2$ is *goldbachian* (since 1000000 is *goldbachian* and $2M_{11} + 2$ is an even integer such that $4 \leq 2M_{11} + 2 \leq 1000000$) $\Big)$].

Definitions.2. For every integer $n \geq 2$, we define $\mathcal{G}'(n)$, g'_n ; $\mathcal{H}(n)$, h_n, $h_{n.1}$; $\mathcal{MC}(n)$, c_n, $c_{n.1}$; $\mathcal{A}(n)$, a_n, $a_{n.1}$; $\mathcal{D}(n)$, d_n, and $d_{n.1}$ as follows: $\mathcal{G}'(n) = \{g'; 1 < g' \leq 2n, \mathit{and}\, g'\, \mathit{is}\, \mathit{goldbachian}\}$, $g'_n = \max_{g' \in \mathcal{G}'(n)} g'$; $\mathcal{H}(n) = \{x; 1 < x < 2n\, \mathit{and}\, x\, \mathit{is}\, \mathit{a}\, \mathit{Sophie}\, \mathit{Germain}\, \mathit{prime}\}$, $h_n = 2 \max_{h \in \mathcal{H}(n)} h$, and $h_{n.1} = 4h_n^{h_n}$ [observing (see **Abstract and Definitions**) that 233 is a Sophie Germain prime, then it becomes immediate to deduce that for every integer $n \geq 233$, $233 \in \mathcal{H}(n)$] ; $\mathcal{MC}(n) = \{x; 1 <$

$x < 2n \, and \, x \, is \, a \, Mersenne \, composite\}$, $c_n = 2 \max_{c \in \mathcal{MC}(n)} c$, and $c_{n.1} = 4c_n^{c_n}$ [observing (see **Abstract and Definitions**) that M_{11} is a Mersenne composite, then it becomes immediate to deduce that for every integer $n \geq M_{11}$, $M_{11} \in \mathcal{MC}(n)$]; $\mathcal{A}(n) = \{x; 1 < x < 2n \, and \, x \, is \, a \, friendly \;\; number\,\}$, $a_n = 2 \max_{a \in \mathcal{A}(n)} a$, and $a_{n.1} = 4a_n^{a_n}$ [observing (see **Abstract and Definitions**) that 284 is a friendly number, then it becomes immediate to deduce that for every integer $n \geq 284$, $284 \in \mathcal{A}(n)$] ; $\mathcal{D}(n) = \{x; 1 < x < 2n \, and \, x \, is \, a \, perfect \, number\}$, $d_n = 2 \max_{d \in \mathcal{D}(n)} d$, and $d_{n.1} = 4d_n^{d_n}$ [observing (see **Abstract and Definitions**) that 496 is a perfect number, then it becomes immediate to deduce that for every integer $n \geq 496$, $496 \in \mathcal{D}(n)$].

Using the previous definitions, let us define:

Definition.3 *(Fundamental.1).* For every integer $n \geq 2$, we put

$$\mathcal{Z}(n.1) = \{h_{n.1}\} \bigcup \{c_{n.1}\} \bigcup \{a_{n.1}\} \bigcup \{d_{n.1}\},$$

where $h_{n,1}$ and $c_{n.1}$ and $a_{n.1}$ and $d_{n.1}$ are introduced in Definitions.**2**.

From Definition.**3** and Definitions.**2**, then the following two assertions are immediate.

Assertion 1. *Let n be an integer ≥ 2. Then:*
(1.0) $g'_{n+1} \leq 2n + 2$.
(1.1) $g'_{n+1} < 2n + 2$ if and only if $g'_{n+1} = g'_n$.
(1.2) $g'_{n+1} = 2n + 2$ if and only if $2n + 2$ is goldbachian.
(1.3) $2n+2$ is goldbachian if and only if $2n$ is goldbachian and $2n+2$ is goldbach.

Assertion 2. *Let n be an integer $\geq M_{11}$; consider $z_{n.1} \in \mathcal{Z}(n.1)$, and look at z_n [Example.*__6__*. If $z_{n.1} = c_{n.1}$, then $z_n = c_n$ and we are playing with the Mersenne composites; if $z_{n.1} = a_{n.1}$, then $z_n = a_n$ and we are playing with the friendly numbers; if $z_{n.1} = d_{n,1}$, then $z_n = d_n$ and we* *are playing with the perfect numbers; and if* $z_{n.1} = h_{n.1}$*, then* $z_n = h_n$ *and we are playing with the Sophie Germain primes]. Then* $232 < z_n < z_{n.1}$*, and* $z_{n.1} > 233^{233} > M_{11}$ *and* $z_{n-1.1} \leq z_{n.1}$.

Now, using the previous definitions, let **(Q.)** be the following statement:

(Q.): For every integer $r \geq M_{11}$, **one and only one** of the following two properties $\mathbf{w(Q.}r\mathbf{)}$ and $\mathbf{x(Q.}r\mathbf{)}$ is satisfied.

$\mathbf{w(Q.}r\mathbf{)}$**:** $2r + 2$ is not goldbach.

$\mathbf{x(Q.}r\mathbf{)}$**:** For every $z_{r.1} \in \mathcal{Z}(r.1)$, we have $z_{r.1} > g'_{r+1}$.

We will see that if for every integer $r \geq M_{11}$, property $\mathbf{x(Q.}r\mathbf{)}$ of statement **(Q.)** is satisfied, then the *Sophie Germain primes problem, the Mersenne composites problem, the friendly numbers problem, and the perfect numbers problem are simultaneously special cases of the Goldbach conjecture*. It is easy to see that property $\mathbf{x(Q.}r\mathbf{)}$ of statement **(Q.)** is satisfied for large values of $r \geq M_{11}$. In this paper, using only the immediate part of the generalized Fermat induction, we prove a Theorem which immediately implies the following result **(E.)**:

(E.): Suppose that statement **(Q.)** is true. Then the Sophie Germain primes problem, the Mersenne composites problem, the friendly numbers problem, the perfect numbers problem and the Goldbach problem are simultaneously true.

Result **(E.)** *helps us to explain why it is natural and not surprising to conjecture that the friendly numbers problem, the perfect numbers problem, the Mersenne composites problem, and the Sophie Germain primes problem are simultaneously special cases of the Goldbach problem.*

3 The proof of Theorem which implies result (E.)

The following Theorem immediately implies result **(E.)** mentionned above.
Theorem 1. *Let $(n, c(n))$ be a couple of integers such that $n \geq M_{11}$ and $c(n)$ is a cache of n. Now suppose that statement* **(Q.)** *is true. We have the following.*
(0.) *If $c(n) \equiv 0 mod[2]$, then $2n + 2 - c(n)$ is goldbachian.*
(1.) *If $c(n) \equiv 1 mod[2]$, then for every $z_{n.1} \in \mathcal{Z}(n.1)$, we have $z_{n.1} > 1 + g'_{n+1} - c(n)$.*

To prove Theorem **1**, we use:

Lemma 1. *Let $(n, c(n))$ be a couple of integers, where $c(n)$ is a cache of n. Suppose that $n = M_{11}$. Then Theorem* **1** *is contented.*
Proof. We have to distinguish two cases (namely case where $c(n)$ is even and case where $c(n)$ is odd).
Case.0. $c(n)$ is even . Clearly $c(n) \equiv 0 mod[2]$ and we have to show that property **(0.)** of Theorem **1** is satisfied by the couple $(n, c(n))$. Recall $n = M_{11}$ and so $2n + 2 = 2M_{11} + 2$; observe that $2M_{11} + 2$ is goldbachian (by Example.**5**); in particular $2n + 2 - c(n)$ is goldbachian [use the definition of goldbachian (see Definitions.**1**) and note (by the previous) that $2n + 2$ is goldbachian (since $n = M_{11}$) and $c(n) \in \{0, 2, ...M_{11} - 1\}$]. So property **(0.)** of Theorem **1** is satisfied by the couple $(n, c(n))$, and Theorem **1** is contented. Case.0 follows.
Case.1. $c(n)$ is odd. Clearly $c(n) \equiv 1 mod[2]$ and we have to show that property **(1.)** of Theorem **1** is satisfied by the couple $(n, c(n))$. Since $n = M_{11}$ and since $2M_{11}+2$ is goldbachian (by Example.**5**), clearly $g'_{n+1} = g'_{M_{11}+1} = 2M_{11} + 2$, $233 \in \mathcal{H}(n)$, $h_n \geq 233$, $h_{n.1} > 233^{233} > 2M_{11} + 2$; $M_{11} \in \mathcal{MC}(n)$, $c_n \geq M_{11}$, $c_{n.1} > M_{11}^{M_{11}} > 2M_{11} + 2$; $284 \in \mathcal{A}(n)$, $a_n \geq 284$, $a_{n.1} > 284^{284} > 2M_{11} + 2$; $496 \in \mathcal{D}(n)$, $d_n \geq 496$, and $d_{n.1} > 496^{496} > 2M_{11} + 2$; clearly $\mathcal{Z}(n.1) = \{h_{n.1}, c_{n.1}, a_{n.1}, d_{n.1}\}$, and using the

previous inequalities and the fact that $g'_{n+1} = g'_{M_{11}+1} = 2M_{11} + 2$, it becomes immediate to see that for every $z_{n.1} \in \mathcal{Z}(n.1)$, we have $z_{n.1} > g'_{n+1}$; in particular for every $z_{n.1} \in \mathcal{Z}(n.1)$, we have $z_{n.1} > 1 + g'_{n+1} - c(n)$. So property **(1.)** of Theorem.**1** is satisfied by the couple $(n, c(n))$, and Theorem.**1** is contented. Case.1 follows, and Lemma **1** immediately follows.

Using Lemma **1** and the meaning of Theorem **1** and the fact $c(n, 2) \in \{0, 1\}$, then it becomes easy to see:

Remark 1. *If Theorem* **1** *is false, then there exists $(n, c(n))$ such that $(n, c(n))$ is a counter-example with n* **minimum** *and $c(n, 2)$* **maximum**.

Consequence 1. *(Application of Remark* **1** *and Lemma* **1***). Suppose that Theorem* **1** *is false, and let $(n, c(n))$ be a counter-example with n* **minimum** *and $c(n, 2)$* **maximum**. *Then $n \geq M_{11} + 1$.*
Proof. Clearly $n \geq M_{11} + 1$ [use Lemma **1**].

Remark 2. *Suppose that Theorem* **1** *is false, and let $(n, c(n))$ be a counter-example with n* **minimum** *and $c(n, 2)$* **maximum**. *We have the following two simple properties (2.0) and (2.1).*
(2.0) [The using of the **minimality** *of n]. Put $u = n - 1$, then, for every $z_{u.1} \in \mathcal{Z}(u.1)$, we have $z_{u.1} > g'_{u+1}$.*

Indeed, let $u = n - 1$ and let $c(u) = j$, where $j \in \{0, 1\}$; now consider the couple $(u, c(u))$ [note that $u < n$, $u \geq M_{11}$ (use Consequence **1***), $c(u)$ is a cache of u, and the couple $(u, c(u))$ clearly exists]. Then, by the* **minimality** *of n, the couple $(u, c(u))$ is not a counter-example to Theorem* **1**. *Clearly $c(u) \equiv j mod[2]$ [because $c(u) = j$, where $j \in \{0, 1\}$], and therefore property* **(j.)** *of Theorem* **1** *is satisfied by the couple $(u, c(u))$ [[Example.***7**. *If $j = 0$ (i.e. if $c(u) = j = 0$), then property* **(0.)** *of Theorem* **1** *is satisfied by the couple $(u, c(u))$; so $2u + 2$ is goldbachian. Example.***8**. *If $j = 1$ (i.e. if $c(u) = j = 1$), then property* **(1.)** *of Theorem* **1** *is satisfied by the couple $(u, c(u))$; so, for every $z_{u.1} \in \mathcal{Z}(u.1)$, we* *have* $z_{u.1} > g'_{u+1}$*]].*

*(*2.1*) [The using of the* **maximality** *of* $c(n, 2)$*: the immediate part of the generalized Fermat induction]. If* $c(n) \equiv 0 mod[2]$*, then for every* $z_{n.1} \in \mathcal{Z}(n.1)$*, we have* $z_{n.1} > g'_{n+1}$*.*

Indeed, if $c(n) \equiv 0 mod[2]$*; clearly* $c(n, 2) = 0$*. Now let the couple* $(n, y(n))$ *such that* $y(n) = 1$*. Clearly* $y(n)$ *is a cache of* n *such that* $y(n, 2) = 1$ *[note that* $n \geq M_{11} + 1$ *(use Consequence* **1***)]. Clearly* $y(n, 2) > c(n, 2)$*, where* $y(n)$ *and* $c(n)$ **are two caches** *of* n *[since* $c(n, 2) = 0$ *and* $y(n, 2) = 1$*, by the previous]; then, by the* **maximality** *of* $c(n, 2)$*, the couple* $(n, y(n))$ *is not a counter-example to Theorem* **1** *[because* $(n, c(n))$ *is a counter-example to Theorem* **1** *such that* n *is minimum and* $c(n, 2)$ *is maximum, and the couple* $(n, y(n))$ *is of the form* $y(n, 2) > c(n, 2)$*, where* $y(n)$ *and* $c(n)$ **are two caches***]. Note that* $y(n) \equiv 1 mod[2]$ *[since* $y(n) = 1$*, by the definition of* $y(n)$*] , and therefore, property* **(1.)** *of Theorem* **1** *is satisfied by the couple* $(n, y(n))$*; so, for every* $z_{n.1} \in \mathcal{Z}(n.1)$*, we have* $z_{n.1} > 1 + g'_{n+1} - y(n)$*, and clearly, for every* $z_{n.1} \in \mathcal{Z}(n.1)$*, we have* $z_{n.1} > g'_{n+1}$ *[because* $y(n) = 1$*].*

Consequence 2. *(Application of Remark* **2***). Suppose that Theorem* **1** *is false, and let* $(n, c(n))$ *be a counter-example with* n **minimum** *and* $c(n, 2)$ **maximum**. *Then we have the following four properties.*

*(*2.0*)* $2n$ *is goldbachian [i.e.* $g'_n = 2n$*].*

*(*2.1*) For every* $z_{n-1.1} \in \mathcal{Z}(n - 1.1)$*, we have* $z_{n-1.1} > g'_n$*.*

*(*2.2*) For every* $z_{n.1} \in \mathcal{Z}(n.1)$*, we have* $z_{n.1} > g'_n$*.*

*(*2.3*) If* $c(n) \equiv 0 mod[2]$*, then* $2n + 2$ *is goldbach.*

Proof. Property (2.0) is easy [indeed consider the couple $(u, c(u))$ such that $u = n - 1$ and $c(u) = 0$, and apply Example.**7** of property (2.0) of Remark **2**]; property (2.1) is also easy [indeed consider the couple $(u, c(u))$ such that $u = n - 1$ and $c(u) = 1$, and apply Example.**8** of property (2.0) of Remark **2**]; and property (2.2) is an immediate consequence of property (2.1) via Assertion **2** [indeed, note that $z_{n-1.1} \leq z_{n.1}$, by using Assertion **2**]. Now to prove Consequence **2** it suffices to show property (2.3). *Fact: $2n+2$ is goldbach.* Indeed, observing [by using property (2.1) of Remark **2**] that for every $z_{n.1} \in \mathcal{Z}(n.1)$, we have $z_{n.1} > g'_{n+1}$, clearly property $\mathbf{x(Q.}n\mathbf{)}$ of statement **(Q.)** is satisfied, and recalling that statement **(Q.)** is true, then we immediately deduce that property $\mathbf{w(Q.}n\mathbf{)}$ of statement **(Q.)** is not satisfied; therefore $2n+2$ is goldbach.

Proof of Theorem 1. We reason by reduction to absurd. Suppose that Theorem **1** is false and let $(n, c(n))$ be a counter-example with n **minimum** and $c(n,2)$ **maximum** [such a couple exists, by Remark.**1**]. Then we observe the following.

*Observation.*1.0. $c(n) \not\equiv 0mod[2]$.

Otherwise,

$$c(n) \equiv 0mod[2] \qquad (1),$$

and clearly

$$2n+2-c(n) \textit{ is not goldbachian} \qquad (2)$$

[indeed note $c(n) \equiv 0mod[2]$ [by congruence *(1)*], and in particular, property **(0.)** of Theorem **1** is not satisfied by the couple $(n, c(n))$; so $2n+2-c(n)$ *is not goldbachian*]. *(2)* immediately implies that

$$2n+2 \textit{ is not goldbachian} \qquad (3)$$

[indeed, recalling that $c(n)$ is a cache of n such that $c(n) \equiv 0mod[2]$ [by congruence *(1)*], clearly $c(n) \geq 0$ and $2n+2-c(n) \geq 4$ [note that $n \geq M_{11}+1$, by Consequence **1**]; now using the previous and the definition of goldbachian via *(2)*, then we immediately deduce that $2n+2$ is not goldbachian]. Now we have the following two simple *Facts*.

Fact.1.0.0. $g'_{n+1} = g'_n$. Indeed, observing [via *(3)*] that $2n+2$ is

not goldbachian, clearly $g'_{n+1} < 2n+2$ [use the definition of g'_{n+1} via the definition of g'_n, and observe (by the previous) that $2n+2$ is not goldbachian] and property (1.1) of Assertion **1** implies that $g'_{n+1} = g'_n$.
Fact.1.0.1. $2n+2$ *is not goldbach.* Otherwise, observing [via property (2.0) of Consequence **2**] that $2n$ is goldbachian, then, using the previous, it immediately follows that $2n+2$ is goldbach and $2n$ is goldbachian; consequently $2n+2$ is goldbachian [use the fact that $2n+2$ is goldbach and $2n$ is goldbachian and apply property (1.3) of Assertion **1**], and this contradicts *(3)*. The *Fact.1.0.1* follows.

These two simple *Facts* made, observing [by *Fact.1.0.1*] that $2n+2$ is not goldbach, clearly property $\mathbf{w(Q.}n\mathbf{)}$ of statement **(Q.)** is satisfied, and recalling that statement **(Q.)** is true, then we immediately deduce that property $\mathbf{x(Q.}n\mathbf{)}$ of statement **(Q.)** is not satisfied; therefore

$$\textit{there exists } z_{n.1} \in \mathcal{Z}(n.1) \textit{ such that } z_{n.1} \leq g'_{n+1} \qquad (4).$$

Now using *Fact.1.0.0*, then *(4)* immediately implies that there exists $z_{n.1} \in \mathcal{Z}(n.1)$ such that $z_{n.1} \leq g'_n$, and this contradicts property (2.2) of Consequence **2**. Observation.1.0 follows.

Observation.1.0 implies that

$$c(n) \equiv 1 mod[2] \qquad (5),$$

and clearly

$$\textit{there exists } z_{n.1} \in \mathcal{Z}(n.1) \textit{ such that } z_{n.1} \leq g'_{n+1} \qquad (6)$$

[indeed note $c(n) \equiv 1 mod[2]$ (by congruence *(5)*), and in particular, property **(1.)** of Theorem **1** is not satisfied by the couple $(n, c(n))$; so there exists $z_{n.1} \in \mathcal{Z}(n.1)$ such that $z_{n.1} \leq 1 + g'_{n+1} - c(n)$, and consequently, there exists $z_{n.1} \in \mathcal{Z}(n.1)$ such that $z_{n.1} \leq g'_{n+1}$, because $c(n) \geq 1$ (since $c(n) \equiv 1 mod[2]$ [by congruence *(5)*], and $c(n)$ is a cache of n)]. *(6)* clearly says that property $\mathbf{x(Q.}n\mathbf{)}$ of statement **(Q.)** is not satisfied, and recalling that statement **(Q.)** is true, then we immediately deduce that property $\mathbf{w}(\mathbf{Q}.n)$ of statement **(Q.)** is satisfied; therefore

$$2n + 2 \ \textit{is not goldbach} \qquad (7).$$

(7) immediately implies that $g'_{n+1} < 2n + 2$; now using property (1.1) of Assertion **1** and the previous inequality, we immediately deduce that

$$g'_{n+1} = g'_n \qquad (8).$$

Now using equality *(8)*, then *(6)* clearly says that there exists $z_{n.1} \in \mathcal{Z}(n.1)$ such that $z_{n.1} \leq g'_n$, and this contradicts property (2.2) of Consequence **2**. Theorem **1** follows.

Remark 3. *Note that to prove Theorem* **1**, *we consider a couple* $(n, c(n))$ *such that* $(n, c(n))$ *is a counter-example with* n **minimum** *and* $c(n, 2)$ **maximum**. *In properties (*2.0*),(*2.1*), and (*2.2*) of Consequence* **2** *(via property (*2.0*) of Remark* **2***), the minimality of* n *is used; and in property (*2.3*) of Consequence* **2** *(via property (*2.1*) of Remark* **2***), the maximality of* $c(n, 2)$ *is used. Consequence* **2** *helps us to give a simple and detailed proof of Theorem* **1**.

Corollary 1. *Suppose that statement* **(Q.)** *is true. Then we have the following four properties.*
*(*1.0*) For every integer* $n \geq 1$, $2n+2$ *is goldbachian [i.e.* $g'_{n+1} = 2n+2$*].*
*(*1.1*) The Goldbach conjecture is true.*
*(*1.2*) For every integer* $n \geq M_{11}$, *and for every* $z_{n.1} \in \mathcal{Z}(n.1)$, *we have* $z_{n.1} > 2n + 2$.
*(*1.3*) The Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.*
Proof. (1.0). It is immediate if $n \in \{1, 2, ..., M_{11}\}$ (since $2M_{11} + 2$ is goldbachian, by using Example.**5** given in Definitions.**1**). If $n \geq M_{11} + 1$,

consider the couple $(n, c(n))$ with $c(n) = 0$. The couple $(n, c(n))$ is of the form $0 \leq c(n) < n$, where $n \geq M_{11}$, $c(n) \equiv 0 mod[2]$, and $c(n)$ is a cache of n. Then property **(0.)** of Theorem **1** is satisfied by the couple $(n, c(n))$. So $2n+2$ is goldbachian [because $c(n) = 0$], and consequently $g'_{n+1} = 2n + 2$.
(1.1). Indeed, the Goldbach conjecture immediately follows, by using property (1.0).
(1.2). Let the couple $(n, c(n))$ be such that $c(n) = 1$. The couple $(n, c(n))$ is of the form $0 \leq c(n) < n$, where $n \geq M_{11}$, $c(n) \equiv 1 mod[2]$, and $c(n)$ is a cache of n. Then property **(1.)** of Theorem **1** is satisfied by the couple $(n, c(n))$. So, for every $z_{n.1} \in \mathcal{Z}(n.1)$, we have $z_{n.1} > g'_{n+1}$ [because $c(n) = 1$]; now observing [by property (1.0)] that $g'_{n+1} = 2n+2$, then, we immediately deduce that for every $z_{n.1} \in \mathcal{Z}(n.1)$, we have $z_{n.1} > 2n + 2$.
(1.3). Indeed, the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite, by using property (1.2) and the definition of $\mathcal{Z}(n.1)$ (see Definition.**3** for the meaning of $\mathcal{Z}(n.1)$).

Using property (1.1) and property (1.3) of Corollary **1**, then the following result **(E.)** becomes immediate.

Result (E.). *Suppose that statement* **(Q.)** *is true. Then, the Goldbach conjecture is true, and moreover, the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.*

Conjecture.1. Statement **(Q.)** is true.

4 Epilogue.

To conjecture that the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture is not surprising. Indeed, let **(Q'.)** *be the following statement*:

(Q'.): For every integer $r \geq M_{11}$, **at most one** of the following two properties **w(Q'.**r**)** and **x(Q'.**r**)** is true.
w(Q'.r**):** $2r + 2$ is not goldbach.
x(P'.r**):** For every $z_{r.1} \in \mathcal{Z}(r.1)$, we have $z_{r.1} > g'_{r+1}$.

Note that statement **(Q'.)**, somewhere, resembles to statement **(Q.)**. More precisely, statement **(Q.)** implies statement **(Q'.)** [**Proof.** In particular, the Goldbach conjecture is true [use property (1.1) of Corollary **1**]; consequently, statement **(Q'.)** is true [use definition of statement **(Q'.)** and the previous]].

Conjecture.2. Statement **(Q.)** and statement **(Q'.)** are equivalent.

Conjecture.**2.** implies that the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture .

Proof. Suppose that conjecture.**2** is true. If the Goldbach conjecture is true, clearly statement **(Q'.)** is true; observing that statement **(Q'.)** and statement **(Q.)** are equivalent, then **(Q.)** is true, and result **(E.)** implies that the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

Conjecture.3. Suppose that statement **(Q'.)** is true. Then the Goldbach conjecture is true, and moreover, the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

Conjecture.**3.** immediately implies that the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture.

Proof. Suppose that conjecture.**3** is true. If the Goldbach conjecture is true, clearly statement **(Q'.)** is true, and in particular the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

Conjecture.4. For every integer $r \geq M_{11}$, property $\mathbf{x(Q'.}r\mathbf{)}$ of statement **(Q'.)** is true [note that property $\mathbf{x(Q'.}r\mathbf{)}$ of statement **(Q'.)** is exactly property $\mathbf{x(Q.}r\mathbf{)}$ of statement **(Q.)**; moreover, it is immediate to see that property $\mathbf{x(Q'.}r\mathbf{)}$ of statement **(Q'.)** is satisfied for large values of $r \geq M_{11}$].

Conjecture.**4.** also implies that the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture.

Proof. Suppose that conjecture.**4** is true. If the Goldbach conjecture is true, clearly, $g'_{n+1} = 2n + 2$, and so for every $z_{n.1} \in \mathcal{Z}(n.1)$, we have

$$z_{n.1} > g'_{n+1} > 2n \qquad (9).$$

Observing that *(9)* is true for every integer $n \geq M_{11}$, then in particular, it results that the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite.

Now, using conjecture.**2** and conjecture.**3** and conjecture.**4** , it becomes natural and not surprising to conjecture the following:

Conjecture.5. The Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are consequences of the Goldbach conjecture. More precisely, the Sophie Germain primes problem, the Mersenne composite numbers problem, the friendly numbers problem and the perfect numbers problem are protected by the umbrella of Goldbach.

Conjecture.6. Let $(n, b(n))$ be a couple of integers such that $n \geq M_{11}$ and $0 \leq b(n) < n$. We have the following.

(0.) If $b(n) \equiv 0 mod[4]$; then $2n + 2 - b(n)$ is goldbachian.
(1.) If $b(n) \equiv 1 mod[4]$; then $h_{n.1} > 1 + g'_{n+1} - b(n)$ and $c_{n.1} > 1 + g'_{n+1} - b(n)$.
(2.) If $b(n) \equiv 2 mod[4]$; then $a_{n.1} > 2 + g'_{n+1} - b(n)$.
(3.) If $b(n) \equiv 3 mod[4]$; then $d_{n.1} > 3 + g'_{n+1} - b(n)$.

It is easy to see that conjecture.**6** simultaneously implies that: not only the Goldbach conjecture is true, but the Sophie Germain primes, the Mersenne composite numbers, the friendly numbers and the perfect numbers are all infinite, and to attack this conjecture, we must consider the generalized Fermat induction.

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An investigation of Kaprekar operation on six-digit numbers computational approach [1]

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Abstract

In this paper we investigate Kaprekar operation when it is carried out on six-digit numbers (not having all digits the same). We prove the operation do one of three things. It will terminate with the number 549945 upon the first iteration, terminate with the number 631764 within the first four iterations, or goes into a seven-element loop within the first fourteen iterations.

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1 Introduction

Let n a six-digit number (not having all digits the same), let uvwxyz be the number obtained by writing the digits of n in descending order, and let $D_1 =$ *uvwxyz - zyxwvu.*

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For D_1 let abcdef, the number obtained by writing the digits of D_1 in descending order, and let D_2 = *abcdef* - *fedcba*. When this operation continue, it called Kaprekar operation.
In this paper we investigate what happens when this operation is carried out on a six-digit number n.

2 Lemmas and Theorems

We start with the following very simple but important lemma.

Lemma 1 *For six-digit number n, if uvwxyz is the number obtained by writing the digits of n in descending order, and* D_1 = *uvwxyz - zyxwvu is the number obtained after the first iteration of Kaprekar operation, then* D_1 = *99999(u - v) +9990(v - y) + 900(w - x).*

Proof.
Now,

$$uvwxyz \; = 100000u \; + 10000v \; + 1000w \; + 100x \; + 10y \; + z$$

and

$$zyxwvu \; = 100000z \; + 10000y \; + 1000x \; + 100w \; + 10v \; + u$$

hence

$$D_1 = 99999(u \text{ - } z) + 9990(v \text{ - } y) + 900(w \text{ - } x).$$

The importance of this simple lemma, is that when applying Kaprekar operation on a number n, then the number obtained after the first iteration does not depends on the numerical value of n, but rather on the three differences u - z, v - y and w - x, where *uvwxyz* is the number obtained by writing digits of n in descending order. This suggests for us to classify the numbers based on these three differences.

Lemma 2 *Let A be the set of all six-digit numbers (not having all digits the same). For n, m belongs to A, let n related to m, ($n \sim m$) iff u -z = $\acute{u}$ - $\acute{z}$, v - y = $\acute{v}$ - $\acute{y}$ and w - x = $\acute{w}$ - $\acute{x}$, where $uvwxyz$ and $\acute{u}\acute{v}\acute{w}\acute{x}\acute{u}\acute{z}$, the two numbers obtained by writing the digits of n and m in descending order respectively then the relation ($\sim$) is an equivalence relation on A.*

We shall write the equivalence classes of this relation on the form $[a, b, c]$, where $[a, b, c]$ represents all numbers $n \in A$ with u- z= a, v - y = b and w- x = c. That is if n belongs to the class [a, b, c], then D_1 in Lemma (1) given by D_1 = 99999(a) + 9990(b) + 900(c).

Theorem 1 *The equivalence relation ($\sim$) has 219 equivalence classes.*

Proof.

Since $z \leq y \leq x \leq w \leq v \leq u$, then $0 \leq w - x \leq v - y \leq u - z$. That is in any class $[a, b, c]$, we have $0 \leq c \leq b, 0 \leq b \leq a$ and $0 \leq a \leq 9$, but since $u \neq z$, then $1 \leq a \leq 9$. Therefore number of equivalence classes N is given by

$$\begin{aligned} N &= \sum_{a=1}^{9} \sum_{b=0}^{a} \sum_{c=0}^{b} 1 \\ &= \sum_{a=1}^{9} \sum_{b=0}^{a} b + 1 \\ &= \sum_{a=1}^{9} (1 + 2 + 3 + \cdots + a + (a + 1)) \\ &= \sum_{a=1}^{9} \frac{(a+1)(a+2)}{2} = 219. \end{aligned}$$

Here is a list of the equivalence classes

[1,1,1],[1,1,0],[1,0,0]

[2,2,2],[2,2,1],[2,2,0],[2,1,1],[2,1,0],[2,0,0]

[3,3,3],[3,3,2],[3,3,1],[3,3,0],[3,2,2],[3,2,1],[3,2,0],[3,1,1],[3,1,0],[3,0,0]

[4,4,4],[4,4,3],[4,4,2],[4,4,1],[4,4,0],[4,3,3],[4,3,2],[4,3,1],[4,3,0],[4,2,2]

[4,2,1],[4,2,0],[4,1,1],[4,1,0],[4,0,0],[5,5,5],[5,5,4],[5,5,3],[5,5,2],[5,5,1]

[5,5,0],[5,4,4],[5,4,3],[5,4,2],[5,4,1],[5,4,0],[5,3,3],[5,3,2],[5,3,1],[5,3,0]

[5,2,2],[5,2,1],[5,2,0],[5,1,1],[5,1,0],[5,0,0],[6,6,6],[6,6,5],[6,6,4],[6,6,3]

[6,6,2],[6,6,1],[6,6,0],[6,5,5],[6,5,4],[6,5,3],[6,5,2],[6,5,1],[6,5,0],[6,4,4]

[6,4,3],[6,4,2],[6,4,1],[6,4,0],[6,3,3],[6,3,2],[6,3,1],[6,3,0],[6,2,2],[6,2,1]
[6,2,0],[6,1,1],[6,1,0],[6,0,0],[7,7,7],[7,7,6],[7,7,5],[7,7,4],[7,7,3],[7,7,2]
[7,7,1],[7,7,0],[7,6,6],[7,6,5],[7,6,4],[7,6,3],[7,6,2],[7,6,1],[7,6,0],[7,5,5]
[7,5,4],[7,5,3],[7,5,2],[7,5,1],[7,5,0],[7,4,4],[7,4,3],[7,4,2],[7,4,1],[7,4,0]
[7,3,3],[7,3,2],[7,3,1],[7,3,0],[7,2,2],[7,2,1],[7,2,0],[7,1,1],[7,1,0],[7,0,0]
[8,8,8],[8,8,7],[8,8,6],[8,8,5],[8,8,4],[8,8,3],[8,8,2],[8,8,1],[8,8,0],[8,7,7]
[8,7,6],[8,7,5],[8,7,4],[8,7,3],[8,7,2],[8,7,1],[8,7,0],[8,6,6],[8,6,5],[8,6,4]
[8,6,3],[8,6,2],[8,6,1],[8,6,0],[8,5,5],[8,5,4],[8,5,3],[8,5,2],[8,5,1],[8,5,0]
[8,4,4],[8,4,3],[8,4,2],[8,4,1],[8,4,0],[8,3,3],[8,3,2],[8,3,1],[8,3,0],[8,2,2]
[8,2,1],[8,2,0],[8,1,1],[8,1,0],[8,0,0],[9,9,9],[9,9,8],[9,9,7],[9,9,6],[9,9,5]
[9,9,4],[9,9,3],[9,9,2],[9,9,1],[9,9,0],[9,8,8],[9,8,7],[9,8,6],[9,8,5],[9,8,4]
[9,8,3],[9,8,2],[9,8,1],[9,8,0],[9,7,7],[9,7,6],[9,7,5],[9,7,4],[9,7,3],[9,7,2]
[9,7,1],[9,7,0],[9,6,6],[9,6,5],[9,6,4],[9,6,3],[9,6,2],[9,6,1],[9,6,0],[9,5,5]
[9,5,4],[9,5,3],[9,5,2],[9,5,1],[9,5,0],[9,4,4],[9,4,3],[9,4,2],[9,4,1],[9,4,0]
[9,3,3],[9,3,2],[9,3,1],[9,3,0],[9,2,2],[9,2,1],[9,2,0],[9,1,1],[9,1,0],[9,0,0]

Upon the investigation of Kaprekar operation when it is carried out, we find out the 219 classes, divided into three different categories as follow:

C_1: Classes whose elements terminates with the number 949945 upon the first iteration, and this category consists of the single class [5,5,0].

C_2: Classes whose elements terminates with the number 631764 within the first four iterations, and this category consists of the seventeen classes [6,3,2], [4,3,2], [6,6,2], [8,6,6], [8,6,4], [4,3,3], [6,6,3], [6,3,3], [7,6,6], [7,6,4], [7,6,0], [7,4,0], [9,6,0], [9,4,0], [8,8,7], [8,8,5], [8,8,3].

C_3: This category consists of all the other 201 classes and the elements in these classes reach the number 851742 within the first fourteen iterations then they goes into a loop of seven elements.

Theorem 2 *The elements of the class* $[5,5,0]$ *terminate with the number 549945 upon the first iteration.*

Proof.

If $n \in [5, 5, 0]$ then $D_1 = 99999(5) + 9990(5) = 549945$. And since $D_1 = 549945$ belongs to the class [5,5,0], then the operation terminate by this number.

Theorem 3 *The elements of the classes in the second category terminate with the number 631764 within the first four iterations.*

Proof.
We divide the classes in this category into four families in this way
$F_1 = \{[6, 3, 2]\}$.
$F_2 = \{[4, 3, 2], [6, 6, 2], [8, 6, 6], [8, 6, 4]\}$.
$F_3 = \{[4, 3, 3], [6, 6, 3], [6, 3, 3], [7, 6, 6], [7, 6, 4], [7, 6, 0], [7, 4, 0], [9, 6, 0], [9, 4, 0]\}$.
$F_4 = \{[8, 8, 7], [8, 8, 5], [8, 8, 3]\}$.
Now if class of a number n is in F_1, then

$$\begin{aligned} D_1 &= 99999(a) + 9990(b) + 900(c) \\ &= 99999(6) + 9990(3) + 900(2) = 631764. \end{aligned}$$

And since 631764 belongs to the class [6,3,2], then the operation terminates upon the first iteration.
If n has class in F_2, then

$$\begin{aligned} D_1 &= 99999(a) + 9990(b) + 900(c) \\ &= 431766, 661734, 865332 \textit{ or } 863532. \end{aligned}$$

Thus, the class of D_1 is [6,3,2]. And since [6,3,2] belongs to F_1, then D_1 terminate with the number 631764 on the first iteration, and hence n terminates with the same number in the second iteration.
If n has class in F_3, then

$$\begin{aligned} D_1 =& 99999(a) + 9990(b) + 900(c) \\ =& 432666, 662634, 632664, 765333, 763533, 759933, \\ & 739953, 959931 \textit{or } 939951. \end{aligned}$$

Thus the class of D_1 is [4,3,2],[6,6,2] or [8,6,4]. And since the class of D_1 belongs to F_2, then the operation terminates with the number 631764 on the third iteration.

Finally, if n has class in F_4, then $D_1 = 886212{,}884412$ or 882612. The class of $D_1 =$ [7,6,4] or [7,6,0]. And since the class of D_1 belongs to F_3 then the operation terminate with the number 631764 in the fourth iteration.

Theorem 4 *The elements of the classes in the third category reaches the number 851742 within the first fourteen iterations, then they runs in a loop of seven numbers.*

Proof.
Similar to Theorem (2) we divide the classes into families, in such a way that if n has class in F_i, then D_1 has class in F_{i-1}. These families are given as follow:

$F_1 = \{$[8,5,2]$\}$.

$F_2 = \{$[6,2,1], [4,2,1], [9,7,6], [9,7,4]$\}$.

$F_3 = \{$[8,5,4], [6,4,3], [6,5,2], [6,5,3], [4,4,3], [4,4,2], [7,5,4], [6,4,2], [7,4,4], [8,4,4]$\}$.

$F_4 = \{$[9,7,7], [8,6,3], [8,7,4], [7,5,0], [8,7,6], [3,3,2], [3,2,1], [9,7,3], [4,2,2], [8,5,0], [6,2,2], [7,7,1], [7,6,2], [7,3,2], [7,2,1], [8,3,3]$\}$.

$F_5 = \{$[1,1,1], [8,6,1], [9,8,1], [3,1,1], [6,3,0], [2,1,1], [4,3,0], [9,6,1], [5,2,1], [7,1,1], [8,8,6], [8,8,4], [8,8,1], [8,3,1], [8,1,1], [9,8,7], [9,8,2], [9,8,8], [9,8,3], [9,7,5], [9,6,2], [9,3,2], [9,3,1], [9,1,1]$\}$.

$F_6 = \{$[8,4,1], [1,1,0], [6,5,4], [9,5,2], [8,6,5], [5,3,3], [7,1,0][4,4,4], [8,5,1], [4,0,0], [5,5,5], [5,3,2],[6,4,4], [7,6,5], [7,1,0], [7,0,0], [9,9,9], [9,9,6], [9,9,4], [9,9,0], [9,7,1], [9,4,2], [9,2,1], [9,1,0]$\}$.

$F_7 = \{$[7,5,1], [8,7,3], [4,1,0], [2,2,1], [8,7,1], [3,3,3], [8,2,1], [3,2,2], [7,6,3], [7,7,2], [4,2,0], [6,1,0],[7,4,1], [9,4,3], [6,2,0], [7,3,3], [7,2,2], [8,7,7], [9,7,2], [9,5,3], [9,5,1], [9,4,1], [9,2,2]$\}$.

F_8 = {[9,7,0], [9,5,5], [2,2,0], [6,4,1], [8,6,0], [4,4,1], [6,5,1], [8,3,0], [5,5,3], [5,5,2], [5,4,3], [5,4,2], [5,3,0], [8,4,2], [8,7,0], [5,1,0], [8,8,0], [8,5,5], [8,4,0], [8,2,0], [9,5,4], [9,4,4], [9,3,0]}

F_9 = {[8,1,0], [2,2,2], [2,1,0], [8,4,3], [3,2,0], [7,2,0], [4,4,0], [5,3,1], [8,7,2], [6,4,0], [6,5,0], [5,2,0], [5,1,1], [6,6,0], [7,7,7], [7,7, 6], [7,7,4], [7,7,3], [7,5,2], [7,4,2], [8,5,3], [8,2,2], [9,8,5], [9,6,5], [9,5,0]}.

F_{10} = {[9,0,0], [2,0,0], [3,3,1], [7,6,1], [3,0,0], [7,5,3], [4,1,1], [5,5,4], [5,5,1], [5,4,4], [5,4,1], [5,4,0], [5,2,2], [6,6,5], [6,5,5], [6,1,1], [7,4,3], [7,3,1], [8,7,5], [8,0,0], [9,8,6], [9,8,4], [8,6,3], [9,5,5], [9,3,3]}.

F_{11} = {[1,0,0], [3,3,0], [8,3,2], [5,0,0], [6,6,6], [6,6,4], [6,0,0], [7,7,5], [7,7,0], [7,3,0], [8,6,2], [9,9,8], [9,9,7], [9,9,3], [9,9,2]}.

F_{12} = {[4,3,1], [6,6,1], [6,3,1], [8,8,8], [8,8,2], [9,9,1], [9,6,6], [9,6,4]}.

F_{13} = {[9,8,0], [9,2,0]}.

F_{14} = {[9,9,5]}

Now if class of n in F_1 , then

$$\begin{aligned} D_1 &= 99999(a) + 9990(b) + 900(c) \\ &= 99999(8) + 9990(5) + 900(2) = 851742. \end{aligned}$$

Then n reach the number 851742 in the first iteration.

If n has class in F_2, then

$$\begin{aligned} D_1 &= 99999(a) + 9990(b) + 900(c) \\ &= 620874, 420876, 975321 \, or \; 975321. \end{aligned}$$

Then the class of D_1 is [8,5,2]. And since class of D_1 belongs F_1, then D_1 reach the number 851742 in the first iteration, and hence n reach the number 851742 in the second iteration.

If n has class in F_3, then D_1 = 853542, 642654, 651744, 652644,442656, 441756, 753543, 641754, 743533, or 843552. The class of D_1 is [6,2,1] or [4,2,1]. And since class of D_1 belongs to F_2, then the number n reach the number 851742 in the third iteration.

If we continue in this fashion we will find if n has class in F_i , then D_1 has class in F_{i-1}, and hence n reach the number 851742 in the i^{-th} iteration.
To emphasize this point we make the final check on F_{13} and F_{14}. If n has class in F_{13} then,

$$\begin{aligned} D_1 &= 99999(a) + 9990(b) + 900(c) \\ &= 979911 \, or \; 919971. \end{aligned}$$

Class of D_1 is [8,8,2] belongs to F_{12}.
In case n has class in F_{14}, then D_1 = 99999(9)+9990(9)+900(5)=994401.
Class of D_1 is [9,8,0] belongs to F_{13} .
Once the elements reaches the number 851742 they start running in the following loop of seven elements.

851742 $\rightarrow$ 750843 $\rightarrow$ 840852 $\rightarrow$ 860832 $\rightarrow$ 862632 $\rightarrow$ 642654 $\rightarrow$ 420878 $\rightarrow$ 851742.

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A subclass of harmonic univalent functions with positive coefficients defined by Dziok-Srivastava operator [1]

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Abstract

In this paper using the Dziok-Srivastava [4] operator, we introduce a subclass of the class $\mathcal{H}$ of complex valued Harmonic univalent functions $f = h + \bar{g}$, where h is the analytic part and g is the co-analytic part of f in $|z| < 1$. Coefficient bounds, extreme points, inclusion results and closure under integral operator for this class are obtained.

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1 Introduction

Harmonic mappings have found applications in many diverse fields such as engineering, aerodynamics and other branches of applied Mathematics. Harmonic mappings in a domain $D \subseteq C$ are univalent complex-valued harmonic functions $f = u + iv$ where both u and v are real harmonic. The important work of Clunie and Sheil-Small [2] on the class consisting of complex-valued harmonic orientation preserving univalent functions f defined on the open unit disk U formed the basis for several investigations on different subclasses of harmonic univalent functions.

In any simply-connected domain D, it is known that [2] we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D [2].

Denote by $\mathcal{H}$ the family of harmonic functions

$$f = h + \bar{g} \tag{1}$$

which are univalent and orientation preserving in the open unit disk $U = \{z : |z| < 1\}$ and f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \bar{g} \in \mathcal{H}$ the analytic functions h and g are given by

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m.$$

Hence

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}, \quad |b_1| < 1 \tag{2}$$

We note that the family $\mathcal{H}$ reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, that is $g \equiv 0$.

For complex numbers $\alpha_1, \dots, \alpha_p$ and $\beta_1, \dots, \beta_q$ $(\beta_j \neq 0, -1, \dots;\ j = 1, 2, \dots, q)$ the generalized hypergeometric function [8] ${}_pF_q(z)$ is defined by

$$(3) \qquad {}_pF_q(z) \equiv {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \dots (\alpha_p)_m}{(\beta_1)_m \dots (\beta_q)_m} \frac{z^m}{m!}$$

$$(p \leq q+1; p, q \in N_0 = N \cup \{0\}; z \in U),$$

where N denotes the set of all positive integers and $(a)_m$ is the Pochhammer symbol defined by

$$(4) \qquad (a)_m = \begin{cases} 1, & m = 0, \\ a(a+1)(a+2)\dots(a+m-1), & m \in N. \end{cases}$$

Dziok and Srivastava [4] introduced an operator in their study of analytic functions associated with generalized hypergeometric functions. This Dziok-Srivastava operator is known to include many well-known operators as special cases.

Let $H(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q) : A \to A$ be a linear operator defined by

$$[(H(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q))(\phi)](z) = z\ {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z) * \phi(z)$$

$$(5) \qquad = z + \sum_{m=2}^{\infty} \Gamma_m a_m z^m$$

where

$$(6) \qquad \Gamma_m = \frac{(\alpha_1)_{m-1} \dots (\alpha_p)_{m-1}}{(\beta_1)_{m-1} \dots (\beta_q)_{m-1}} \frac{1}{m-1!}$$

and $\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q$ are positive real numbers, such that $p \leq q+1$; $p, q \in N \cup \{0\}$, and $(a)_m$ is the familiar Pochhammer symbol.

The linear operator $H(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q)$ or $H_q^p[\alpha_1, \beta_1]$ in short, is the Dziok-Srivastava operator ([4] & [12]), which includes several well known operators.

The Dziok-Srivastava operator when extended to the harmonic function $f = h + \bar{g}$ is defined by

$$H_q^p[\alpha_1, \beta_1]f(z) = H_q^p[\alpha_1, \beta_1]h(z) + \overline{H_q^p[\alpha_1, \beta_1]g(z)} \tag{7}$$

Denote by $V_{\mathcal{H}}$ the subclass of $\mathcal{H}$ consisting of functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{m=2}^{\infty} |a_m| z^m, \quad g(z) = \sum_{m=1}^{\infty} |b_m| z^m, \quad |b_1| < 1 \tag{8}$$

Motivated by earlier works of [1, 3, 6, 7, 10, 11] on harmonic functions, we introduce here a new subclass $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ of $V_{\mathcal{H}}$ using Dziok-Srivastava operator extended to harmonic functions.

We denote by $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$, the subclass of $V_{\mathcal{H}}$, consisting of functions of the form (8) satisfying the condition

$$Re \left\{ \begin{array}{l} \alpha \left(\dfrac{H_q^p[\alpha_1, \beta_1]h(z) + H_q^p[\alpha_1, \beta_1]g(z)}{z} \right) \\ +(H_q^p[\alpha_1, \beta_1]h(z))' + (H_q^p[\alpha_1, \beta_1]g(z))' - \alpha \end{array} \right\} < \beta$$

where $\alpha \geq 0$, $1 < \beta \leq 2$.

For $p = q + 1$, $\alpha_2 = \beta_1, \ldots, \alpha_p = \beta_q$, $\alpha_1 = 1$, $\alpha = 0$ the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ reduces to the class $R_{\mathcal{H}}(\beta)$ studied in [3]. Further if the co-analytic part of $f = h + \bar{g}$ is zero that is $g \equiv 0$, the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ reduces to the class studied in [13].

In this paper extreme points, inclusion results and closure under integral operator for the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ are obtained.

2 Main Results

Theorem 1. *A function f of the form (8) is in $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ if and only if*

$$\sum_{m=2}^{\infty} (\alpha + m)\Gamma_m |a_m| + \sum_{m=1}^{\infty} (\alpha + m)\Gamma_m |b_m| \leq \beta - 1 \tag{9}$$

Proof. Let $\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|\leq\beta-1.$
It suffices to prove that

$$\left|\frac{\alpha\left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z)+H_q^p[\alpha_1,\beta_1]g(z)}{z}\right)+(H_q^p[\alpha_1,\beta_1]h(z))'+(H_q^p[\alpha_1,\beta_1]g(z))'-\alpha-1}{\alpha\left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z)+H_q^p[\alpha_1,\beta_1]g(z)}{z}\right)+(H_q^p[\alpha_1,\beta_1]h(z))'+(H_q^p[\alpha_1,\beta_1]g(z))'-\alpha-(2\beta-1)}\right|<1,\quad z\in U.$$

we have

$$\left|\frac{\alpha\left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z)+H_q^p[\alpha_1,\beta_1]g(z)}{z}\right)+(H_q^p[\alpha_1,\beta_1]h(z))'+(H_q^p[\alpha_1,\beta_1]g(z))'-\alpha-1}{\alpha\left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z)+H_q^p[\alpha_1,\beta_1]g(z)}{z}\right)+(H_q^p[\alpha_1,\beta_1]h(z))'+(H_q^p[\alpha_1,\beta_1]g(z))'-\alpha-(2\beta-1)}\right|$$

$$=\left|\frac{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1}+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}}{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1}+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}+1-(2\beta-1)}\right|$$

$$=\left|\frac{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1}+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}}{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1}+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}-2(\beta-1)}\right|$$

$$\leq \left| \frac{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1} + \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}}{2(\beta-1) - \sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1} - \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}} \right|$$

$$\leq \left| \frac{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m| + \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|}{2(\beta-1) - \sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m| - \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|} \right|$$

which is bounded above by 1, by hypothesis and the sufficient part is proved.

Conversely, suppose that

$$Re \left\{ \begin{array}{c} \alpha \left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z) + H_q^p[\alpha_1,\beta_1]g(z)}{z} \right) \\ +(H_q^p[\alpha_1,\beta_1]h(z))' + (H_q^p[\alpha_1,\beta_1]g(z))' - \alpha \end{array} \right\} < \beta,$$

which is equivalent to

$$Re \left\{ \sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1} + \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1} + 1 \right\} < \beta.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z to be real and let $z \to 1^-$, we obtain

$$\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m| + \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m| \leq \beta - 1,$$

which gives the necessary part. This completes the proof of the theorem. □

We now determine the extreme points of the closed convex hulls of $R_{\mathcal{H}}([\alpha_1,\beta_1],\alpha,\beta)$ denoted by *clco* $R_{\mathcal{H}}([\alpha_1,\beta_1],\alpha,\beta)$.

Theorem 2. *A function* $f(z) \in clco\ R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ *if and only if*

$$f(z) = \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z)) \tag{10}$$

where $h_1(z) = z$, $h_m(z) = z + \frac{\beta-1}{(\alpha+m)\Gamma_m} z^m$; $(m \geq 2)$, $g_m(z) = z + \frac{\beta-1}{(\alpha+m)\Gamma_m} \overline{z}^{-m}$; $(m \geq 1)$ *and* $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$, $X_m \geq 0$ *and* $Y_m \geq 0$. *In particular, the extreme points of* $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ *are* $\{h_m\}$ *and* $\{g_m\}$.

Proof. For functions f of the form (10) write

$$\begin{aligned} f(z) &= \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z)) \\ &= \sum_{m=1}^{\infty} (X_m + Y_m) z + \sum_{m=2}^{\infty} \frac{\beta - 1}{(\alpha + m)\Gamma_m} X_m z^m + \sum_{m=1}^{\infty} \frac{\beta - 1}{(\alpha + m)\Gamma_m} Y_m z^{-m} \\ &= z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^{-m}, \end{aligned}$$

where

$$A_m = \frac{\beta - 1}{(\alpha + m)\Gamma_m} X_m, \quad \text{and} \quad B_m = \frac{\beta - 1}{(\alpha + m)\Gamma_m} Y_m$$

Therefore,

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} A_m + \sum_{m=1}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} B_m & \\ = \sum_{m=2}^{\infty} X_m + \sum_{m=1}^{\infty} Y_m & \\ = 1 - X_1 \leq 1, & \end{aligned}$$

and hence $f(z) \in clco\ R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.

Conversely, suppose that $f(z) \in clco\ R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.
Setting

$$\begin{aligned} X_m &= \frac{(\alpha+m)\Gamma_m}{\beta-1} A_m; \quad (m \geq 2), \\ Y_m &= \frac{(\alpha+m)\Gamma_m}{\beta-1} B_m; \quad m \geq 1 \end{aligned}$$

where $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$. We have

$$\begin{aligned} f(z) &= z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^{-m}, \quad A_m, B_m \geq 0 \\ &= z + \sum_{m=2}^{\infty} \frac{\beta-1}{(\alpha+m)\Gamma_m} X_m z^m + \sum_{m=1}^{\infty} \frac{\beta-1}{(\alpha+m)\Gamma_m} Y_m z^{-m} \\ &= z + \sum_{m=2}^{\infty} (h_m(z) - z) X_m + \sum_{m=1}^{\infty} (g_m(z) - z) Y_m \\ &= \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z)) \end{aligned}$$

as required. □

Theorem 3. *Each function in the class* $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ *maps a disk* U_r *where* $r < \inf_m \left\{ \frac{1}{m(\beta-1-(\alpha+1)|b_1|} \right\}^{\frac{1}{m-1}}$ *onto convex domains for* $\beta > 1 + (\alpha+1)|b_1|$.

Proof. Let $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ and let r be fixed, $0 < r < 1$. Then

$r^{-1}f(rz) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ and we have

$$\begin{aligned}\sum_{m=2}^{\infty} m^2(|a_m| + |b_m|)r^{m-1} &= \sum_{m=2}^{\infty} m(|a_m| + |b_m|)(mr^{m-1}) \\ &\leq \sum_{m=2}^{\infty} m(|a_m| + |b_m|) \\ &\leq \sum_{m=2}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1}(|a_m| + |b_m|) \\ &\leq \beta - 1 - (\alpha+1)|b_1| \\ &\leq 1\end{aligned}$$

provided

$$mr^{m-1} \leq \frac{1}{\beta - 1 - (\alpha+1)|b_1|}$$

or

$$r < \inf_m \left\{ \frac{1}{m(\beta - 1 - (\alpha+1)|b_1|} \right\}^{\frac{1}{m-1}}.$$

This completes the proof of theorem 3. □

For our next theorem, we need to define the convolution of two harmonic functions.

For harmonic functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} |a_m| z^m + \sum_{m=1}^{\infty} |b_m| z^{-m}$$

and

$$F(z) = z + \sum_{m=2}^{\infty} |A_m| z^m + \sum_{m=1}^{\infty} |B_m| z^{-m},$$

we define their convolution

$$(f * F)(z) = f(z) * F(z) = z + \sum_{m=2}^{\infty} |a_m A_m| z^m + \sum_{m=1}^{\infty} |b_m B_m| z^{-m} \tag{11}$$

Using this definition, we show that the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ is closed under convolution.

Theorem 4. *For* $1 < \beta \leq \delta \leq 2$*, let* $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$ *and* $F \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$*. Then* $f * F \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta) \subseteq R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$*.*

Proof. Let $f(z) = z + \sum_{m=2}^{\infty} |a_m| z^m + \sum_{m=1}^{\infty} |b_m| z^{-m} \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$ and $F(z) = z + \sum_{m=2}^{\infty} |A_m| z^m + \sum_{m=1}^{\infty} |B_m| z^{-m} \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.
The convolution $(f * F)$ is given by (11).
We note that, for $F \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$, $|A_m| \leq 1$ and $|B_m| \leq 1$. Now we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |a_m||A_m| + \sum_{m=1}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |b_m||B_m| \\
&\quad \leq \sum_{m=2}^{\infty} \frac{(\alpha + m)}{\beta - 1} |a_m| + \sum_{m=1}^{\infty} \frac{(\alpha + m)}{\beta - 1} |b_m| \\
&\quad \leq 1, \quad (f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)
\end{aligned}$$

Therefore $f * F \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta) \subseteq R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$. □

Next, we show that $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ is closed under convex combinations of its members.

Theorem 5. *The class* $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ *is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, let $f_i \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$, where

$$f_i(z) = z + \sum_{m=2}^{\infty} |a_{m,i}| z^m + \sum_{m=1}^{\infty} |b_{m,i}| z^{-m}.$$

Then by theo 1, we have

$$\sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |a_{m,i}| + \sum_{m=1}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |b_{m,i}| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{m=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{m,i}| \right) z^m + \sum_{m=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{m,i}| \right) z^{-m}$$

Then by theo 1, we have

$$\begin{aligned} &\sum_{m=2}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} \left(\sum_{i=1}^{\infty} t_i |a_{m,i}| \right) + \sum_{m=1}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} \left(\sum_{i=1}^{\infty} t_i |b_{m,i}| \right) \\ &\quad = \sum_{i=1}^{\infty} t_i \left(\sum_{m=2}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} |a_{m,i}| + \sum_{m=1}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} |b_{m,i}| \right) \\ &\quad \leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

Therefore, $\sum_{i=1}^{\infty} t_i f_i(z) \in R_H([\alpha_1, \beta_1], \alpha, \beta)$. □

Following Ruscheweyh [9], the δ-neighborhood of f is the set

$$\begin{aligned} N_\delta(f) = \Bigg\{ F : F(z) = z + \sum_{m=2}^{\infty} |A_m| z^m + \sum_{m=1}^{\infty} |B_m| \bar{z}^m \text{ and} \\ \sum_{m=2}^{\infty} m(|a_m - A_m| + |b_m - B_m| + |b_1 - B_1| \leq \delta \Bigg\} \end{aligned}$$

Theorem 6. *Let $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ and $\delta = \beta - 1 - \alpha|b_1|$. Then $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta) \subset N_\delta(I)$, where I is the identity function $I(z) = z$.*

Proof. Let $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.

We have

$$\begin{aligned}|b_1| + \sum_{m=2}^{\infty} m(|a_m| + |b_m|) \\ &\leq |b_1| + \sum_{m=2}^{\infty} (\alpha + m)\Gamma_m(|a_m| + |b_m|) \\ &\leq |b_1| + \beta + \alpha - (1+\alpha)(1+|b_1|) \\ &= \beta - 1 - \alpha|b_1|.\end{aligned}$$

Hence $f(z) \in N_\delta(I)$. □

3 Integral Operator

Now, we examine a closure property of the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f) = \frac{c+1}{Z^c} \int_0^z t^{c-1} f(t)dt, \quad c > -1.$$

Theorem 7. *Let* $f(z) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$. *Then* $L_c(f(z)) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.

Proof. From the representation of $L_c(f(z))$, it follows that

$$\begin{aligned} L_c(f) &= \frac{c+1}{Z^c} \int_0^z t^{c-1}(h(t) + \overline{g(t)})dt \\ &= \frac{c+1}{Z^c} \left(\int_0^z t^{c-1} \left(t + \sum_{m=2}^{\infty} a_m t^m \right) dt + \overline{\int_0^z t^{c-1} \left(\sum_{m=1}^{\infty} b_m t^m \right) dt} \right) \\ &= z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^m, \end{aligned}$$

where, $A_m = \dfrac{c+1}{c+n} a_m$, $B_m = \dfrac{c+1}{c+n} b_m$.

Therefore,

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left(\frac{(\alpha+m)}{\beta-1}\left(\frac{c+1}{c+n}\right)|a_m|+\frac{(\alpha+m)}{\beta-1}\left(\frac{c+1}{c+n}\right)|b_m|\right)\Gamma_m \\
&\qquad\leq \sum_{m=1}^{\infty}\left(\frac{(\alpha+m)}{\beta-1}|a_m|+\frac{(\alpha+m)}{\beta-1}|b_m|\right)\Gamma_m \\
&\qquad\leq 1,
\end{aligned}$$

since $f(z) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$, therefore by theo 1, $L_c(f(z)) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$. □

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Stability of a common fixed point iterative procedure involving four selfmaps of a metric space [1]

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Abstract

The purpose of this paper is to establish stability results for (S,T)-iterative procedure involving two pairs of weakly compatible selfmaps of a metric space. Our results generalize those of Singh et al[18] and others.

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Key words and Phrases: Stability, (S,T)-iteration, common fixed point.

1 Introduction and Preliminaries

Let T be a selfmap of a metric space X with the set $\{x \in X : Tx = x\}$ of fixed points of T containing at least one member. Let the sequence $\{x_n\}$

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converging to the fixed point of T in X be generated by the iterative process $f(T, x_n)$. Then the iterative process $f(T, x_n)$ is said to be T-stable if and only if a sequence $\{y_n\}$ in X, approximately close to $\{x_n\}$, converges to the same fixed point of X. After Harder and Hicks[2,3] first gave the formal definition of the stability of general iterative procedures, various authors have studied several special cases of the general iterative procedure over many years. Among such authors are Berinde[1], Imoru and Olatinwo[5], Jachymski[6,7], Matkowski and Singh[10], Osilike[12] and Rhoades[17]. In 2005, Singh et al [18] introduced the stability of Jungck and Jungck-Mann iterative procedures for a pair of Jungck-Osilike-type maps on an arbitrary set with values in a metric space.

Let Y be an arbitrary nonempty set and (X, d) a metric space. Let $S, T : Y \longrightarrow X$ and $TY \subseteq SY$. For any $x_0 \in Y$, consider

$$Sx_{n+1} = f(T, x_n), \quad n = 0, 1, ... \tag{1}$$

The stability of this general procedure was developed by Singh in [18]. He observed that putting $Y = X$ and $f(T, x_n) = Tx_n$, yields the Jungck iterative procedure (or, J-iteration), namely, $Sx_{n+1} = Tx_n, \ \ n = 0, 1, ...$ Jungck iterative procedure was much earlier introduced in 1976, and it gives the Picard iterative procedure when S is taken as the identity map on X.

Definition 1.1. [Singh et al.(2005)] Let $S, T : Y \longrightarrow X, \ TY \subseteq SY$, and z a coincidence point of T and S, that is, $Sz = Tz = p$(say). For any $x_0 \in Y$, let the sequence $\{Sx_n\}$, generated by the iterative procedure (1), converges to p. Let $\{Sy_n\}$ in X be an arbitrary sequence, and set

$$\epsilon_n = d(Sy_{n+1}, f(T, y_n)), \quad n = 0, 1, ...$$

Then the iterative procedure $f(T, x_n)$ will be called (S, T)-stable if and only if $\lim_n \epsilon_n = 0$ implies $\lim_n Sy_n = p$.

This basic result for the stability of J-iterations given below is proved in Singh et al[18].

Theorem 1.2. *Let S and T be maps on an arbitrary set Y with values in a metric space X such that* $TY \subseteq SY$*, and SY or TY is a complete subspace of X. Let z be a coincidence point of T and S, that is, Sz=Tz=p(say). Let* $x_0 \in Y$ *and let the sequence* $\{Sx_n\}$*, generated by* $Sx_{n+1} = Tx_n, \;\; n = 0, 1, ...$ *If the pair (S,T) is a J-contraction, then*
$d(p, Sy_{n+1}) \leq d(p, Sx_{n+1}) + k^{n+1}d(Sx_0, Sy_0) + \sum_{i=0}^{n} k^{n+1}\epsilon_n$*; further,*
$\lim_n Sy_n = p$ *if and only if* $\lim_n \epsilon_n = 0$.

Let A, B, S and T be mappings of a metric space (X, d) into itself such that $AX \subseteq TX$ and $BX \subseteq SX$. Then for an arbitrary $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. For this point x_1 we can choose a point x_2 such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$(2) \qquad y_{2n} = Ax_{2n} = Tx_{2n+1} \;\; \textit{and} \;\; y_{2n+1} = Sx_{2n+2} = Bx_{2n+1},$$

for all $n = 0, 1, 2, ...$
This iterative sequence has been used by many authors (see [4], [13], [14]) to establish existence of unique common fixed points for some classes of contractions in both metric and Banach spaces. For example, Pathak et al.[14,15] proved that, the sequence converges to the unique fixed point of the four maps given that some contractive conditions are satisfied and that the pairs (A, S) and (B, T) are weakly compatible. Two maps are weakly compatible if they commute at their coincidence points. For more on compatibility and its weaker forms, the reader may see [4], [8], [9], [13], [14] and

[15].

The iterative procedure (2) is considered numerically stable if and only if a sequence $\{z_n\}$ in X, approximately close to $\{y_n\}$, converges to the common fixed point $p \in X$ of A, B, S and T.
In this scheme we define

$$\epsilon_{2k} = d(Bz_{2k+1}, Sz_{2k}) \ \ and \ \ \epsilon_{2k+1} = d(Az_{2k}, Tz_{2k+1}) \tag{3}$$

for $k = 1, 2, ...$

Motivated by Definition 1.1 above, we shall say that the iterative procedure (2) is stable with respect to the mappings A, B, S and T if and only if

$$\lim_{k\to\infty} \epsilon_k = 0 \Leftrightarrow \lim_{n\to\infty} Sz_{2n} = \lim_{n\to\infty} Tz_{2n+1} = p.$$

Beginning with the contractive definition

$$d(Ax, By) \leq kd(Sx, Ty), \quad k \in [0, 1), \tag{4}$$

we shall investigate stability of the (S, T)-iterations for some classes of contractive mappings.

2 Main results

Theorem 2.1. *Let A,B,S and T be mappings of a metric space (X,d) into itself such that $AX \subseteq TX$ and $BX \subseteq SX$, and SY or TY a complete subspace of X. Let p be a common fixed point of A,B,S,T, and the sequence $\{y_n\}$ in X generated by (2), for $x_0 \in X$ and $n = 0, 1, 2, ...,$ converge to p.*

Let $\{z_n\}$ *be in* X, *and define*

$$\epsilon_{2n} = d(Bz_{2n+1}, Sz_{2n}) \quad and \quad \epsilon_{2n+1} = d(Az_{2n}, Tz_{2n+1}), \quad n = 0, 1, ...$$

If A,B,S,T satisfy (4) for all $x, y \in X$, *then*

$$\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} Sz_{2n} = \lim_{n\to\infty} Tz_{2n+1} = p.$$

Proof. Suppose $\lim_{n\to\infty} \epsilon_n = 0, \quad n = 0, 1, 2....$ Using (2), (4) and the triangle inequality,

$$\begin{aligned} d(p, Tz_{2n+1}) &\leq d(Tz_{2n+1}, Az_{2n}) + d(Az_{2n}, Bx_{2n+1}) + d(Bx_{2n+1}, p) \\ &\leq \epsilon_{2n+1} + kd(Sz_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, p) \\ &\leq \epsilon_{2n+1} + k[d(Sz_{2n}, Bz_{2n+1}) + d(Bz_{2n+1}, Tx_{2n+1})] + d(Bx_{2n+1}, p) \\ &= \epsilon_{2n+1} + k\epsilon_{2n} + kd(Bz_{2n+1}, Ax_{2n}) + d(Bx_{2n+1}, p) \\ &\leq \epsilon_{2n+1} + k\epsilon_{2n} + k^2 d(Sx_{2n}, Tz_{2n+1}) + d(Bx_{2n+1}, p) \end{aligned}$$

Letting $n \to \infty$, we have $\epsilon_n = 0, Sx_{2n} = 0$ and $Bx_{2n+1} = 0$ so that $(1 - k^2)d(p, Tz_{2n+1}) \leq 0$.
But $1 - k^2 > 0$. Therefore, $Tz_{2n+1} = p$ as $n \to \infty$.
Since $\{Az_{2n}\} \subseteq \{Tz_{2n+1}\}$, we also have $Az_{2n} = p$ as $n \to \infty$.

Further,

$$\begin{aligned} d(p, Sz_{2n}) &\leq d(Sz_{2n}, Bz_{2n+1}) + d(Bz_{2n+1}, Az_{2n}) + d(p, Az_{2n}) \\ &\leq \epsilon_{2n} + kd(Sz_{2n}, Tz_{2n+1}) + d(p, Az_{2n}) \end{aligned}$$

As $n \to \infty$,

$$d(p, Sz_{2n}) \leq kd(Sz_{2n}, p)$$

so that, since $1-k>0$, $Sz_{2n}=p$ as $n\to\infty$.
Therefore,

$$\lim_{k\to\infty}\epsilon_k=0\Rightarrow\lim_{n\to\infty}Sz_{2n}=\lim_{n\to\infty}Tz_{2n+1}=p.$$

Conversely, suppose

$$\lim_{n\to\infty}Sz_n=\lim_{n\to\infty}Tz_n=p.$$

Then, by $\{Az_{2n}\}\subseteq\{Tz_{2n+1}\}$ and $\{Bz_{2n+1}\}\subseteq\{Sz_{2n}\}$, we also have

$$\lim_{n\to\infty}Az_{2n}=\lim_{n\to\infty}Bz_{2n+1}=p.$$

Therefore, $\lim_{n\to\infty}\epsilon_n=0,\quad n=0,1,2....$ This completes the proof.

Corollary 2.2. *Let A,B,S be mappings of a metric space (X,d) into itself such that $AX\cup BX\subseteq SX$, and SX a complete subspace of X. Let p be a common fixed point of A,B,S, and the sequence $\{y_n\}$ in X generated by $x_0\in X$ and*

(5) $y_{2n}=Ax_{2n}=Sx_{2n+1}$ *and* $y_{2n+1}=Sx_{2n+2}=Bx_{2n+1},\quad n=0,1,2,...$

converge to p.
Let $\{z_n\}$ be in X, and define

$$\epsilon_{2n}=d(Bz_{2n+1},Sz_{2n})\ \text{ and }\ \epsilon_{2n+1}=d(Az_{2n},Sz_{2n+1}),\quad n=0,1,....$$

If A,B,S satisfy

$$d(Ax,By)\le kd(Sx,Sy),\quad k\in[0,1),$$

for all $x,y\in X$, then

$$\lim_{n\to\infty}\epsilon_n=0\Leftrightarrow\lim_{n\to\infty}Sz_n=p.$$

Proof. Let $S = T$ in Theorem 2.1.

Remark 2.3. If we put $A = B$ into Theorem 2.1., the iterative scheme (5) becomes the Picard iteration and A becomes an S-contraction.

It is important to note that if we put $a = 1$ and $\varphi(u) = ku$ for all $u \in \Re$ in the definition of parametrically $\varphi(\epsilon, \delta; a)$-contraction introduced by Pathak et al in [15], we obtain (4). Hence, our assumption on the existence of common fixed point for A, B, S, T is in order, especially when the pairs (A, S) and (B, T) are weakly compatible.

In what follows, we obtain similar results for the classes of contractive mappings given below.

Let A, B, S, T be selfmaps of a metric space X. For all $x, y \in X$, there exist constants $\gamma, \beta \in [0, \frac{1}{2})$ such that

$$d(Ax, By) \leq \beta[d(Sx, Ax) + d(Ty, By)] \tag{6}$$

and

$$d(Ax, By) \leq \gamma[d(Sx, By) + d(Ty, Ax)]. \tag{7}$$

Theorem 2.4. *Let A,B,S and T be mappings of a metric space (X,d) into itself such that $AX \subseteq TX$ and $BX \subseteq SX$, and SY or TY a complete subspace of X. Suppose p is a common fixed point of A,B,S,T, and the sequence $\{y_n\}$ in X generated by (2), for $x_0 \in X$ and $n = 0, 1, 2, ...$, converges to p. Let $\{z_n\}$ be in X, and define*

$$\epsilon_{2n} = d(Bz_{2n+1}, Sz_{2n}) \quad \textit{and} \quad \epsilon_{2n+1} = d(Az_{2n}, Tz_{2n+1}), \quad n = 0, 1, ...$$

If A,B,S,T satisfy at least one of (6) and (7) for all $x, y \in X$ and $\gamma, \beta \in [0, \frac{1}{2})$, then

$$\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} Sz_{2n} = \lim_{n\to\infty} Tz_{2n+1} = p.$$

Proof. Firstly, assuming (6) holds. Suppose $\epsilon_n = 0$ as $n \to \infty$.
With the triangle inequality and the iterative scheme (2), we obtain

$$\begin{aligned}
d(Az_{2n}, Sz_{2n}) &\leq d(Az_{2n}, Bz_{2n+1}) + d(Bz_{2n+1}, Sz_{2n}) \\
&\leq \beta[d(Sz_{2n}, Az_{2n}) + d(Tz_{2n+1}, Bz_{2n+1})] + \epsilon_{2n} \\
&= \epsilon_{2n} + \beta d(Az_{2n}, Sz_{2n}) + \beta d(Tz_{2n+1}, Bz_{2n+1}) \\
(1-\beta) d(Az_{2n}, Sz_{2n}) &\leq \epsilon_{2n} + \beta[d(Tz_{2n+1}, Az_{2n}) + d(Az_{2n}, Bz_{2n+1})] \\
&= \epsilon_{2n} + \beta\epsilon_{2n+1} + \beta d(Az_{2n}, Bz_{2n+1}) \\
&\leq \epsilon_{2n} + \beta\epsilon_{2n+1} + \beta[d(Az_{2n}, Sz_{2n}) + d(Sz_{2n}, Bz_{2n+1})]
\end{aligned}$$

Therefore,

$$(1-2\beta) d(Az_{2n}, Sz_{2n}) \leq (1+\beta)\epsilon_{2n} + \beta\epsilon_{2n+1}.$$

Since $1 - 2\beta > 0$, we have $Az_{2n} = Sz_{2n}$ as $n \to \infty$.
Putting $x = z_{2n},\ y = x_{2n+1}$ into (6), as $n \to \infty$, $Az_{2n} = p$.
By the inclusions $AX \subseteq TX$ and $BX \subseteq SX$, we have

$$\lim_{n\to\infty} Sz_{2n} = \lim_{n\to\infty} Az_{2n} = \lim_{n\to\infty} Tz_{2n+1} = p.$$

Conversely, let $\lim_{n\to\infty} Sz_{2n} = \lim_{n\to\infty} Tz_{2n+1} = p$. Then $\lim_{n\to\infty} Az_{2n} = \lim_{n\to\infty} Bz_{2n+1}$ as well.
Consequently,

$$\lim_{n\to\infty} \epsilon_{2n} = \lim_{n\to\infty} \epsilon_{2n+1} = 0.$$

In a similar manner, when (7) holds we obtain

$$d(Az_{2n}, Sz_{2n}) \leq (1+\gamma)\epsilon_{2n} + \gamma\epsilon_{2n+1}$$

and the rest of the proof is easy.

The summary of Theorems 2.1 and 2.4 is stated below.

Corollary 2.5. *Let A,B,S and T be mappings of a metric space (X,d) into itself such that $AX \subseteq TX$ and $BX \subseteq SX$, and SY or TY a complete subspace of X. Suppose p is a common fixed point of A,B,S,T, and the sequence $\{y_n\}$ in X generated by (2), for $x_0 \in X$ and $n = 0, 1, 2, ...$, converges to p. Let $\{z_n\}$ be in X, and define*

$$\epsilon_{2n} = d(Bz_{2n+1}, Sz_{2n}) \;\; and \;\; \epsilon_{2n+1} = d(Az_{2n}, Tz_{2n+1}), \quad n = 0, 1, ...$$

If A,B,S,T satisfy

$$d(Ax, By) \leq \tag{8}$$

$$max\{\alpha d(Sx, Ty),\; \beta[d(Sx, Ax) + d(Ty, By)],\; \gamma[d(Sx, By) + d(Ty, Ax)]\}$$

for all $x, y \in X, \quad 0 \leq \alpha < 1,\; 0 \leq \gamma, \beta < \frac{1}{2}$, *then*

$$\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} Sz_{2n} = \lim_{n\to\infty} Tz_{2n+1} = p.$$

Remark 2.6. Putting $A = B$ and $S = T = Id$, where Id is the identity mapping of X, in (8) yields the Zamfirescu mapping. It is worth emphasizing that the Zamfirescu mapping is a nice generalization of several contractractive definitions in the literature. Interested reader may see [1], [16] and [19].

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G-Loewner chains and parabolic starlike mappings in several complex variables [1]

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Abstract

Let f be a locally univalent function on the unit disc and $Q : \mathbb{C}^n \to \mathbb{C}$ be a homogenous polynomial of degree 2. We consider the normalized extension of f to the Euclidean unit ball $B^n \subseteq \mathbb{C}$ given by

$$[\Phi_{n,Q}(f)](z) = (f(z_1) + f'(z_1)Q(\hat{z}), \sqrt{f'(z_1)}\hat{z})$$

where $z = (z_1, \hat{z}) \in B^n$. This operator was recently introduced by Muir. In the case $Q \equiv 0$ this operator reduces to the well known Roper-Suffridge extension operator. By using the method of Loewner chain, we prove that if f can be embedded as the first element of g-Loewner chain on the unit disc, where $g = \frac{1}{q}$, then $F = \Phi_{n,Q}(f)$ can be embedded as the first element of g-Loewner chain on B^n. Moreover, let f be a parabolic starlike mapping on U, then $F = \Phi_{n,Q}(f)$ is parabolic starlike mapping on B^n if and only if $\|Q\| \leq \frac{1}{4}$.

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1 Introduction

Let $\mathbb{C}^n$ be the vector space of n-complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $< z, w >= \sum_{k=1}^{\infty} z_k \bar{w}_k$ and Euclidean norm $\|z\| = < z, z >^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n by B^n. In the case of one complex variable, B^1 is denoted by U. It is convenient, if $n \geq 2$, to write a vector $z \in \mathbb{C}^n$ as $z = (z_1, \hat{z})$, where $z_1 \in \mathbb{C}$ and $\hat{z} \in \mathbb{C}^{n-1}$.

Let Q_n denote the set of all homogenous polynomials $Q : \mathbb{C}^n \to \mathbb{C}$ of degree 2. That is, $Q(\lambda z) = \lambda^2 Q(z)$ for all $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. We know that Q_n is a Banach space with the norm

$$\|Q\| = \sup_{z \in \mathbb{C}^n \setminus \{0\}} \frac{|Q(z)|}{\|z\|^2}, \quad Q \in Q_n.$$

Let $H(B^n, \mathbb{C}^n)$ denote the topological vector space of all holomorphic mappings $F : B^n \to \mathbb{C}^n$. If $F \in H(B^n)$, we say that F is normalized if $F(0) = 0$ and $DF(0) = I$, where DF is the Fréchet differential of F and I is the identity operator on $\mathbb{C}^n$. Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n, and $S_1 = S$ is the classical family of univalent mappings of U.

A map $f \in S(B^n)$ is said to be convex if its image is convex domain in $\mathbb{C}^n$, and starlike if the image is a starlike domain with respect to 0. We denote the classes of normalized convex and starlike mappings on B^n respectively by $K(B^n)$ and $S^*(B^n)$.

In 1995, Roper and Suffridge [17] introduced an extension operator which gives a way of extending a(locally) univalent function on the unit disc to a(locally) univalent mapping of B^n into $\mathbb{C}^n$.
For fixed $n \geq 2$, the Roper-Suffridge extension operator (see [5] and [17]) in the function

$$[\Phi_n(f)](z) = (f(z_1), \sqrt{f'(z_1)}\hat{z}), \quad f \in S_1, z \in B^n$$

The branch of the power function is chosen so that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

The following results illustrate the important and usefulness of the Roper-Suffridge extension operator

$$\Phi_n(K_1) \subseteq K_n, \quad \Phi_0(S_1^*) \subseteq S_n^*,$$

The first was proved by K. A. Roper and T. J. Suffridge when they introduced their operator [17], while the second result was given by I. Graham and G. Kohr [6]. Until now, it is difficult to constant the concrete convex mappings, starlike mappings on B^n. By making use of the Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator. A good treatment of further applications of the Roper-Suffridge extension operator can be found in the recent book by Graham and Kohr [5].

For a function $f \in S$, we introduce the quantity

$$\Lambda_f(z) = \frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)} - \bar{z}, \tag{1.1}$$

for $z \in U$. Now, fix $z \in U$. The disk automorphism transform is denoted by ψ. In other words,

$$\psi(w) = \frac{z-w}{1-\bar{z}w}.$$

Consider the Koebe transform of f with respect to disk automorphism ψ by the form

$$g(w) = \frac{f(\psi(w)) - f(\psi(0))}{f'(\psi(0))\psi'(0)}$$

for $w \in U$. Clearly, $g \in U$, and a simple calculation shows that $g''(0) = -2\Lambda_f(z)$. It then follows that g has a power series expansion of the form

$$g(w) = w - \Lambda_f(z)w^2 + \mathcal{O}(\|w\|^3),$$

for $w \in U$. The well-known coefficient bound for the second coefficient of a function in S gives

$$|\Lambda_f(z)| \leq 2,$$

for $f \in S$ and $z \in U$.

Definition 1.1 *Let $Q \in Q_{n-1}$. For any $f \in S$, define the operator $\Phi_Q(f) : B^n \to \mathbb{C}^n$ by*

$$[\Phi_Q(f)](z) = (f(z_1) + f'(z_1)Q(\hat{z}), \sqrt{f'(z_1)}\hat{z}), \quad z = (z_1, \hat{z}) \in B^n,$$

we choose the branch of the power function such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

Recently Muir [13] proved the following results:

Theorem 1.2 *If $Q \in Q_{n-1}$, then $\Phi_{n,Q}(K) \subseteq K(B^n)$ if and only if $\|Q\| \leq \frac{1}{2}$.*

Theorem 1.3 *If $Q \in Q_{n-1}$, then $\Phi_{n,Q}(S^*) \subseteq S^*(B^n)$ if and only if $\|Q\| \leq \frac{1}{4}$.*

Let

$$\mathcal{M}_g = \{h \in H(B^n) : h(0) = 0, Dh(0) = I_n, \\ < h(z), \frac{z}{\|z\|^2} > \in g(U), z \in B^n \backslash \{0\}\}$$

For $g(\xi) = \frac{1+\xi}{1-\xi}$, $\xi \in U$, we obtain the well known set $\mathcal{M}_g = \mathcal{M}$ of mapping with "positive real part on B^n", i.e.

$$\mathcal{M} = \{h \in H(B^n) : h(0) = 0, Dh(0) = I_n, \\ Re < h(z), \frac{z}{\|z\|^2} >> 0, z \in B^n \backslash \{0\}\}$$

Now, we give the definition of parabolic starlike mappings on B^n (see [9]). Let

$$q(\eta) = 1 + \frac{4}{\pi^2}(\log \frac{1 + \sqrt{\eta}}{1 - \sqrt{\eta}})^2. \tag{1.2}$$

Then q is a biholomorphic mapping from U onto domain Ω, where

$$\Omega = \{w = u + iv : v^2 < 4u\} = \{w : |w - 1| < 1 + Rew\}.$$

we not that Ω is a parabolic region in the right half- plane.

Definition 1.4 *Let f be a normalized locally biholomorphic mapping on B^n, we say that f is a parabolic starlike mapping if*

$$< [Df(z)]^{-1}f(z), \frac{z}{\|z\|} >\in g(U), \quad z \in B^n\setminus\{0\}$$

where $g = \frac{1}{q}$.

Next, we recall the definition of subordination and Loewner chain. For various results related to the Loewner chain in $\mathbb{C}^n$, the reader may consult $[1-5, 8, 10, 14, 20]$.

Let $f, g \in H(B^n)$, we say that f is subordinate to g (and write $f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$. If g is biholomorphic on B^n, this is equvalent to requiring that $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.

Definition 1.5 *A mapping $f : B^n \times [0, +\infty) \to \mathbb{C}^n$ is called Loewner chain if $f(., t)$ is biholomorphic on B^n, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$, and $f(., s) \prec f(., t)$ wherever $0 \leq s \leq t < \infty$ and $z \in B^n$.*

The above subordination condition is equivalent to the fact that there exists a unique biholomorphic Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$, such that

$$f(z, s) = f(v(z, s, t), t), \quad z \in B^n, \quad t \geq s \geq 0.$$

The authors [4], [8] (see also [5, Theorem 8.1.6]; cf. [14] and [16]) obtained the following sufficient condition for a mapping to be a Loewner chain.

Lemma 1.6 *Let $h = h(z, t) : B^n \times [0, +\infty) \to \mathbb{C}^n$ satisfy the following conditions:*

(i) $h(., t) \in \mathcal{M}$ for $t \geq 0$;

(ii) $h(z, .)$ is measurable on $[0, +\infty)$ for $z \in B^n$.

Let $f = f(z,t) : B^n \times [0,+\infty) \to \mathbb{C}^n$ *be a mapping such that* $f(.,t) \in H(B^n)$, $f(0,t) = 0$, $Df(0,t) = e^t I_n$ *for* $t \geq 0$, *and* $f(z,.)$ *is locally absolutely continuous on* $[0,+\infty)$ *locally uniformly with respect to* $z \in B^n$. *Assume that*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad a.e.\ t \geq 0, \forall z \in B^n.$$

Further, assume that exists an increasing sequence $\{t_m\}_{m\in\mathbb{N}}$ *such that* $t_m > 0$, $t_m \to \infty$ *and*

$$\lim_{m\to\infty} e^{t_m} f(z,t_m) = F(z),$$

locally uniformly on B^n. *Then* $f(z,t)$ *is a Loewner chain.*

Remark 1.7 *In the case of one complex variable if* $f(\xi,t)$ *is a Loewner chain, then it is well known that* $\{e^{-t}f(.,t)\}_{t\geq 0}$ *is a normal family on* U, *and there exists function* $p = p(\xi,t)$ *such that* (*see* [15/85]) $p(0) = 0$, $Rep(\xi) > 0$ *for* $t \geq 0$, *and* $p(\xi,.)$ *is a measurable on* $[0,+\infty)$ *for* $\xi \in U$ *and* (*see* [35/85])

$$\frac{\partial f}{\partial t}(\xi,t) = \xi f'(\xi,t)p(\xi,t), \quad a.e.\ t \geq 0, \forall \xi \in U.$$

Remark 1.8 *In higher dimensions, Graham, Kohr and Kohr [8] (see also [5]), proved that if* $f(z,t)$ *is a Loewner chain on* B^n, *then* $f(z,.)$ *is locally Lipschitz on* $[0,+\infty)$ *locally uniformly with respect to* $z \in B^n$. *Also, then exists a mapping* $h = h(z,t)$, *which satisfies the conditions* (*i*) *and* (*ii*) *Lemma 1.6 such that* (*see [4]*)

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad a.e.\ t \geq 0, \forall z \in B^n. \tag{1.3}$$

Now, we are able to recall the notions of g-Loewner chain and g-parametric representation (cf. [4]). For our purpose, we consider these notion only for $g = \frac{1}{q}$.

Definition 1.9 *A Loewner chain $f(z,t)$ is called g-Loewner chain (cf. [4, 10/44]) if $\{e^{-t}f(.,t)\}_{t\geq 0}$ is a normal family on B^n and the mapping $h = h(z,t)$ which occurs in the Loewner differential equation (1.3) satisfies the condition $h(.,t) \in \mathcal{M}_g$ for a.e. $t \geq 0$.*

Definition 1.10 *Let $f : B^n \to \mathbb{C}^n$ be a normalized holomorphic mapping. we say that f has g-parametric representation if there exists a g-Loewner chain $f(z,t)$ such that $f = f(,.0)$.*

Let $S_g^0(B^n)$ be the set of all mapping f such that there exists a g-Loewner chain $f(z,t)$ such that $f = f(,.0)$. When $f \in S_g^0(B^n)$, we also say that f has g-parametric representation on B^n [4]. If $g(\xi) = \frac{1+\xi}{1-\xi}$, then $S_g^0(B^n)$ reduce to the set $S^0(B^n)$ of mapping which have parametric representation on B^n. Clearly $S_g^0(B^n) \subset S^0(B^n) \subset S(B^n)$. On the other hand, we remark that, in several complex variables, there exist mappings which can be imbedded in Loewner chain without having parametric representation ([4, Example 2.12]).

2 Some Lemmas

In order to prove the main results, we need the following lemmas.

Lemma 2.1 *[9]. Let $g = \frac{1}{q}$. Then $g(U)$ is starlike with respect to* 1.

Lemma 2.2 *[21]. Let p be a holomorphic function on U. If Re $p(z) > 0$ and $p(0) > 0$, then*

$$|p'(z)| \leq \frac{2Re\ p(z)}{1-|z|^2}. \tag{2.1}$$

Lemma 2.3 *[21]. Let p be a normalized biholomorphic function on U. Then*

$$|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}| \leq 4. \tag{2.2}$$

3 Main Results

Theorem 3.1 *Let* $Q : \mathbb{C}^{n-1} \to \mathbb{C}^n$ *be a homogeneous polynomial of degree* 2 *such that* $\|Q\| \leq \frac{1}{4}$. *Also, assume* $f \in S$ *can be embedded as the first element of a g-Loewner chain on* U, *where* $g = \frac{1}{q}$. *Then* $F = \Phi_{n,Q}(f)$ *can be embedded as the first element of a g-Loewner chain on* B^n.

Proof: Let $f(z_1, t)$ be a g-Loewner chain such that $f(z_1) = f(z_1, 0)$ for $z_1 \in U$. Let $F(z,t)$ be the map defined by

$$F(z,t) = \left(f(z_1,t) + Q(\hat{z}) f'(z_1,t), \hat{z} e^{\frac{t}{2}} (f'(z_1,t))^{\frac{1}{2}} \right), \tag{3.1}$$

for $z = (z_1, \hat{z}) \in B^n$ and we choose the branch of the power function such that $(f'(z_1,t))^{\frac{1}{2}} |_{z_1=0} = e^{\frac{t}{2}}$ for $t \geq 0$. For $\|Q\| \leq \frac{1}{4}$, we know that $F(z,t)$ a Loewner chain (see [12]).

Since $f(z_1,t)$ is a Loewner chain, $f(z_1,t)$ is locally absolutely continuous on $[0, +\infty)$, locally uniformly with respect to $z_1 \in U$, and there exists the function $p(z_1,t)$ that is holomorphic on U and measurable in $t \geq 0$, with $p(0,t) = 1$, $Re p(z_1,t) > 0$ for $z_1 \in U$, and such that (see [15])

$$\frac{\partial f}{\partial t}(z_1,t) = z_1 f'(z_1,t) p(z_1,t), \quad a.e.\ t \geq 0, \forall z_1 \in U.$$

The fact that $f(z_1,t)$ is a g-Loewner chain is equivalent to the condition

$$|p(z_1,t) - 1| < 1, \quad a.e.\ t \geq 0, \forall z_1 \in U.$$

The mapping $h = h(z,t)$ which occurs in the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t) h(z,t), \quad a.e.\ t \geq 0, \forall z \in B^n.$$

is given by

$$h(z,t) = \left(z_1 p(z_1,t) - Q(\hat{z}), \frac{\hat{z}}{2} \left(1 + p(z_1,t) + z_1 p'(z_1,t) + Q(\hat{z}) \frac{f''(z_1,t)}{f'(z_1,t)} \right) \right)$$

for all $z \in B^n$ and $t \geq 0$.

We will show that $h(.,t) \in \mathcal{M}_g$ for a.e. $t \geq 0$, which is equivalent to

$$\left| \frac{1}{\|z\|^2} < h(z,t), z > -1 \right| < 1, \quad a.e.\ t \geq 0, \forall z \in B^n \backslash \{0\}.$$

If $z = (z_1, 0)$ then

$$\left| \frac{1}{\|z\|^2} < h(z,t), z > -1 \right| = |p(z_1,t) - 1| < 1, \quad a.e.\ t \geq 0,$$

in view of the fact that $f(z_1,t)$ is a g-Loewner chain. Hence, it suffices to assume that $z = (z_1, \hat{z}) \in B^n \backslash \{0\}$ with $\hat{z} \neq 0$. Then it is easy to see that $h(.,t)$ is a holomorphic in a neighborhood of each point $z = (z_1, \hat{z}) \in \bar{B}^n$ with $\hat{z} \neq 0$, and in view of the minimum principal for harmonic functions, it enough to prove that

$$| < h(z,t), z > -1| \leq 1, \ \ a.e.\, t \geq 0, \forall z = (z_1, \hat{z}) \in \mathbb{C}^n, \ |z_1|^2 + \|\hat{z}\|^2 = 1, \hat{z} \neq 0.$$

By elementary computation, we obtain that

$$< h(z,t), z > = \frac{1+|z_1|^2}{2} p(z_1,t) + \frac{1-|z_1|^2}{2}[z_1 p'(z_1,t)] + \frac{1-|z_1|^2}{2} + Q(\hat{z})\{\frac{1-|z_1|^2}{2}\frac{f''(z_1,t)}{f'(z_1,t)} - \bar{z}_1\}.$$

Therefor, using the fact that $\|Q\| \leq \frac{1}{4}$,we need to prove that

$$| < h(z,t), z > -1| \leq 1, \quad a.e.\ t \geq 0, \forall z = (z_1, \hat{z}) \in \partial B^n \ , \hat{z} \neq 0,$$

or, equivalently

$$Re < h(z,t), z > \geq \frac{1}{2}, \quad a.e.\ t \geq 0, \forall z = (z_1, \hat{z}) \in \partial B^n \ , \hat{z} \neq 0.$$

On the other hand, since $e^{-t} f(.,t) \in S$ (see [12]), $t \geq 0$, it is well known that (by using of Lemma 2.3)

$$\left| \frac{1-|z_1|^2}{2}\frac{f''(z_1,t)}{f'(z_1,t)} - \bar{z}_1 \right| \leq 2 \quad z_1 \in U, \ t \geq 0. \tag{3.2}$$

Since $p(0,t)=1$, $Rep(z_1,t)>0$ for $z_1\in U$, it follows that (see e.g. [5])

$$|p'(z_1,t)| \le \frac{2}{1-|z_1|^2} Rep(z_1,t) \quad z_1\in U,\ t\ge 0. \tag{3.3}$$

(see e.g. [15]); from this we obtain

$$Re\,(z_1 p'(z_1,t)) \ge -\frac{2|z_1|}{1-|z_1|^2} Rep(z_1,t) \quad z_1\in U,\ t\ge 0. \tag{3.4}$$

Fix $t\ge 0$ and let $z=(z_1,\hat{z})\in\partial B^n$ with $\hat{z}\neq 0$. Using this inequality together with (3.2), we obtain

$$\begin{aligned} Re < h(z,t), z >= & \frac{1+|z_1|^2}{2} Rep(z_1,t) + \frac{1-|z_1|^2}{2} Re[z_1 p'(z_1,t)] + \frac{1-|z_1|^2}{2} \\ & + Re\left[Q(\hat{z})\{\frac{1-|z_1|^2}{2}\frac{f''(z_1,t)}{f'(z_1,t)} - \bar{z}_1\}\right] \\ & \ge \frac{(1-|z_1|)^2}{2} Rep(z_1,t) + \frac{1-|z_1|^2}{2} - 2(1-|z_1|^2)\|Q\| \ge \frac{1}{2}, \end{aligned}$$

whenever $\|Q\|\le\frac{1}{4}$.

Finally, it remains to prove that $\{e^{-t}F(.,t)\}_{t\ge 0}$ is a normal family on B^n. Indeed, since $\{e^{-t}f(.,t)\}_{t\ge 0}$ is a normal family on U, there exists a sequence (t_m) such that $t_m\to\infty$, $t_m>0$ and $e^{-t_m}f(z_1,t_m)\to w(z_1)$ locally uniformly on U as $m\to\infty$. It is clear that $w\in S$, in view of Hurwitz's theorem. Then it is easy to see that $\lim_{m\to\infty} e^{-t_m}F(z,t_m) = W(z)$ locally uniformly on B^n as $m\to\infty$, where $W(z)=\Phi_{n,Q}(w)$, thus $\{e^{-t}F(.,t)\}_{t\ge 0}$ is also normal family on B^n.

Combining the above argument and taking into account Lemma 1.6, we deduce that $F(z,t)$ is a g-Loewner chain as desired. This completes the proof.

In view of Theorem 3.1, we obtain the following particular cases. This result was obtained in [12], in the case $g(\xi)=\frac{1-\xi}{1+\xi}$.

Corollary 3.2 *Let $Q:\mathbb{C}^{n-1}\to\mathbb{C}^n$ be a homogeneous polynomial of degree 2 such that $\|Q\|\le\frac{1}{4}$ and $f:U\to\mathbb{C}$ has g-parametric representation, then $F=\Phi_{n,Q}(f)\in S_g^0(B^n)$ where $g=\frac{1}{q}$.*

Proof: Since f has g-parametric representation, there exists g-Loewner chain $f(\xi,t)$ such that $f = f(.,0)$. In view of Theorem 3.1, $F(z,t)$ given by (3.1) is a g-Loewner chain. Since $\{e^{-t}F(.,t)\}_{t\geq 0}$ is also normal family on B^n by the proof of Theorem 3.1 and $F = F(.,0)$ we deduce that $F = \Phi_{n,Q}(f) \in S_g^0(B^n)$, as desired. This completes the proof.

In the next Theorem, we show that if $Q \in Q_{n-1}$ and $f \in S$ is a parabolic starlike mapping on U, then $F = \Phi_{n,Q}(f)$ is a parabolic starlike mapping on B^n if and only if $\|Q\| \leq \frac{1}{4}$.

Theorem 3.3 *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}^n$ be a homogeneous polynomial of degree 2 and let $f : U \to \mathbb{C}$ be a normalized locally univalent function, which satisfies the condition*

$$\left|\frac{z_1 f'(z_1)}{f(z_1)} - 1\right| < 1, \qquad z_1 \in U. \tag{3.5}$$

Also let $F = \Phi_{n,Q}(f)$, then

$$\left|\frac{\|z\|^2}{< DF^{-1}(z)F(z), z >} - 1\right| < 1 \qquad z \in B^n \backslash \{0\},$$

and hence F is parabolic starlike mapping on B^n if and only if $\|Q\| \leq \frac{1}{4}$.

Proof: Let $f \in S$ is a parabolic starlike mapping on U and $\|Q\| \leq \frac{1}{4}$. Since $Re < DF^{-1}(z)F(z), z >> 0$ parabolic starlike mappings are starlike mappings by Suffridge [18]. Therefor $f(\xi,t) = e^t f(\xi)$ is a g-Loewner chain where $g = \frac{1}{q}$, $|\xi| < 1$. In view of Theorem 3.1, we deduce that $F(z,t)$ given by (3.1) is a g-Loewner chain and the fact that $\|Q\| \leq \frac{1}{4}$. On the other hand, since

$$F(z,t) = \left(e^t f(z_1) + Q(\hat{z})e^t f'(z_1), \hat{z}e^t\sqrt{f'(z_1)}\right) = e^t \Phi_{n,Q}(f)(z),$$

for $z_1 \in B^n$, $t \geq 0$. We deduce that $\Phi_{n,Q}(f)$ is a parabolic starlike mapping on B^n.

For the converse, suppose that $\|Q\| \geq \frac{1}{4}$, let $f(\xi) = \frac{\xi}{1-\xi}$, $\xi \in U$. Obviously f is a parabolic starlike mapping on U. Let $F = \Phi_{n,Q}(f)$. For $z = (z_1, \hat{z}) \in B^n$, by simple calculations we have (see [**?**] Theorem 4.1)

$$DF^{-1}(z)F(z) = \begin{pmatrix} \frac{f(z_1)}{f'(z_1)} - Q(\hat{z}) \\ \\ -\frac{1}{2}\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\hat{z} + \frac{1}{2}\frac{f''(z_1)}{f'(z_1)}Q(\hat{z})\hat{z} + \hat{z} \end{pmatrix}$$

therefor

$$\begin{aligned} < DF(z)^{-1}F(z), z >&= \frac{f(z_1)}{f'(z_1)}\bar{z}_1 - \bar{z}_1 Q(\hat{z}) \\ &\quad + \|\hat{z}\|^2 \left(1 - \frac{1}{2}\frac{f(z_1)f''(z_1)}{[f'(z_1)]^2}) + \frac{1}{2}\frac{f''(z_1)}{f'(z_1)}Q(\hat{z})\right). \end{aligned} \tag{3.6}$$

Now, let $u \in \partial B^{n-1}$ be such that $Q(u) = -\|Q\|$, and $z = (r, \sqrt{1-r^2}u) \in \partial B$ for $r \in (0, 1)$. A simple calculation reveals that $\Lambda_f(z_1) = 1$, and $Q(\hat{z}) = -(1-r^2)\|Q\|$. Therefore, by using (3.6), we have

$$\begin{aligned} Re < DF(z)^{-1}F(z), z > &= r^2(1-r) - (1-r^2)\|Q\| + (1-r^2)(1-r) \\ &= \frac{r^2}{1+r}(1-r^2) - (1-r^2)\|Q\| + \frac{(1-r^2)^2}{1+r} \\ &= (\frac{1}{1+r} - \|Q\|)(1-r^2) \end{aligned} \tag{3.7}$$

For r sufficiently close to 1 and $\|Q\| \geq 1/4$, the above relation will be negative (therefor less than $\frac{1}{2}$) , proving that F is not a parabolic starlike mapping on B^n. This complete the proof. The following corollary was obtained by Muir [13] in the case of $g(\xi) = \frac{1-\xi}{1+\xi}$ (in the case of $Q \equiv 0$ see [7]).

Corollary 3.4 *Let $n \geq 2$ and $Q \in Q_{n-1}$. Then $\Phi_{n,Q}(S_1^*) \subseteq S_n^*$ if and only if $\|Q\| \leq \frac{1}{4}$.*

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A class of univalent functions obtained by a general multiplier transformation [1]

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Abstract

In this paper we introduce a multiplier transformation denoted by $I_p(\beta, m, n; \lambda, l)$. Using this transformation, the class of univalent functions denoted by $S_p^{\alpha}(\beta, m, n; \lambda, l)$ is introduced. An integral operator denoted by $L(f)$ is also introduced and it allows the proof of an inclusion relation for this class.

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Key words and phrases: Analytic function, differential subordination, multiplier transformation, integral operator.

1 Introduction and preliminaries

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} \mid |z| < 1\}$,

$$A_n = \{f \in \mathcal{H} \mid f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\},$$

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with $A_1 = A$,

$$S = \{f \in A \mid f \text{ is univalent in } U\},$$

$$K = \left\{ f \in A \mid \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\},$$

the class of convex functions in U,

$$A(p,n) = \left\{ f \in \mathcal{H} \mid f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k,\ p, n \in \mathbb{N} = \{1, 2, \dots, \},\ z \in U \right\}.$$

In particular, we set

$$A(p,1) = A_p = \left\{ f \in \mathcal{H} \mid f(z) = z^p + \sum_{k=p+1} a_k z^k,\ p \in \mathbb{N},\ z \in U \right\}$$

and $A(1,1) = A$.

For two functions $f, g \in A(p,n)$, the Hadamard product (or convolution) $(f * g)(z)$ is defined, as usual, by

$$(f * g)(z) := z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k = (g * f)(z).$$

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Definition 1. [13, p. 16] Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant (1). (Note that the best dominant is unique up to a rotation of U).

In order to prove our main results we shall make use of the following lemmas.

Lemma A. (Hallenbeck and Ruscheweyh [13, p. 71]) *Let h be a convex function in U, with $h(0) = a$, $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad z \in U$$

then

$$p(z) \prec q(z) \prec h(z), \quad z \in U$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt.$$

The function q is convex and it is the best dominant.

2 Main results

Definition 2. For $l, \lambda, \beta \in \mathbb{R}$, $l \geq 0$, $\beta \geq 0$, $\lambda \geq 0$, $n, p \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $f \in A(p, n)$, we define the multiplier transformation $I_p(\beta, m, n; \lambda, l)$ on $A(p, n)$ by the following infinite series:

(2) $$I_p(\beta, m, n; \lambda, l)f(z)$$

$$:= z^p + \sum_{k=p+n}^{\infty} \left[\frac{(1-\lambda)(p-k) + l + \dfrac{(m+k-1)!}{m!(k-1)!}}{p + l - k + \dfrac{(m+k-1)!}{m!(k-1)!}} \right]^{\beta} \frac{(m+k-1)!}{m!k!} a_k z^k,$$

$z \in U$.

Remark 1. For $\beta \in \mathbb{N} \cup \{0\}$, $m = 1$, $p = 1$, $l = 0$, the operator was introduced and studied by Al-Oboudi [3] which reduces to the Sălăgean differential operator [15] for $\lambda = 1$. The operator $I_1(\beta, 1, n; 1, l)f(z)$ was studied recently by Cho and Srivastava [7] and Cho and Kim [8]. The operator $I_1(\beta, 1, n; 1, 1)f(z)$ was studied by Uralegaddi and Somanatha [17], the operator $I_1(\beta, 1, n; \lambda, 0)f(z)$ was introduced by Acu and Owa [1] and the operator $I_p(\beta, 1, n; 1, l)f(z)$ was investigated recently by Kumar, Taneja and Ravichandran [10]and Srivastava et al. [16]. The operator $I_p(0, 1, n; \lambda, l)f(z)$ was introduced and studied recently by A. Cătaş [6] and the operator $I_p(\beta, 1, n; 1, 0)f(z)$ was studied in [4], [9] and [12].

Properties. By using Definition 2, the following properties can be easily obtained:

P_1.
$$\left[p + l + \frac{(m+k-1)!}{m!(k-1)!} - k\right] I_p(\beta+1, m, n; \lambda, l)f(z)$$
$$= \left[p(1-\lambda) + l + \frac{(m+k-1)!}{m!(k-1)!} - k\right] I_p(\beta, m, n; \lambda, l)f(z)$$
$$+\lambda z I_p'(\beta, m, n; \lambda, l)f(z), \quad z \in U.$$

P_2.
$$I_p(\beta_1, m, n; \lambda, l)(I_p(\beta_2, m, n; \lambda, l)f(z))$$
$$= I_p(\beta_2, m, n; \lambda, l)(I_p(\beta_1, m, n; \lambda, l)f(z)).$$

P_3. If we let
$$\varphi^{\beta}_{p,m,n;\lambda,l}(z) :=$$
$$:= z^p + \sum_{k=p+n}^{\infty} \left[\frac{(1-\lambda)(p-k) + l + \dfrac{(m+k-1)!}{m!(k-1)!}}{p + l - k + \dfrac{(m+k-1)!}{m!(k-1)!}}\right]^{\beta} \frac{(m+k-1)!}{m!k!} z^k$$
then
$$I_p(\beta, m, n; \lambda, l)f(z) = (f * \varphi^{\beta}_{p,m,n;\lambda,l})(z).$$

For $p = 1$, $r, t \in \mathbb{R}$, $r + t = 1$, $f, g \in A_n$ we have:

P_4.
$$I_1(\beta, m, n; \lambda, l)[rf(z) + tg(z)] = rI_1(\beta, m, n; \lambda, l)f(z)$$
$$+tI_1(\beta, m, n; \lambda, l)g(z), \quad z \in U.$$

P_5. $\quad I_1(\beta, m, n; \lambda, l)[zf'(z)] = zI_1'(\beta, m, n; \lambda, l)f(z), \quad z \in U.$

Definition 3. Let $0 \le \alpha < 1$, $m \in \mathbb{N} \cup \{0\}$, $p, n \in \mathbb{N}$, $l, \lambda, \beta \in \mathbb{R}$, $l \ge 0$, $\lambda \ge 0$, $\beta \ge 0$. A function $f \in A(p, n)$ is said to be in the class $S_p^\alpha(\beta, m, n; \lambda, l)$ if it satisfies the following inequality

$$\operatorname{Re} I_p'(\beta, m, n; \lambda, l)f(z) > \alpha, \quad z \in U, \tag{3}$$

where $I_p(\beta, m, n; \lambda, l)f(z)$ is the operator given by (2).

Remark 2. For $\beta = 0$, $m = 1$, $\alpha = 0$ we obtain a univalence criterion, $\operatorname{Re} f'(z) > 0$, [14].

For $\beta = 0$, $m = 1$ and $0 \le \alpha < 1$ we obtain the class of univalent functions

$$R_\alpha = \{f \in A : \ \operatorname{Re} f'(z) > \alpha, \ z \in U, \ 0 \le \alpha < 1\}.$$

The new introduced class was obtained earlier by A. Cătaş [6], using a different method, namely subordination technique.

Theorem 1. *If* $0 \le \alpha < 1$, $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $l, \lambda, \beta \in \mathbb{R}$, $l \ge 0$, $\lambda > 0$, $\beta \ge 0$, $f \in A(1, n) = A_n$,

$$\frac{E(m, l, k)}{\lambda} > 0,$$

where

$$E(m, l, k) = 1 + l - k + \frac{(m + k - 1)!}{m!(k - 1)!}$$

then

$$S_1^\alpha(\beta + 1, m, n; \lambda, l) \subset S_1^\alpha(\delta, m, n; \lambda, l), \tag{4}$$

where

$$\delta = \delta(\alpha, m, n, l, k, \lambda) \tag{5}$$

$$= 2\alpha - 1 + 2(1 - \alpha)\frac{E(m, l, k)}{\lambda n} \cdot \sigma\left[\frac{E(m, l, k)}{\lambda n}\right]$$

and

$$\sigma(x)=\int_0^z \frac{t^{x-1}}{1+t}dt,\quad z\in U. \tag{6}$$

Proof. Let $f\in S_1^{\alpha}(\beta+1,m,n;\lambda,l)$. From Definition 3 we have

$$\operatorname{Re} I_1'(\beta+1,m,n;\lambda,l)f(z)>\alpha,\quad 0\le\alpha<1,\ z\in U. \tag{7}$$

From property P_1 we have:

$$I_1(\beta+1,m,n;\lambda,l)f(z) \tag{8}$$
$$=\frac{\lambda z I_1'(\beta,m,n;\lambda,l)f(z)+[E(m,l,k)-\lambda]\, I_1(\beta,m,n;\lambda,l)f(z)}{E(m,l,k)}$$

Differentiating (8), we obtain

$$I_1'(\beta+1,m,n;\lambda,l)f(z)=I_1'(\beta,m,n;\lambda,l)f(z) \tag{9}$$
$$+\frac{\lambda}{E(m,l,k)}z\cdot I_1''(\beta,m,n;\lambda,l)f(z),\quad z\in U$$

We let

$$p(z)=I_1'(\beta,m,n;\lambda,l)f(z)=1+b_nz^n+\dots,\quad z\in U, \tag{10}$$

and we deduce $p(0)=1$, $p\in\mathcal{H}[1,n]$.

Using (10) in (9), we have

$$I_1'(\beta+1,m,n;\lambda,l)f(z) \tag{11}$$
$$=p(z)+\frac{\lambda}{E(m,l,k)}zp'(z),\quad z\in U.$$

Then (7) becomes

$$\operatorname{Re}\left[p(z)+\frac{\lambda}{E(m,l,k)}zp'(z)\right]>\alpha,\quad z\in U \tag{12}$$

which is equivalent to

$$p(z)+\frac{\lambda}{E(m,l,k)}zp'(z)\prec h(z) \tag{13}$$

$$= \frac{1+(2\alpha-1)z}{1+z}, \quad z \in U.$$

Since $p(0) = h(0) = 1$, by using Lemma A, we have

$$p(z) \prec q(z) \tag{14}$$

$$= \frac{E(m,l,k)}{\lambda n z^{\frac{E(m,l,k)}{\lambda n}}} \int_0^z t^{\frac{E(m,l,k)}{\lambda n}-1} \cdot \frac{1+(2\alpha-1)t}{1+t} dt$$

$$= 2\alpha - 1 + 2(1-\alpha)\frac{E(m,l,k)}{\lambda n} \cdot \frac{1}{z^{\frac{E(m,l,k)}{\lambda n}}} \cdot \sigma(x)$$

where σ is given by (6).

The function q is convex and is the best dominant.

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\operatorname{Re} p(z) > \inf_{|z|<1} \operatorname{Re}\{q(z)\} = \operatorname{Re} q(1) \tag{15}$$

$$= 2\alpha - 1 + 2(1-\alpha)\frac{E(m,l,k)}{\lambda n}\sigma\left(\frac{E(m,l,k)}{\lambda n}\right).$$

Using (10) in (15), we obtain

$$\operatorname{Re} I_1'(\beta, m, n; \lambda, l) f(z) > \delta = \delta(\alpha, l, k, m, n)$$

$$= 2\alpha - 1 + \frac{2(1-\alpha)\,[E(m,l,k)]}{\lambda n}\sigma\left[\frac{E(m,l,k)}{\lambda n}\right].$$

From Definition 3, we have $f \in S_1^{\alpha}(\delta, m, n; \lambda, l)$. Since $f \in S_1^{\alpha}(\beta + 1, m, n; \lambda, l)$, we obtain $S_1^{\alpha}(\beta+1, m, n; \lambda, l) \subset S_1^{\alpha}(\beta, m, n; \lambda, l)$.

Definition 4. Let $0 \le \alpha < 1$, $m \in \mathbb{N} \cup \{0\}$, $p, n \in \mathbb{N}$, $l, \lambda, \beta \in \mathbb{R}$, $\lambda \ge 1$, $\beta \ge 0$, $f \in A(p,n)$. We denote by $L : A(p,n) \to A(p,n)$ the integral operator defined by $L(f) = F$, where F is given by

$$F(z) = \frac{p + E(m,l,k) - 1}{\lambda z^{\frac{p(1-\lambda)+E(m,l,k)-1}{\lambda}}} \int_0^z f(t) \cdot t^{\frac{p(1-\lambda)+E(m,l,k)-1}{\lambda}-1} dt \tag{23}$$

Remark. 1) $F(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k$, $z \in U$.

2) For $p = 1$, $l = 0$, $m = 1$, $\lambda = 1$, $f \in A_n$,

$$F(z) = \int_0^z \frac{f(t)}{t},$$

we obtain the Alexander operator [2].

3) For $p = 1$, $m = 1$, $\lambda = 1$, $l = 1$, $f \in A_n$,

$$F(z) = \frac{2}{z} \int_0^z f(t)dt,$$

we obtain the Libera operator [3].

4) For $p = 1$, $l \geq 0$, $m = 1$, $\lambda = 1$, $f \in A_n$,

$$F(z) = \frac{l+1}{z^l} \int_0^z f(t)t^{l-1}dt,$$

we obtain the Bernardi operator [5].

5) For $p \in \mathbb{N}$, $l > 0$, $m = 1$, $\lambda \geq 1$, $f \in A_p$,

$$F(z) = \frac{p+l}{\lambda z^{\frac{p(1-\lambda)+l}{\lambda}}} \int_0^z f(t)t^{\frac{p(1-\lambda)+l}{\lambda}}dt$$

was defined in [6].

Theorem 2. *Let $0 \leq \alpha < 1$, $m \in \mathbb{N} \cup \{0\}$, $p = 1$, $n \in \mathbb{N}$, $l, \lambda, \beta \in \mathbb{R}$, $\lambda \geq 1$, $\beta \geq 0$ and $f \in A_n$. Then f belongs to the class $S_1^{\alpha}(\beta, m, n; \lambda, l)$ if and only if F defined by (23) belong to the class $S_1^{\alpha}(\beta + 1, m, n; \lambda, l)$.*

Proof. (i) If we let $f \in S_1^{\alpha}(\beta, m, n; \lambda, l)$, then from Definition 3, we have:

$$\mathrm{Re}\, I_1'(\beta, m, n; \lambda, l)f(z) > \alpha, \quad z \in U. \tag{24}$$

From (23) we have

$$z^{\frac{(1-\lambda)+E(m,l,k)-1}{\lambda}} F(z) \tag{25}$$

$$= [E(m,l,k)] \cdot \int_0^z f(t) \cdot t^{\frac{(1-\lambda)+E(m,l,k)-1}{\lambda}-1} dt.$$

Differentiating (25) we obtain

(26) $$[E(m,l,k)-\lambda]\, F(z) + \lambda z F'(z)$$
$$= [E(m,l,k)-\lambda]\, f(z).$$

Since $F \in A_n$, $f \in A_n$, using P_4 and P_5 we have

(27) $$[E(m,l,k)-\lambda]\, I_1(\beta,m,n;\lambda,l)F(z)$$
$$+\lambda z[I_1(\beta,m,n;\lambda,l)F(z)]$$
$$= [E(m,l,k)-\lambda]\, I_1(\beta,m,n;\lambda,l)f(z), \quad z \in U.$$

Using P_1, (27) becomes

(28) $$[E(m,l,k)-\lambda]\, I_1(\beta+1,m,n;\lambda,l)F(z)$$
$$= [E(m,l,k)-\lambda]\, I_1(\beta,m,n;\lambda,l)f(z), \quad z \in U,$$

which is equivalent to

(29) $$I_1(\beta+1,m,n;\lambda,l)F(z) = I_1(\beta,m,n;\lambda,l)f(z), \quad z \in U.$$

From (29) we deduce

(30) $$\operatorname{Re} I_1'(\beta+1,m,n;\lambda,l)F(z) = \operatorname{Re} I_1'(\beta,m,n;\lambda,l)f(z), \quad z \in U.$$

From (24) we have

(32) $$\operatorname{Re} I_1'(\beta+1,m,n;\lambda,l)F(z) > \alpha, \quad z \in U$$

which implies $F \in S_1^\alpha(\beta+1,m,n;\lambda,l)$.

(ii) If we suppose that $F \in S_1^\alpha(\beta+1,m,n;\lambda,l)$, then using (30), we also get $f \in S_1^\alpha(\beta,m,n;\lambda,l)$.

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Legendre-Zhang's Conjecture & Gilbreath's Conjecture and Proofs Thereof [1]

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Abstract

If reduce limits which contain odd primes by a half for Legendre's conjecture, then there is at least an odd prime within the either half likewise, this is exactly the Legendre-Zhang's conjecture. We shall first prove the Legendre-Zhang's conjecture by mathematical induction with the aid of two number axes' positive half lines whose directions reverse from each other. Successively, prove the Gilbreath's conjecture by mathematical induction with the aid of the got result.

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Key words and phrases: Legendre-Zhang's conjecture, Mathematical induction, Number axis's positive half line, Odd prime points, PLS, $RPLS_P$, SCRP AB, Gilbreath's conjecture.

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1 Basic Concepts

The Gilbreath's conjecture was first suggested in 1958 by the American mathematician and amateur magician Norman L. Gilbreath following some doodling on a napkin.

He started by writing down the first few primes.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,. . .

Under these he put their differences:

1, 2, 2, 4, 2, 4, 2, 4, 6, 2, . . .

Under these he put the unsigned difference of the differences.

1, 0, 2, 2, 2, 2, 2, 2, 4, . . .

And he continued this process of finding iterated differences:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, . . .
1, 2, 2, 4, 2, 4, 2, 4, 6, 2, . . .
1, 0, 2, 2, 2, 2, 2, 2, 4, . . .
1, 2, 0, 0, 0, 0, 0, 2, . . .
1, 2, 0, 0, 0, 0, 2, . . .
1, 2, 0, 0, 0, 2, . . .
1, 2, 0, 0, 2, . . .
1, 2, 0, 2, . . .
1, 2, 2, . . .
1, 0, . . .
1, . . .

The Gilbreath's Conjecture is that the numbers in the first column except for first number of first rank are all one.

The Legendre's conjecture is known as an age-old problem. Is there always a prime between n^2 and $(n+1)^2$ for every positive integer n?

If odd numbers between n^2 and $(n+1)^2$ are divided into two parts by n(n+1) except for n=1, then Legendre-Zhang's conjecture asserts that there is always an odd prime in either such part. Manifestly the Legendre-Zhang's conjecture is better than the Legendre's conjecture.

Please, see the sequence of integers in relation to the Legendre-Zhang's conjecture as listed below.

1(1+1), 3, $2^{2,}$ 5, 2(2+1), 7, $3^{2,}$11, 3(3+1), 13, $4^{2,}$17, 19, 4(4+1), 23, $5^{2,}$29, 5(5+1), 31, $6^{2,}$ 37, 41, 6(6+1), 43, 47, $7^{2,}$ 53, 7(7+1), 59, 61, $8^{2,}$67, 71, 8(8+1), 73, 79, $9^{2,}$83, 89, 9(9+1), 97, $10^{2,}$ *101, 103, 107, 109,* 10(10+1), 113, $11^{2,}$ *127, 131,* 11(11+1), *137, 139,* $12^{2,}$*149, 151,* 12(12+1), *157, 163, 167,* $13^{2,}$ *173, 179, 181,* 13(13+1), *191, 193,* $14^{2,}$ *197, 199,* 14(14+1), *211, 223,* $15^{2,}$*227, 229, 233, 239,* 15(15+1), *241, 251,* 16^2, ...

Thus, the Legendre-Zhang's conjecture states concretely that there is at least an odd prime between n(n+1) and $(n+1)^2$;and there is at least an odd prime between $(n+1)^2$and (n+1)(n+2), where n is a nature number. When n is an odd number, there is following a series of numbers.

There are $(n+1)/2$ odd numbers between n(n+1) and $(n+1)^2$;

There are $(n+1)/2$ odd numbers between $(n+1)^2$ and (n+1)(n+2);

There are $(n+1)/2$ odd numbers between (n+1)(n+2) and $(n+2)^2$;

There are $(n+1)/2$ odd numbers between $(n+2)^2$ and (n+2)(n+3) ;

There are $(n+3)/2$ odd numbers between (n+2)(n+3) and $(n+3)^2$;

There are $(n+3)/2$ odd numbers between $(n+3)^2$ and (n+3)(n+4);

There are $(n+3)/2$ odd numbers between (n+3)(n+4) and $(n+4)^2$;

There are $(n+3)/2$ odd numbers between $(n+4)^2$ and (n+4)(n+5) ;

There are $(n+5)/2$ odd numbers between (n+4)(n+5) and $(n+5)^2$

...

Above has mentioned that odd numbers between n^2 and $(n+1)^2$ are divided into two parts by n(n+1), then we regard every part as a special segment. So begin with odd number 3 alone at 1 special segment between 1(1+1) and 2^2, afterwards, add an odd number at each special segment after per consecutive four special segments. For example, 1(1+1) _1_ 2^2 _1_ 2(2+1) _1_ 3^2 _1_ 3(3+1) _2_ 4^2 _2_ 4(4+1) _2_ 5^2 _2_ 5(5+1) _3_ 6^2 _3_ 6(6+1) _3_ 7^2 _3_ 7(7+1) _4_ 8^2 _4_ 8(8+1) _4_ 9^2 _4_ 9(9+1) _5_ 10^2 _5_ 10(10+1) _5_ 11^2 _5_ 11(11+1) _6_ 12^2 _6_ 12(12+1) _6_ 13^2 _6_ 13(13+1) _7_ 14^2 _7_ 14(14+1) _7_ 15^2 _7_ 15(15+1) ...

In addition, we stipulate that odd number $(n+2)^2$ is added into special segment between $(n+2)^2$ and (n+2)(n+3), where n is an odd number.

Obviously, all special segments both exist within the sequence of positive integers, and exist at the number axis's positive half line.

We number the ordinal number of seriate each and every special segment by seriate natural numbers $\geq$1, whether special segments exist in the series of positive integers, or they exist at the number axis's positive half line. Then, numbers of odd numbers at (1+4t), at (2+4t) and at (3+4t) is all 1+t; yet the number of odd points at (4+4t) is 2+t, where $t \geq 0$.

Namely, numbers of odd numbers at 1, at 2 and at 3 special segment is all 1;

Numbers of odd points at 4, at 5, at 6 and at 7 special segment is all 2;

Numbers of odd points at 8, at 9, at 10 and at 11 special segment is all 3;

Numbers of odd points at 12, at 13, at 14 and at 15 special segment is all 4;

Numbers of odd points at 16, at 17, at 18 and at 19 special segment is all 5 . . .

We shall prove indirectly the Legendre-Zhang's conjecture with the aid of odd points at positive half line of the number axis, thereinafter.

On purpose of watching convenience, for the number axis's positive half line, we let it begins with odd point 3, and it is marked merely odd points, and the length between every two consecutive odd points is just the same. We term a distance whereby each odd prime point and odd point 3 act as two endmost points "a prime length". "PL" is abbreviated from "prime length", and "PLS" denotes the plural of PL. From odd prime point 3 to odd point 3 according to the definition is a PL as well, nevertheless, its length is equal to zero.

We use two positive half lines of number axes, yet their directions reverse from each other, and endmost point 3 of either half line can coincide with any odd point of another. For example, endmost point 3 of either half line coincides with odd point P of another. Please, see first illustration below:

3 5 7 P-4 P-2 P

P P-2 P-4 7 5 3

2 First Illustration

We term PLS at the leftward direction's half line "reverse PLS", and RPLS is abbreviated from reverse PLS.

RPLS whereby any odd point P at the rightward direction's half line acts as a common right endmost point are written as $RPLS_P$. One within $RPLS_P$ is written as a RPL_P.

When P=3, the RPL is written as RPL_3. Namely RPL_3 is under the case that odd point 3 at the leftward direction's half line coincides with odd point 3 at the rightward direction's half line.

The common right endmost point of $RPLS_P$ is odd point P, and part left endmost points of $RPLS_P$coincide monogamously with part odd prime points at line segment 3P of the rightward direction's half line.

At line segment 3P, odd prime points at the rightward direction's half line and left endmost points of $RPLS_P$ at the leftward direction's half line assume always one-to-one bilateral symmetry whereby the center point of line segment 3P acts as the symmetric center.

At the rightward direction's half line, begin with an odd point B, leftwards take seriatim each odd point as a common right endmost point of $RPLS_{B-2f}$, then, part left endmost points of $RPLS_{B-2f}$ monogamously coincide with part odd prime points at line segment 3B of the rightward direction's half line, where f =0, 1, 2, 3, $\ldots$ c, $\ldots$ in proper order.

Suppose that f increases orderly to c, and left endmost points of $\sum RPLS_{B-2f}[0\leq f\leq c]$ just monogamously coincide with all odd prime points at line segment 3B of the rightward direction's half line, then, we consider line segment B(B-2c) as shortest line segment of common right endmost points of RPLS$_{B-2f}$ at line segment 3B. "SCRP" is abbreviated from "shortest line segment of common right endmost points". So, such line segment B(B-2c) is written as SCRP B(B-2c) within line segment 3B too. Let A=B-2c, SCRP B(B-2c) is exactly SCRP AB.

If there is only a CRP at a shorter line segment which begins with odd point 3, then the odd point is an odd prime point surely, because odd prime point 3 can only coincide with the left endmost point of a RPL whereby an odd prime point acts as the right endmost point.

At line segment 3B, we regard odd prime points which first coincide with left endmost points of RPLS$_{B-2f}$ as special odd prime points which left endmost points of RPLS$_{B-2f}$ coincide with, where 0≤f≤c. Manifestly, all odd prime points which monogamously coincide with left endmost points of RPLS$_B$are namely special odd prime points which left endmost points of RPLS$_B$coincide with. Yet special odd prime points which left endmost points of RPLS$_A$coincide with are totally different from special odd prime points which left endmost points of $\sum$RPLS$_X$ [B ≥ X ≥A+2] coincide with.

Thus, all odd prime points at line segment 3B are namely special odd prime points which left endmost points of $\sum$ RPLS$_{B-2f}$ [0≤f≤c] coincide with.

For example, when B=95, there are four odd points at SCRP AB, then A=89. For the distribution of odd prime points which monogamously coincide with left endmost points of RPLS$_F$, where F=95, 93, 91 and 89, please, see following second illustration:

Odd prime point: 19 31 37 61 67 79

Left end points of RPLS$_{95}$: 79 67 61 37 31 19

Odd prime point: 7 13 17 23 29 37 43 53 59 67 73 79 83 89

Left end points of RPLS$_{93}$: 89 83 79 73 67 59 53 43 37 29 23 17 13 7

Odd prime point: 5 11 23 41 47 53 71 83 89

Left end points of $RPLS_{91}$: 89 83 71 53 47 41 23 11 5
Odd prime point: 3 13 19 31 61 73 79 89
Left end points of $RPLS_{89}$: 89 79 73 61 31 19 13 3

3 Second Illustration

Let us continue to leftwards take odd point A-2 as a common right endmost point of $RPLS_{A-2}$. Since left endmost points of $\sum RPLS_{B-2f}[0\leq f\leq c]$ just monogamously coincide with all odd prime points at line segment 3B of the rightward direction's half line, yet odd prime points which left endmost points of $\sum RPLS_{B-2e}[1\leq e\leq c]$ coincide with are constant, therefore, special odd prime points which left endmost points of $RPLS_{A-2}$ coincide with belong within odd prime points which left endmost points of $RPLS_B$coincide with.

Since odd prime point 3 can only coincide with the left endmost point of a RPL whereby an odd prime point at the rightward direction's half line acts as the right endmost point, therefore, provided B is an only odd prime point, and others are all odd composite points at SCRP AB, then A-2 is an odd prime point. In addition, there is SCRP (A-2)(B-2) within line segment 3(B-2).

Since odd prime point 3 can only coincide with the left endmost point of a RPL whereby an odd prime point at the rightward direction's half line acts as the right endmost point, thus any SCRP contains at least an odd prime point. Provided there is only a CRP at a shorter line segment which begins with 3, then this CRP is an odd prime point too.

If A is an only odd prime point, and others are all odd composite points at SCRP AB, then B+2 is an odd prime point. So, there is SCRP A(B+2) within line segment 3(B+2). Here, SCRP A(B+2) within line segment 3(B+2) is an odd point more than SCRP AB within line segment 3B. Manifestly, the odd point is exactly odd prime point B+2.

Factual proof, that the aforesaid case is unique under these circumstances that numbers of odd points at two SCRP which share at least an odd point are inequality from each other.

Conversely, we may too begin with odd point A, rightwards take seriatim each odd point as a common right endmost point of RPLS$_{A+2f}$, when part left endmost points of RPLS$_{A+2f}$ monogamously coincide just with all odd prime points at line segment 3(A+2f), where f $\geq$ 0, then there is SCRP A(A+2f) within line segment 3(A+2f). In reality, this is known as A+2f =B, so SCRP A(A+2f) is exactly SCRP AB. But, at here, special odd prime points which left endmost points of RPLS$_{A+2f}$ coincide with are determined according to from left to right odd points as common right endmost points of RPLS$_{A+2f}$.

Take seriatim each odd point as a common right endmost point of RPLS, whether the order is from right to left, or from left to right, there is always at least a special odd prime point which coincides with a left endmost point of RPLS whereby each odd point at SCRP AB acts as a common right endmost point.

Since the order of taking common right endmost points of RPLS can be each other's reverse directions, so special odd prime points which coincide with left endmost points of RPLS whereby an identical odd point acts as the common right endmost point are not all alike on two directions.

There are *(B-A+2)/2* odd points at SCRP AB. One complete SCRP must contain all odd points at the SCRP. Basically, a complete SCRP is able to be substituted by all odd points at the SCRP. Instance SCRP AB, SCRP AB is a complete SCRP within line segment 3B, and *(B-A+2)/2* odd points at SCRP AB can substitute for SCRP AB. If reduce any odd point at SCRP AB, then the remainder is not a SCRP.

4 The Proof of the Legendre-Zhang's Conjecture

Since there is at least an odd prime point at any complete SCRP, or when there is only a CRP at a shorter line segment which begins with 3, the CRP is an odd prime point according to preceding proof, thus let us first prove that there is a complete SCRP at y special segment by mathematical induction, where y is any natural number.

(1). When y=y_1=1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20, each number of odd points at y_1 special segment in the proper order is orderly *1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5* and 6. Also, each number of odd points at each SCRP which contains most right odd point within y_1 special segment in the proper order is orderly *1, 1, 1, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 4, 4, 4* and 6.

This shows that there is a CRP alone or a complete SCRP within y_1 special segment.

(2). When y=k, suppose that there is a complete SCRP within k special segment, where k $\geq$ 20.

(3). When y=k+1, prove that there is a complete SCRP within (k+1) special segment too.

Proof. Since there is a complete SCRP within k special segment, then the complete SCRP contains at least an odd prime point.

Since odd prime point 3 can only coincide with the left endmost point of a RPL whereby an odd prime point at the rightward direction's half line acts as the right endmost point, thus, there is at least an odd prime point at the complete SCRP within k special segment.

Suppose that most right or unique odd prime point at k special segment is P_m, then, let a complete SCRP within k special segment contain P_m. Moreover, suppose that the left endmost point of the complete SCRP is odd point E, and its right endmost point is odd point F. Of course, P_m at the here may become to E or F.

There are *(F-E+2)/2* odd points at SCRP EF. Also let F+2=G, and 2F-E+2=H or 2F-E+4=H, then line segment GH is exactly SCRP GH within line segment 3H. Either the number of odd points at SCRP GH within line segment 3H is equal to the number of odd points at SCRP EF within line segment 3F, or SCRP GH within line segment 3H is an odd point more than SCRP EF within line segment 3F. But also, SCRP GH within line segment 3H contains at least an odd prime point. Undoubtedly, the odd prime point exists on the right side of k special segment.

From the preceding exposition, we know that either the number of odd points at k special segment is equal to the number of odd points at (k+1) special segment, or (k+1) special segment is an odd point more than k special segment. Therefore, SCRP GH exists within (k+1) special segment. Consequently, there is at least an odd prime point at (k+1) special segment.

Start from a proven special segment to prove the next special segment which adjoins the proven special segment for each once, then after via infinite times, there are infinitely many proven special segments. Namely, let y is equal to each and every natural number, then inevitably reach a conclusion that there is at least an odd prime point at every special segment.

Or rather, there is at least an odd prime between n(n+1) and $(n+1)^2$;and there is at least an odd prime between $(n+1)^2$and (n+1)(n+2), where n expresses each and every nature number.

Consequently the Legendre-Zhang's conjecture does hold water by proof.

5 The Proof of the Gilbreath's Conjecture

First, we number the ordinal number of each and every odd prime from small to great. Namely, we regard odd prime 3 as 1 odd prime, and write odd prime 3 down P_1; also regard odd prime 5 as 2 odd prime, and write odd prime 5 down P_2 ... and so on and so forth, reckon odd prime P_m as m odd prime.

On the supposition that m is the ordinal number of most right odd prime at a row of consecutive odd primes which begin 3, then the number of the consecutive primes from even prime 2 to odd prime P_m is m+1 altogether.

Let us review the set rule of the Gilbreath's Conjecture, namely after write down consecutive primes which began with 2 into a row, count out a difference of every two adjacent primes, and put each difference underneath the left prime, and continued this process of finding iterated unsigned differences.

After proven the Legendre-Zhang's conjecture, let us successively prove the Gilbreath's conjecture by mathematical induction with the aid of the result have gained, as follows.

(1). When m=20, P_m=73, all operational results and the arrangement thereof according to the set rule suit the Gilbreath's conjecture. Please, see following rows and columns.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73 ...

1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 8, 6, 6, 2 ...

1, 0, 2, 2, 2, 2, 2, 2, 4, 4, 2, 2, 2, 2, 0, 2, 2, 0, 4 ...

1, 2, 0, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 2, 2, 0, 2, 4 ...

1, 2, 0, 0, 0, 0, 2, 2, 2, 2, 0, 0, 2, 0, 2, 2 2 ...

1, 2, 0, 0, 0, 2, 0, 0, 0, 2, 0, 2, 2, 2, 0, 0 ...

1, 2, 0, 0, 2, 2, 0, 0, 2, 2, 2, 0, 0, 2, 0 ...

1, 2, 0, 2, 0, 2, 0, 2, 0, 0, 2, 0, 2, 2 ...

1, 2, 2, 2, 2, 2, 2, 2, 0, 2, 2, 2, 0 ...

1, 0, 0, 0, 0, 0, 0, 2, 2, 0, 0, 2 ...

1, 0, 0, 0, 0, 0, 2, 0, 2, 0, 2 ...

1, 0, 0, 0, 0, 2, 2, 2, 2, 2 ...

1, 0, 0, 0, 2, 0, 0, 0, 0 ...
1, 0, 0, 2, 2, 0, 0, 0 ...
1, 0, 2, 0, 2, 0, 0 ...
1, 2, 2, 2, 2, 0 ...
1, 0, 0, 0, 2 ...
1, 0, 0, 2 ...
1, 0, 2 ...
1, 2 ...
1 ...

When P_m=73, 73 exists at 14 special segment due to $1(1+1)_2^2$as 1 special segment, $2^2\_2(2+1)$ as 2 special segment, $2(2+1)\_3^2$ as 3 special segment... $8^2\_8(8+1)$ as 14 special segment, just right, 73 exists between 8^2 and 8(8+1).

(2). When m=c, suppose that all operational results and the arrangement thereof according to the set rule suit the Gilbreath's conjecture, where c≥20.

(3). When m=c+1, prove that all operational results and the arrangement thereof according to the set rule suit the Gilbreath's conjecture too.

Proof. Let us arrange 2 and from small to great odd numbers which begin with 3 into a row, and reckon the row as first row. Then put each difference of every two adjacent integers underneath each left integer. Furthermore, we continue this process of finding iterated unsigned differences. Evidently, except for first integer in first row, numbers in the first column are all one.

If delete odd composite numbers which contain prime factor 3 in first row, afterwards make operating and arranging unsigned differences like the aforementioned way of doing, then except for first integer in first row, numbers in the first column are all one.

If continue to delete odd composite numbers which contain prime factor 5 in first row, afterwards make operating and arranging unsigned differences like the aforementioned way of doing, then except for first integer in first row, numbers in the first column are all one.

. . .

If continue to delete odd composite numbers which contain prime factor P_a in first row, and suppose that except for first integer in first rank, numbers in the first column are all one.

Why numbers in the first column except for first integer in first row are all one? Because, the biggest difference of two adjacent odd numbers in first row after delete odd composite numbers which contain prime factors from 3 to P_a is always smaller than the sum of unsigned differences on the left of the biggest difference. If subtract the biggest difference from the right part of the all unsigned differences, then, the remainder is exactly a proven series of integers.

From proven the Legendre-Zhang's conjecture, we know that there is at least an odd prime at each and every special segment.

Suppose that odd prime P_c exists at y special segment. If odd prime P_{c+1}exists at y special segment too, then the number of odd numbers between p_c and p_{c+1} is less than the number of odd numbers at y special segment.

If odd prime P_{c+1} exists at (y+1) special segment, then the number of odd numbers between p_c and p_{c+1} is less than the number of odd numbers at y special segment plus (y+1) special segment.

From preceding basic concepts, we know that the most front and most behind two special segments within consecutive four special segments differ by one of odd number.

Thus, y and (y+1) special segments is one or two odd numbers more than (y-1) and (y-2) special segments.

Therefore, odd numbers between p_c and p_{c+1} are not more than odd numbers at (y-1) and (y-2) special segments.

In addition, there is at least an odd prime at every special segment, including (y-1) and (y-2) special segments.

Thus, either the difference of p_c from $p_{c+1}$is less than the sum of unsigned differences of every two adjacent odd primes at (y-1) special segment, or the difference of p_c from $p_{c+1}$is less than the sum of unsigned differences of every two adjacent odd primes at (y-1) and (y-2) special segments.

Therefore, when m=c+1, after above-mentioned either sum minus corresponding a difference, the remainder at first row is a proven series of primes.

Consequently, when m=c+1, all operational results and the arrangement thereof according to the set rule suit the Gilbreath's conjecture too.

Proceed from a proven conclusion to add a larger adjacent odd prime for each once, then via infinite times, namely let m to equal each and every natural number, or rather, let 2 and all odd primes are putting in first row, afterwards make all operational results and the arrangement thereof according to the set rule, then we reach inevitably a conclusion that except for first integer in first row, the numbers in the first column are all one. Namely the Gilbreath's conjecture holds water by proof.

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A certain subclass of p-valently analytic functions with negative coefficients [1]

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Abstract

In this paper, we introduce a subclass $P^*(p, A, B, \alpha, j)$ of analytic and p-valent functions with negative coefficients. We obtain coefficient estimates, distortion theorem , closure theorems and radii of close-to-convexity, starlikeness and convexity of order $\phi\,(0 \le \phi < p)$ for this class. We also obtain class preserving integral operators for this class. Furthermore, several results for the modified Hadamard products of functions belonging to the class $P^*(p, A, B, \alpha, j)$ are also given.

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1 Introduction

Let $A(p)$ denote the class of functions of the form:

$$(1.1) \qquad f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} (p \in N = \{1, 2, ...\})$$

which are analytic and p-valent in the open unit disc $U = \{z \in C$ and $|z| < 1\}$. Let $f(z)$ and $g(z)$ be analytic in U. Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $f(z) = g(w(z)) (z \in U)$. We denote this subordination by $f(z) \prec g(z)$. A function $f(z) \in A(p)$ is said to be in the class $R_{p,j}(\alpha)$ if it satisfies the following inequality:

$$(1.2) \qquad \mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \alpha \quad (z \in U; 0 \leq \alpha < \delta(p, j); 0 \leq j \leq p),$$

where

$$(1.3) \qquad \delta(p, j) = \frac{p!}{(p-j)!} = \begin{cases} p(p-1)......(p-j-1) & (j \neq 0) \\ 1 & (j = 0). \end{cases}$$

The class $R_{p,j}(\alpha)$ was studied by Saitoh ([11] , [12] and [13]), Patel and Mohanty [10] and Srivastava et al. ([16] and [17]) (see also Nunokawa [8]).

For A, B fixed, $-1 \leq A < B \leq 1, 0 < B \leq 1, 0 \leq \alpha < \delta(p, j), 0 \leq j \leq p$ and $p \in N$, we say that $f(z) \in A(p)$ is in the class $P(p, A, B, \alpha, j)$ if it satisfies the following subordination condition:

$$(1.4) \qquad \frac{1}{(\delta(p, j) - \alpha)}\left(\frac{f^{(j)}(z)}{z^{p-j}} - \alpha\right) \prec \frac{1 + Az}{1 + Bz} \, (z \in U),$$

or , equivalently , $f(z) \in P(p, A, B, \alpha, j)$ if and only if

$$(1.5) \qquad \left|\frac{\frac{f^{(j)}(z)}{z^{p-j}} - \delta(p, j)}{B\frac{f^{(j)}(z)}{z^{p-j}} - [\delta(p, j)B + (A - B)(\delta(p, j) - \alpha)]}\right| < 1 (z \in U).$$

We note that:

(i) $P(p, A, B, 0, 1) = S_p(A, B)$ (Chen [4]);

(ii) $P(p, \beta A, \beta B, \alpha, 1) = S_p(\alpha,\beta, A, B)(0 \le \alpha < p, 0 < \beta \le 1)$ (Aouf [3]);

(iii) $P(p, -1, 1, \alpha, 1) = S_p(\alpha)(0 \le \alpha < p)$ (Owa [9]);

(iv) $P(1, A, B, 0, 1) = R(A, B)$(Mehrok [7]);

(v) $P(p, A, B, 0, j) = R_{p,j}(1, A, B)(0 \le j \le p)$ and $P(p, -1, 1, \alpha, j) = R_{p,j}(1, \alpha)(0 \le \alpha < \delta(p, j); 0 \le j \le p)$(Srivastava et al.[17]).

Also we note that :

(i) $P(p, \alpha, A, B, p) = P_p(\alpha, A, B)$

$$= \left\{ f(z) \in A(p) : \left| \frac{f^{(p)}(z) - p!}{Bf^{(p)}(z) - [p!B + (A - B)(p! - \alpha)]} \right| < 1, \right.$$

$$(z \in U,\ -1 \le A < B \le 1\,,\ 0 < B \le 1,\ 0 \le \alpha < p!,\ p \in N\,); \tag{1.6}$$

(ii) $P(p, \alpha, -\beta, \beta, p) = P_p(\alpha, \beta)$

$$= \left\{ f(z) \in A(p) : \left| \frac{f^{(p)}(z) - p!}{f^{(p)}(z) + p! - 2\alpha} \right| < \beta, \right.$$

$$(z \in U,\quad 0 \le \alpha < p!\,,\ 0 < \beta \le 1,\ \ p \in N\,)\}\,; \tag{1.7}$$

(iii) $P(p, 0, A, B, p) = R_p(A, B)$

$$= \left\{ f(z) \in A(p) : f^{(p)}(z) \prec p!\, \frac{1 + Az}{1 + Bz}, z \in U \right\}. \tag{1.8}$$

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} (a_{p+n} \ge 0; p \in N). \tag{1.9}$$

Further, we define the classes $P^*(p, A, B, \alpha, j), R^*_{p,j}(\alpha), P^*_p(\alpha, A, B)$, $P^*_p(\alpha, \beta)$ and $R^*_p(A, B)$ by:

$$P^*(p, A, B, \alpha, j) = P(p, A, B, \alpha, j) \cap T(p); \tag{1.10}$$

(1.11) $$R^*_{p,j}(\alpha) = R_{p,j}(\alpha) \cap T(p);$$

(1.12) $$P^*_p(\alpha, A, B) = P_p(\alpha, A, B) \cap T(p);$$

(1.13) $$P^*_p(\alpha, \beta) = P_p(\alpha, \beta) \cap T(p);$$

and

(1.14) $$R^*_p(A, B) = R_p(A, B) \cap T(p).$$

We note that, by specializing the parameters A, B, α, p and j, we obtain the following subclasses studied by various authors:

(i) $P^*(p, A, B, \alpha, 1) = P^*(p, A, B, \alpha)$ (Aouf [1]),

(ii) $P^*(p, -\beta, \beta, \alpha, 1) = P^*_p(\alpha, \beta)(0 \le \alpha < p, 0 < \beta \le 1)$ (Aouf [2]);

(iii) $P^*(p, A, B, 0, 1) = P^*(p, A, B)$ (Shukla and Dashrath [15]);

(iv) $P^*(p, -1, 1, \alpha, 0) = F_p(1, \alpha p)(0 \le \alpha < 1)$ and $P^*(p, -1, 1, \alpha, 1) = F_p(1, \alpha)(0 \le \alpha < p)$(Lee et al. [6]);

(v) $P^*(1, -\beta, \beta, \alpha, 1) = P^*(\alpha, \beta)(0 \le \alpha < 1; 0 < \beta \le 1)$ (Gupta and Jain[5]).

2 Coefficient Estimates

Theorem 1 *Let the function* $f(z)$ *be defined by* (1.9). *Then* $f(z) \in P^*(p, A, B, \alpha, j)$ *if and only if*

(2.1) $$\sum_{n=1}^{\infty}(1 + B)\delta(p + n, j)a_{p+n} \le (B - A)(\delta(p, j) - \alpha).$$

Proof. Assume that the inequality (2.1) holds true and let $|z| = 1$. Then we have

$$\left|\frac{f^{(j)}(z)}{z^{p-j}} - \delta(p, j)\right| - \left|B\frac{f^{(j)}(z)}{z^{p-j}} - [\delta(p, j)B + (A - B)(\delta(p, j) - \alpha)]\right|$$

$$= \left|-\sum_{n=1}^{\infty}\delta(p + n, j)a_{p+n}z^n\right| -$$

$$\left|(B-A)(\delta(p,j)-\alpha)-\sum_{n=1}^{\infty}B\delta(p+n,j)\,a_{p+n}z^n\right|$$

$$\leq\sum_{n=1}^{\infty}(1+B)\delta(p+n,j)a_{p+n}-(B-A)(\delta(p,j)-\alpha)\leq 0,$$

by hypothesis. Hence, by the maximum modulus theorem, we have $f(z)\in P^*(p,A,B,\alpha,j)$.

Conversely, suppose that

$$\left|\frac{\dfrac{f^{(j)}(z)}{z^{p-j}}-\delta(p,j)}{B\dfrac{f^{(j)}(z)}{z^{p-j}}-[\delta(p,j)B+(A-B)(\delta(p,j)-\alpha)]}\right|$$

$$=\left|\frac{-\sum\limits_{n=1}^{\infty}\delta(p+n,j)a_{p+n}z^n}{(A-B)(\delta(p,j)-\alpha)-\sum\limits_{n=1}^{\infty}B\delta(p+n,j)a_{p+n}z^n}\right|<1(z\in U).$$

Since $|\mathrm{Re}(z)|\leq|z|$ for all z, we have

(2.2)
$$\mathrm{Re}\left\{\frac{\sum\limits_{n=1}^{\infty}\delta(p+n,j)a_{p+n}z^n}{(A-B)(\delta(p,j)-\alpha)-\sum\limits_{n=1}^{\infty}B\delta(p+n,j)a_{p+n}z^n}\right\}<1(z\in U).$$

Choosing values of z on the real axis so that $\dfrac{f^{(j)}(z)}{z^{p-j}}$ is real. Upon clearing the denominator in (2.2) and letting $z\to 1^-$ through real values , we obtain

$$\sum_{n=1}^{\infty}(1+B)\delta(p+n,j)a_{p+n}\quad\leq(B-A)(\delta(p,j)-\alpha), \tag{2.3}$$

which leads us at once to (2.1). This completes the proof of Theorem 1.

Corollary 1 *Let the function* $f(z)$ *defined by* (1.9) *be in the class* $P^*(p,A,B,\alpha,j)$. *Then we have*

$$a_{p+n} \le \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)}(p,n\in N). \tag{2.4}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = z^p - \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)}z^{p+n}(p,n\in N). \tag{2.5}$$

3 Distortion Theorems

Theorem 2 *Let the function* $f(z)$ *defined by* (1.9) *be in the class* $P^*(p,A,B,\alpha,j)$. *Then for* $|z| = r < 1$,

$$r^p - \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)}r^{p+1} \le |f(z)| \le$$

$$r^p + \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)}r^{p+1} \tag{3.1}$$

and

$$pr^{p-1} - \frac{(B-A)(\delta(p,j)-\alpha)(p+1)}{(1+B)\delta(p+1,j)}r^p \le \left|f^{'}(z)\right| \le$$

$$pr^{p-1} + \frac{(B-A)(\delta(p,j)-\alpha)(p+1)}{(1+B)\delta(p+1,j)}r^p. \tag{3.2}$$

The equality in (3.1) *and* (3.2) *are attained for the function* $f(z)$ *given by*

$$f(z) = z^p - \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)}z^{p+1}(p\in N). \tag{3.3}$$

Proof. Since $f(z) \in P^*(p,A,B,\alpha,j)$, in view of Theorem 1, we have

$$(1+B)\delta(p+1,j)\sum_{n=1}^{\infty} a_{p+n} \le \sum_{n=1}^{\infty}(1+B)\delta(p+n,j)a_{p+n} \le (B-A)(\delta(p,j)-\alpha),$$

which evidently yields

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)}. \tag{3.4}$$

Consequently, for $|z| = r < 1$, we obtain

$$\begin{aligned} |f(z)| &\leq r^p + r^{p+1}\sum_{n=1}^{\infty} a_{p+n} \\ &\leq r^p + \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)} r^{p+1} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq r^p - r^{p+1}\sum_{n=1}^{\infty} a_{p+n} \\ &\geq r^p - \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)} r^{p+1} \end{aligned}$$

which prove the assertion (3.1) of Theorem 2. Also from Theorem 1, it follows that

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \leq \frac{(B-A)(\delta(p,j)-\alpha)(p+1)}{(1+B)\delta(p+1,j)}. \tag{3.5}$$

Consequently, for $|z| = r < 1$, we have

$$\begin{aligned} \left|f'(z)\right| &\leq pr^{p-1} + \sum_{n=1}^{\infty}(p+n)a_{p+n}\, r^{p+n-1} \\ &\leq pr^{p-1} + r^p \sum_{n=1}^{\infty}(p+n)a_{p+n} \\ &\leq pr^{p-1} + \frac{(B-A)(\delta(p,j)-\alpha)(p+1)}{(1+B)\delta(p+1,j)} r^p \end{aligned}$$

and

$$\begin{aligned}\left|f'(z)\right| &\geq pr^{p-1} - \sum_{n=1}^{\infty}(p+n)a_{p+n}\, r^{p+n-1} \\ &\geq pr^{p-1} - r^{p}\sum_{n=1}^{\infty}(p+n)a_{p+n} \\ &\geq pr^{p-1} - \frac{(B-A)(\delta(p,j)-\alpha)(p+1)}{(1+B)\delta(p+1,j)} r^{p},\end{aligned}$$

which prove the assertion (3.2) of Theorem 2. Finally, it is easy to see that the bounds in (3.1) and (3.2) are attained for the function $f(z)$ given already by (3.3).

Corollary 2 *Let the function $f(z)$ defined by* (1.9) *be in the class $P^*(p, A, B, \alpha, j)$, then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{(1+B)\delta(p+1,j) - (B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)}. \tag{3.6}$$

The result is sharp, with the external function $f(z)$ given by (3.3).

Theorem 3 *If a function $f(z)$ defined by* (1.9) *is in the class $P^*(p, A, B, \alpha, j)$, then*

$$\left\{\frac{p!}{(p-m)!} - \frac{(B-A)(\delta(p,j)-\alpha)(p+1-j)!}{(1+B)(p+1-m)!}|z|\right\}|z|^{p-m} \leq \left|f^{(m)}(z)\right|$$

$$\leq \left\{\frac{p!}{(p-m)!} + \frac{(B-A)(\delta(p,j)-\alpha)(p+1-j)!}{(1+B)(p+1-m)!}|z|\right\}|z|^{p-m} \tag{3.7}$$

$$(z \in U;\ 0 \leq \alpha < \delta(p,j);\ 0 \leq j \leq p\ ;\ m \in N_0 = N \cup \{0\};$$

$$p > m;\ -1 \leq A < B \leq 1\ ;\ 0 < B \leq 1).$$

The result is sharp for the function $f(z)$ given by (3.3).

Proof. In view of Theorem 1, we have

$$\frac{(1+B)\delta(p+1,j)}{(B-A)(\delta(p,j)-\alpha)(p+1)!}\sum_{n=1}^{\infty}(p+n)!\ a_{p+n}$$

$$\leq \sum_{n=1}^{\infty}\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}\ a_{p+n}\ \leq 1,$$

which readily yields

$$\sum_{n=1}^{\infty}(p+n)!a_{p+n}\leq\frac{(B-A)(\delta(p,j)-\alpha)(p+1-j)!}{(1+B)}\ . \tag{3.8}$$

Now, by differentiating both sides of (1.9) m times, we have

$$f^{(m)}(z)=\frac{p!}{(p-m)!}z^{p-m}-\sum_{n=1}^{\infty}\frac{(p+n)!}{(p+n-m)!}a_{p+n}\ z^{p+n-m}$$

$$(p,n\in N\ ;\ m\in N_0\ ;\quad p>m) \tag{3.9}$$

and Theorem 3 would follows from (3.8) and (3.9). Finally, it is easy to see that the bounds in (3.7) are attained for the function $f(z)$ given by (3.3).

4 Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 4 *Let the function $f(z)$ defined by* (1.9) *be in the class $P^{*}(p,A,B,\alpha,j)$, then $f(z)$ is p-valent close-to-convex of order $\phi\,(0\leq\phi<p)$ in $|z|<r_1$, where*

$$r_1=\inf_n\left\{\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}(\frac{p-\phi}{p+n})\right\}^{\frac{1}{n}}\ (n\geq 1). \tag{4.1}$$

The result is sharp, with the external function $f(z)$ given by (2.5).

Proof. We must show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \phi$ for $|z| < r_1$. We have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{n=1}^{\infty}(p+n)a_{p+n}\,|z|^n\,.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \phi$ if

$$\sum_{n=1}^{\infty}\left(\frac{p+n}{p-\phi}\right)a_{p+n}\,|z|^n \leq 1. \tag{4.2}$$

Hence, by Theorem 1, (4.2) will be true if

$$\left(\frac{p+n}{p-\phi}\right)|z|^n \leq \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}$$

or if

$$|z| \leq \left\{\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}\left(\frac{p-\phi}{p+n}\right)\right\}^{\frac{1}{n}} \quad (n \geq 1). \tag{4.3}$$

The theorem follows easily from (4.3).

Theorem 5 *Let the function* $f(z)$ *defined by* (1.9) *be in the class* $P^*(p,A,B,\alpha,j)$, *then* $f(z)$ *is p-valent starlike of order* $\phi\,(0 \leq \phi < p)$ *in* $|z| < r_2$, *where*

$$r_2 = \inf_n \left\{\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}\left(\frac{p-\phi}{p+n-\phi}\right)\right\}^{\frac{1}{n}} \quad (n \geq 1). \tag{4.4}$$

The result is sharp, with the external function $f(z)$ *given by* (2.5).

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \phi$ for $|z| < r_2$. Using similar arguments as given by Theorem 4, we can get the result.

Corollary 3 *Let the function $f(z)$ defined by* (1.9) *be in the class $P^*(p, A, B, \alpha, j)$, then $f(z)$ is p-valent convex of order $\phi\,(0 \leq \phi < p)$ in $|z| < r_3$, where*

$$(4.5) \quad r_3 = \inf_n \left\{ \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} \left(\frac{p(p-\phi)}{(p+n)(p+n-\phi)} \right) \right\}^{\frac{1}{n}} \ (n \geq 1).$$

The result is sharp, with the external function $f(z)$ given by (2.5).

5 Extreme Points

From Theorem 1, we see that the class $P^*(p, A, B, \alpha, j)$, is closed under convex linear combinations, which enables us to determine the extreme points for this class.

Theorem 6 *Let*

$$(5.1) \qquad f_p(z) = z^p$$

and

$$(5.2) \qquad f_{p+n}(z) = z^p - \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)} z^{p+n} (p, n \in N).$$

Then $f(z) \in P^*(p, A, B, \alpha, j)$ if and only if it can be expressed in the form

$$(5.3) \qquad f(z) = \sum_{n=0}^{\infty} \lambda_{p+n}\, f_{p+n}(z),$$

where $\lambda_{p+n} \geq 0$ and $\sum\limits_{n=0}^{\infty} \lambda_{p+n} = 1$.

Proof. Suppose that

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n}\, f_{p+n}(z)$$

$$(5.4) \qquad = z^p - \sum_{n=1}^{\infty} \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)} \lambda_{p+n}\, z^{p+n}.$$

Then it follows that

$$\sum_{n=1}^{\infty} \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} . \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)} \lambda_{p+n}$$

$$(5.5) \qquad = \sum_{n=1}^{\infty} \lambda_{p+n} = 1 - \lambda_p \le 1.$$

Therefore, by Theorem 1, $f(z) \in P^*(p, A, B, \alpha, j)$. Conversely, assume that the function $f(z)$ defined by (1.9) belongs to the class $P^*(p, A, B, \alpha, j)$. Then

$$(5.6) \qquad a_{p+n} \le \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)} (p, n \in N).$$

Setting

$$(5.7) \qquad \lambda_{p+n} = \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} (p, n \in N)$$

and

$$(5.8) \qquad \lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{p+n},$$

we see that $f(z)$ an be expressed in the form (5.3). This completes the proof of Theorem 6.

Corollary 4 *The extreme points of the class* $P^*(p, A, B, \alpha, j)$ *are the functions* $f_p(z) = z^p$ *and*

$$f_{p+n}(z) = z^p - \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)} z^{p+n} (p, n \in N).$$

6 Integral Operators

Theorem 7 *Let the function* $f(z)$ *defined by* (1.9) *be in the class* $P^*(p, A, B, \alpha, j)$, *and let* c *be a real mumber such that* $c > -p$. *Then the function* $F(z)$ *defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{6.1}$$

also belongs to the class $P^*(p, A, B, \alpha, j)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}, \tag{6.2}$$

where

$$b_{p+n} = (\frac{c+p}{c+p+n}) a_{p+n}.$$

Therefore, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} b_{p+n} \\ = & \sum_{n=1}^{\infty} \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} (\frac{c+p}{c+p+n}) a_{p+n} \\ \leq & \sum_{n=1}^{\infty} \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} a_{p+n} \leq 1 \end{aligned}$$

since $f(z) \in P^*(p,A,B,\alpha, j)$. Hence, by Theorem 1, $F(z) \in P^*(p, A, B, \alpha, j)$.

Theorem 8 *Let* c *be a real number such that* $c > -p$. *If* $F(z) \in P^*(p, A, B, \alpha, j)$, *then the function* $f(z)$ *defined in* (6.1) *is p-valent in* $|z| < R_p^*$, *where*

$$R_p^* = \inf_n \left\{ (\frac{c+p}{c+p+n}) \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} (\frac{p}{p+n}) \right\}^{\frac{1}{n}} \; (n \in N). \tag{6.3}$$

The result is sharp.

Proof. Suppose that

$$F(z) = z^p - \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} (a_{p+n} \geq 0).$$

It follows then from (6.1) that

$$\begin{aligned} f(z) &= \frac{z^{1-c}[z^c F(z)]^{'}}{c+p} (c > -p) \\ &= z^p - \sum_{n=1}^{\infty} (\frac{c+p+n}{c+p}) a_{p+n} z^{p+n}. \end{aligned}$$

To prove Theorem 8, it suffices to show that $\left|\frac{f^{'}(z)}{z^{p-1}} - p\right| \leq p$ for $|z| < R_p^*$. Now

$$\begin{aligned} \left|\frac{f^{'}(z)}{z^{p-1}} - p\right| &= \left| - \sum_{n=1}^{\infty} (\frac{c+p+n}{c+p})(p+n) a_{p+n} z^n \right| \\ &\leq \sum_{n=1}^{\infty} (\frac{c+p+n}{c+p})(p+n) a_{p+n} \left|z\right|^n . \end{aligned}$$

Thus $\left|\frac{f^{'}(z)}{z^{p-1}} - p\right| \leq p$ if

$$\sum_{n=1}^{\infty} (\frac{c+p+n}{c+p})(\frac{p+n}{p}) a_{p+n} \left|z\right|^n \leq 1. \tag{6.4}$$

But Theorem 1 confirms that

$$\sum_{n=1}^{\infty} \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} a_{p+n} \leq 1. \tag{6.5}$$

Thus (6.4) will be satisfied if

$$(\frac{p+n}{p})(\frac{c+p+n}{c+p}) a_{p+n} \left|z\right|^n \leq \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} (n \in N),$$

or if

$$|z| \leq \left\{ \left(\frac{c+p}{c+p+n}\right) \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)} \left(\frac{p}{p+n}\right) \right\}^{\frac{1}{n}} \ (n \in N). \tag{6.6}$$

The required result follows now from (6.6).The result is sharp for the function

$$f(z) = z^p - \frac{(c+p+n)(B-A)(\delta(p,j)-\alpha)}{(c+p)(1+B)\delta(p+n,j)} z^{p+n} (p, n \in N). \tag{6.7}$$

7 Modified Hadamard Products

Let the functions $f(z)(\nu = 1, 2)$ be defined by

$$f_\nu(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,\nu} \ z^{p+n} (a_{p+n,\nu} \geq 0; \nu = 1, 2). \tag{7.1}$$

Then the modified Hadamard product (or convolituon) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1} a_{p+n,2} z^{p+n}. \tag{7.2}$$

Theorem 9 *Let the functions $f_\nu(z)(\nu = 1, 2)$ defined by* (7.1) *be in the class $P^*(p, A, B, \alpha, j)$. Then $(f_1 * f_2)(z) \in P^*(p, A, B, \gamma, j)$, where*

$$\gamma = \delta(p,j) - \frac{(B-A)(\delta(p,j)-\alpha)^2}{(1+B)\delta(p+1,j)}. \tag{7.3}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [14], we need to find the largest γ such that

$$\sum_{n=1}^{\infty} \frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\gamma)} a_{p+n,1} a_{p+n,2} \leq 1. \tag{7.4}$$

Since

$$\sum_{n=1}^{\infty}\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}a_{p+n,1}\le 1 \tag{7.5}$$

and

$$\sum_{n=1}^{\infty}\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}a_{p+n,2}\le 1. \tag{7.6}$$

Therefore, by the Cauchy- Schwarz inequality, we have

$$\sum_{n=1}^{\infty}\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}\sqrt{a_{p+n,1}a_{p+n,2}}\le 1. \tag{7.7}$$

Thus it is sufficient to show that

$$\frac{1}{(\delta(p,j)-\gamma)}a_{p+n,1}a_{p+n,2}\le\frac{1}{(\delta(p,j)-\alpha)}\sqrt{a_{p+n,1}a_{p+n,2}}(n\ge 1), \tag{7.8}$$

that is, that

$$\sqrt{a_{p+n,1}a_{p+n,2}}\le\frac{(\delta(p,j)-\gamma)}{(\delta(p,j)-\alpha)}(n\ge 1). \tag{7.9}$$

Note that

$$\sqrt{a_{p+n,1}a_{p+n,2}}\le\frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)}(n\ge 1). \tag{7.10}$$

Consequently, we need only to prove that

$$\frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+n,j)}\le\frac{(\delta(p,j)-\gamma)}{(\delta(p,j)-\alpha)}(n\ge 1) \tag{7.11}$$

or, equivalently, that

$$\gamma\le\delta(p,j)-\frac{(B-A)(\delta(p,j)-\alpha)^2}{(1+B)\delta(p+n,j)}(n\ge 1). \tag{7.12}$$

Since $D(n)$ defined by

$$D(n)=\delta(p,j)-\frac{(B-A)(\delta(p,j)-\alpha)^2}{(1+B)\delta(p+n,j)} \tag{7.13}$$

is an increasing function of $n\,(n \geq 1)$, letting $n = 1$ in (7.13), we obtain

$$\gamma \leq \delta(p,j) - \frac{(B-A)(\delta(p,j)-\alpha)^2}{(1+B)\delta(p+1,j)}, \tag{7.14}$$

which completes the proof of Theorem 9. Finally, by taking the functions

$$f_\nu(z) = z^p - \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)} z^{p+1} (\nu = 1,2\,;\, p \in N) \tag{7.15}$$

we can see that the result is sharp.

Corollary 5 *For the functions $f_\nu(z)(\nu = 1,2)$ as in Theorem 9, we have*

$$h(z) = z^p - \sum_{n=1}^{\infty} \sqrt{a_{p+n,1}a_{p+n,2}}\, z^{p+n} \tag{7.16}$$

belongs to the class $P^*(p,A,B,\alpha,j)$. The result follows from the inequality (7.7)*. It is sharp for the same functions as in Theorem 9.*

Using arguments similar to those in the proof of Theorem 9, we obtain the following result.

Theorem 10 *Let the function $f_1(z)$ defined by* (7.1) *be in the class $P^*(p,A,B,\alpha,j)$ and the function $f_2(z)$ defined by* (7.1) *be the class $P^*(p,A,B,\tau,j)$, then $(f_1 * f_2)(z) \in P^*(p,A,B,\zeta,j)$, where*

$$\zeta = \delta(p,j) - \frac{(B-A)(\delta(p,j)-\alpha)(\delta(p,j)-\tau)}{(1+B)\delta(p+1,j)}. \tag{7.17}$$

The result is the best possible for the functions $f_\nu(z)(\nu = 1,2)$ given by

$$f_1(z) = z^p - \frac{(B-A)(\delta(p,j)-\alpha)}{(1+B)\delta(p+1,j)} z^{p+1} (p \in N) \tag{7.18}$$

and

$$f_2(z) = z^p - \frac{(B-A)(\delta(p,j)-\tau)}{(1+B)\delta(p+1,j)} z^{p+1} (p \in N). \tag{7.19}$$

Corollary 6 *Let each of the functions* $f_\nu(z)(\nu = 1, 2, 3)$ *defined by* (7.1) *be in the class* $P^*(p, A, B, \alpha, j)$. *Then* $(f_1 * f_2 * f_3)(z) \in P^*(p, A, B, \xi, j)$, *where*

$$\xi = \delta(p, j) - \frac{(B - A)^2(\delta(p, j) - \alpha)^3}{[(1 + B)\delta(p + 1, j)]^2}. \tag{7.20}$$

The result is the best possible for the functions $f_\nu(z)(\nu = 1, 2, 3)$ given by

$$f_\nu(z) = z^p - \frac{(B - A)(\delta(p, j) - \alpha)}{(1 + B)\delta(p + 1, j)} z^{p+1} \, (p \in N). \tag{7.21}$$

Proof. Under the hypothesis of Corollary 6, we find from Theorem 9 that $(f_1 * f_2)(z) \in P^*(p, A, B, \gamma, j)$, where γ is given by (7.3).Now by making use of Theorem 10 , we get $(f_1 * f_2 * f_3)(z) \in P^*(p, A, B, \xi, j)$, where

$$\xi = \delta(p, j) - \frac{(B - A)(\delta(p, j) - \alpha)(\delta(p, j) - \gamma)}{(1 + B)\delta(p + 1, j)},$$

$$= \delta(p, j) - \frac{(B - A)(\delta(p, j) - \alpha)^3}{[(1 + B)\delta(p + 1, j)]^2},$$

which completes the proof of Corollary 6 .

Theorem 11 *Let the functions* $f_\nu(z)(\nu = 1, 2)$ *defined by* (7.1) *be in the class* $P^*(p, A, B, \alpha, j)$. *Then the function*

$$h(z) = z^p - \sum_{n=1}^{\infty} (a_{p+n,1}^2 + a_{p+n,2}^2) \, z^{p+n} \tag{7.22}$$

belongs to the class $P^*(p, A, B, \phi, j)$, *where*

$$\phi = \delta(p, j) - \frac{2(B - A)(\delta(p, j) - \alpha)^2}{(1 + B)\delta(p + 1, j)}. \tag{7.23}$$

The result is sharp for the functions $f_\nu(z)(\nu = 1, 2)$ *defined by* (7.15).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{n=1}^{\infty}\left[\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}\right]^2 a_{p+n,\nu}^2$$
$$\leq \left[\sum_{n=1}^{\infty}\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}a_{p+n,\nu}\right]^2 \leq 1(\nu=1,2) \tag{7.24}$$

$$(f_\nu(z)\in P^*(p,A,B,\alpha,j)\ (\nu=1,2))\ ,$$

we have

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}\right]^2(a_{p+n,1}^2+a_{p+n,2}^2)\leq 1. \tag{7.25}$$

Therefore, we need to find the largest ϕ such that

$$\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\phi)}\leq$$

$$\frac{1}{2}\left[\frac{(1+B)\delta(p+n,j)}{(B-A)(\delta(p,j)-\alpha)}\right]^2(n\geq 1) \tag{7.26}$$

that is,

$$\varphi\leq\delta(p,j)-\frac{2(B-A)(\delta(p,j)-\alpha)^2}{(1+B)\delta(p+n,j)}(n\geq 1). \tag{7.27}$$

Since

$$\Psi(n)=\delta(p,j)-\frac{2(B-A)(\delta(p,j)-\alpha)^2}{(1+B)\delta(p+n,j)}$$

is an increasing function of $n\,(n\geq 1)$, we readily have

$$\phi\leq\Psi(1)=\delta(p,j)-\frac{2(B-A)(\delta(p,j)-\alpha)^2}{(1+B)\delta(p+1,j)},$$

and Theorem 11 follows at once.

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A note on multi Poly-Euler numbers and Bernoulli polynomials [1]

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Abstract

In this paper we introduce the generalization of Multi Poly-Euler polynomials and we investigate some relationship involving Multi Poly-Euler polynomials. Obtaining a closed formula for generalization of Multi Poly-Euler numbers therefore seems to be a natural and important problem.

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1 Introduction

In the 17th century a topic of mathematical interst was finite sums of powers of integers such as the series $1 + 2 + ... + (n - 1)$ or the series

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$1^2+2^2+...+(n-1)^2$.The closed form for these finite sums were known ,but the sum of the more general series $1^k+2^k+...+(n-1)^k$was not.It was the mathematician Jacob Bernoulli who would solve this problem.Bernoulli numbers arise in Taylor series in the expansion

$$(1) \qquad \tfrac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \tfrac{x^n}{n!} \ .$$

and we have,
(2)

$$S_m(n) = \sum_{k=1}^{n} k^m = 1^m + 2^m + \cdots + n^m = \tfrac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k \, n^{m+1-k} \ .$$

and we have following matrix representation for Bernoulli numbers(for $n \in \mathbf{N}$),[1-4].

$$(3) \qquad B_n = \frac{(-1)^n}{(n-1)!} \begin{vmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \frac{1}{n} & \frac{1}{n+1} \\ 1 & 1 & 1 & \cdots 1 & 1 \\ 0 & 2 & 3 & \cdots n-1 & n \\ 0 & 0 & \binom{3}{2} & \cdots \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & \ddots \ \vdots & \vdots \\ 0 & 0 & 0 & \cdots \binom{n-1}{n-2} & \binom{n}{n-2} \end{vmatrix} .$$

Euler on page 499 of [5], introduced Euler polynomials, to evaluate the alternating sum

$$(4) \qquad A_n(m) = \sum_{k=1}^{m} (-1)^{m-k} k^n = m^n - (m-1)^n + ... + (-1)^{m-1} 1^n \ .$$

The Euler numbers may be defined by the following generating functions

$$(5) \qquad \tfrac{2}{e^t+1} \ = \ \sum_{n=0}^{\infty} E_n \tfrac{t^n}{n!} \ .$$

and we have following folowing matrix representation for Euler numbers, [1,2,3,4].

$$(6)\qquad E_{2n}=(-1)^n(2n)!\begin{vmatrix}\frac{1}{2!} & 1 & & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix}.$$

The poly-Bernoulli polynomials have been studied by many researchers in recent decade. The history of these polynomials goes back to Kaneko. The poly-Bernoulli polynomials have wide-ranging application from number theory and combinatorics and other fields of applied mathematics. One of applications of poly-Bernoulli numbers that was investigated by Chad Brewbaker in [6,7,8,9], is about the number of $(0,1)$-matrices with n-rows and k columns. He showed the number of $(0,1)$-matrices with n-rows and k columns uniquely reconstructable from their row and column sums are the poly-Bernoulli numbers of negative index $B_n^{(-k)}$. Let us briefly recall poly-Bernoulli numbers and polynomials. For an integer $k \in \mathbf{Z}$, put

$$(7)\qquad \mathrm{Li}_k(z)=\textstyle\sum_{n=1}^{\infty}\frac{z^n}{n^k}.\ .$$

which is the k-th polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. The name of the function come from the fact that it may alternatively be defined as the repeated integral of itself . The formal power series can be used to define Poly-Bernoulli numbers and polynomials. The polynomials $B_n^{(k)}(x)$ are said to be poly-Bernoulli polynomials if they satisfy,

$$(8)\qquad \frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}\ .$$

In fact, Poly-Bernoulli polynomials are generalization of Bernoulli polynomials, because for $n \leq 0$, we have,

$$(-1)^n B_n^{(1)}(-x) = B_n(x) \ . \tag{9}$$

Sasaki,[10], Japanese mathematician, found the Euler type version of these polynomials, In fact, he by using the following relation for Euler numbers,

$$cosht = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n \ . \tag{10}$$

found a poly-Euler version as follows

$$\frac{Li_k(1-e^{-4t})}{4tcosht} = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!} \ . \tag{11}$$

Moreover, he by defining the following L-function, interpolated his definition about Poly-Euler numbers.

$$L_k(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{Li_k(1-e^{-4t})}{4(e^t+e^{-t})} dt \ . \tag{12}$$

and Sasaki showed that

$$L_k(-n) = (-1)^n n \frac{E_{n-1}^{(k)}}{2} \ . \tag{13}$$

But the fact is that working on such type of generating function for finding some identities is not so easy. So by inspiration of the definitions of Euler numbers and Bernoulli numbers, we can define Poly-Euler numbers and polynomials as follows which also A.Bayad [11], defined it by following method in same times.

Definition 1 *(Poly-Euler polynomials):The Poly-Euler polynomials may be defined by using the following generating function,*

$$\frac{2Li_k(1-e^{-t})}{1+e^t} e^{xt} = \sum_{n=0}^{\infty} \mathbf{E}_n^{(k)} \frac{t^n}{n!} \ . \tag{14}$$

If we replace t by $4t$ and take $x = 1/2$ and using the definition $cosht = \frac{e^t+e^{-t}}{2}$, we get the Poly-Euler numbers which was introduced by Sasaki and Bayad and also we can find same interpolating function for them (with some additional constant coefficient).

The generalization of poly-logarithm is defined by the following infinite series

$$Li_{(k_1,k_2,...,k_r)}(z) = \sum_{m_1,m_2,...,m_r} \frac{z^{m_r}}{m_1^{k_1}...m_r^{k_r}} \quad . \tag{15}$$

which here in summation $(0 < m_1 < m_2 < ...m_r)$.

Kim-Kim [12], one of student of Taekyun Kim introduced the Multi poly- Bernoulli numbers and proved that special values of certain zeta functions at non-positive integers can be described in terms of these numbers. The study of Multi poly-Bernoulli numbers and their combinatorial relations has received much attention in [6-13]. The Multi Poly-Bernoulli numbers may be defined as follows

$$\frac{Li_{(k_1,k_2,...,k_r)}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{n=0}^{\infty} B_n^{(k_1,k_2,...,k_r)} \frac{t^n}{n!} \quad . \tag{16}$$

So by inspiration of this definition we can define the Multi Poly-Euler numbers and polynomials .

Definition 2 *Multi Poly-Euler polynomials* $\mathbf{E}_n^{(k_1,...,k_r)}(x)$, $(n = 0, 1, 2, ...)$ *are defined for each integer* $k_1, k_2, ..., k_r$ *by the generating series*

$$\frac{2Li_{(k_1,...,k_r)}(1-e^{-t})}{(1+e^t)^r} e^{rxt} = \sum_{n=0}^{\infty} \mathbf{E}_n^{(k_1,...,k_r)}(x) \frac{t^n}{n!} \quad . \tag{17}$$

and if $x = 0$, then we can define Multi Poly-Euler numbers $\mathbf{E}_n^{(k_1,...,k_r)} = \mathbf{E}_n^{(k_1,...,k_r)}(0)$

Now we define three parameters a, b, c, for Multi Poly-Euler polynomials and Multi Poly-Euler numbers as follows.

Definition 3 *Multi Poly-Euler polynomials* $\mathbf{E}_n^{(k_1,\dots,k_r)}(x,a,b)$, $(n=0,1,2,\dots)$ *are defined for each integer* $k_1, k_2, \dots, k_r$ *by the generating series*

$$\frac{2Li_{(k_1,\dots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{rxt} = \sum_{n=0}^{\infty} \mathbf{E}_n^{(k_1,\dots,k_r)}(x,a,b)\frac{t^n}{n!} \quad . \tag{18}$$

In the same way, and if $x = 0$, then we can define Multi Poly-Euler numbers with a, b parameters $\mathbf{E}_n^{(k_1,\dots,k_r)}(a,b) = \mathbf{E}_n^{(k_1,\dots,k_r)}(0,a,b)$.

In the following theorem, we find a relation between $\mathbf{E}_n^{(k_1,\dots,k_r)}(a,b)$ and $\mathbf{E}_n^{(k_1,\dots,k_r)}(x)$

Theorem 1 *Let* $a, b > 0$, $ab \neq \pm 1$ *then we have*

$$\mathbf{E}_n^{(k_1,k_2,\dots,k_r)}(a,b) = \mathbf{E}_n^{(k_1,k_2,\dots,k_r)}\left(\frac{lna}{lna+lnb}\right)(lna+lnb)^n \quad . \tag{19}$$

Proof. By applying the Definition 2 and Definition 3,we have

$$\begin{aligned}\frac{2Li_{(k_1,\dots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r} &= \sum_{n=0}^{\infty} \mathbf{E}_n^{(k_1,\dots,k_r)}(a,b)\frac{t^n}{n!} \\ &= e^{rt\ln a}\frac{2Li_{(k_1,\dots,k_r)}(1-e^{-t\ln ab})}{(1+e^{t\ln ab})^r}\end{aligned}$$

So, we get

$$\frac{2Li_{(k_1,\dots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r} = \sum_{n=0}^{\infty} \mathbf{E}_n^{(k_1,\dots,k_r)}\left(\frac{\ln a}{\ln a+\ln b}\right)(\ln a+\ln b)^n\frac{t^n}{n!}$$

Therefore, by comparing the coefficients of t^n on both sides, we get the desired result.

Now, In next theorem, we show a shortest relationship between $\mathbf{E}_n^{(k_1,k_2,\dots,k_r)}(a,b)$ and $\mathbf{E}_n^{(k_1,k_2,\dots,k_r)}$.

Theorem 2 *Let $a, b > 0$, $ab \neq \pm 1$ then we have*

$$\mathbf{E}_n^{(k_1,k_2,\ldots,k_r)}(a,b) = \sum_{i=0}^{n} r^{n-i}(\ln a + \ln b)^i (\ln a)^{n-i} \binom{n}{i} \mathbf{E}_i^{(k_1,k_2,\ldots,k_r)} \quad . \tag{20}$$

Proof. By applying the Definition 2, we have,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbf{E}_n^{(k_1,\ldots,k_r)}(a,b)\frac{t^n}{n!} &= \frac{2Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r} \\
&= e^{rt\ln a}\frac{2Li_{(k_1,\ldots,k_r)}(1-e^{-t\ln ab})}{(1+e^{t\ln ab})^r} \\
&= \left(\sum_{k=0}^{\infty}\frac{r^k t^k (\ln a)^k}{k!}\right)\left(\sum_{n=0}^{\infty}\mathbf{E}_n^{(k_1,\ldots,k_r)}(\ln a+\ln b)^n\frac{t^n}{n!}\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j} r^{j-i}\frac{\mathbf{E}_j^{(k_1,\ldots,k_r)}(\ln a+\ln b)^i(\ln a)^{j-i}}{i!(j-i)!}t^j\right)
\end{aligned}$$

So, by comparing the coefficients of t^n on both sides , we get the desired result.

By applying the definition 2, by simple manipulation, we get the following corollary

Corollary 1 *For non-zero numbers a, b, with $ab \neq -1$ we have*

$$\mathbf{E}_n^{(k_1,\ldots,k_r)}(x;a,b) = \sum_{i=0}^{n} \binom{n}{i} r^{n-i}\mathbf{E}_i^{(k_1,\ldots,k_r)}(a,b)x^{n-i} \quad . \tag{21}$$

Furthermore, by combinig the results of Theorem 2, and Corollary 1, we get the following relation between generalization of Multi Poly-Euler polynomials with a, b parameters $\mathbf{E}_n^{(k_1,\ldots,k_r)}(x;a,b)$, and Multi Poly-Euler numbers $\mathbf{E}_n^{(k_1,\ldots,k_r)}$.

(22)
$$\mathbf{E}_n^{(k_1,\ldots,k_r)}(x;a,b)=\sum_{k=0}^{n}\sum_{j=0}^{k}r^{n-k}\binom{n}{k}\binom{k}{j}(\ln a)^{k-j}(\ln a+\ln b)^{j}\mathbf{E}_j^{(k_1,\ldots,k_r)}x^{n-k}\ .$$

Now, we state the "Addition formula" for generalized Multi Poly-Euler polynomials

Corollary 2 *(Addition formula) For non-zero numbers a,b, with $ab\neq-1$ we have*

$$\mathbf{E}_n^{(k_1,\ldots,k_r)}(x+y;a,b)=\sum_{k=0}^{n}\binom{n}{k}r^{n-k}\mathbf{E}_k^{(k_1,\ldots,k_r)}(x;a,b)y^{n-k}\ . \tag{23}$$

Proof. We can write

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{E}_n^{(k_1,\ldots,k_r)}(x+y;a,b)\frac{t^n}{n!}&=\frac{2Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^{t})^{r}}e^{(x+y)rt}\\&=\frac{2Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^{t})^{r}}e^{xrt}e^{yrt}\\&=\left(\sum_{n=0}^{\infty}\mathbf{E}_n^{(k_1,\ldots,k_r)}(x;a,b)\frac{t^n}{n!}\right)\left(\sum_{i=0}^{n}\frac{y^ir^i}{i!}t^i\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}r^{n-k}y^{n-k}\mathbf{E}_k^{(k_1,\ldots,k_r)}(x;a,b)\right)\frac{t^n}{n!}\end{aligned}$$

So, by comparing the coefficients of t^n on both sides , we get the desired result.

2 Explicit formula for Multi Poly-Euler polynomials

Here we present an explicit formula for Multi Poly-Euler polynomials.

Theorem 3 *The Multi Poly-Euler polynomials have the following explicit formula*

(24)

$$\mathbf{E}_n^{(k_1,k_2,\ldots,k_r)}(x)=\sum_{i=0}^{n}\sum_{\substack{0\le m_1\le m_2\le\ldots\le m_r\\ c_1+c_2+\ldots=r}}\sum_{j=0}^{m_r}\frac{2(rx-j)^{n-i}r!(-1)^{j+c_1+2c_2+\cdots}(c_1+2c_2+v\ldots)^i\binom{m_r}{j}\binom{n}{i}}{(c_1!c_2!\ldots)(m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r})}.$$

Proof. We have

$$Li_{(k_1,k_2,\ldots,k_r)}(1-e^{-t})e^{rxt}=\sum_{0\le m_1\le m_2\le\ldots\le m_r}\frac{(1-e^{-t})^{m_r}}{m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r}}e^{rxt}$$

$$=\sum_{0\le m_1\le m_2\le\ldots\le m_r}\frac{1}{m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r}}\sum_{j=0}^{m_r}(-1)^j\binom{m_r}{j}\sum_{n\ge0}(rx-j)^n\frac{t^n}{n!}$$

$$=\sum_{n\ge0}\left(\sum_{0\le m_1\le m_2\le\ldots\le m_r}\sum_{j=0}^{m_r}\frac{(-1)^j(rx-j)^n\binom{m_r}{j}}{m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r}}\right)\frac{t^n}{n!}.$$

On the other hand,

$$\left(\frac{1}{1+e^t}\right)^r=\left(\sum_{n\ge0}(-1)^n e^{nt}\right)^r$$

$$=\sum_{c_1+c_2+\ldots=r}\frac{r!(-1)^{c_1+2c_2+\ldots}}{c_1!c_2!\ldots}e^{t(c_1+2c_2+\ldots)}$$

$$=\sum_{c_1+c_2+\ldots=r}\frac{r!(-1)^{c_1+2c_2+\ldots}}{c_1!c_2!\ldots}\sum_{n\ge0}(c_1+2c_2+\ldots)^n\frac{t^n}{n!}$$

$$=\sum_{n\ge0}\left(\sum_{c_1+c_2+\ldots=r}\frac{r!(-1)^{c_1+2c_2+\ldots}(c_1+2c_2+\ldots)^n}{c_1!c_2!\ldots}\right)\frac{t^n}{n!}.$$

Hence,

$$\frac{2Li_{(k_1,k_2,\ldots,k_r)}(1-e^{-t})}{(1+e^t)^r}e^{rxt}=2Li_{(k_1,k_2,\ldots,k_r)}(1-e^{-t})e^{rxt}\left(\frac{1}{1+e^t}\right)^r$$

$$= \left(\sum_{n\geq 0} \left(\sum_{0\leq m_1\leq m_2\leq \ldots\leq m_r} \sum_{j=0}^{m_r} \frac{(-1)^j (rx-j)^n \binom{m_r}{j}}{m_1^{k_1} m_2^{k_2} \ldots m_r^{k_r}} \right) \frac{t^n}{n!} \right) \times$$

$$\times \left(\sum_{n\geq 0} \left(\sum_{c_1+c_2+\ldots=r} \frac{r!(-1)^{c_1+2c_2+\cdots}(c_1+2c_2+\ldots)^n}{c_1!c_2!\ldots} \right) \frac{t^n}{n!} \right)$$

$$= 2\sum_{n\geq 0}\sum_{i=0}^{n} \left(\sum_{0\leq m_1\leq m_2\leq \ldots\leq m_r} \sum_{j=0}^{m_r} \frac{(-1)^j (rx-j)^{n-i} \binom{m_r}{j}}{m_1^{k_1} m_2^{k_2} \ldots m_r^{k_r}} \right) \frac{t^{n-i}}{(n-i)!} \times$$

$$\times \left(\sum_{c_1+c_2+\ldots=r} \frac{r!(-1)^{c_1+2c_2+\cdots}(c_1+2c_2+\ldots)^i}{c_1!c_2!\ldots} \right) \frac{t^i}{i!}$$

$$= 2\sum_{n\geq 0}\sum_{i=0}^{n} \sum_{\substack{0\leq m_1\leq m_2\leq ..\leq m_r \\ c_1+c_2+\ldots=r}} \sum_{j=0}^{m_r} \frac{(rx-j)^{n-i} r!(-1)^{j+c_1+2c_2+\cdots}(c_1+2c_2+\ldots)^i \binom{m_r}{j}\binom{n}{i}}{(c_1!c_2!\ldots)(m_1^{k_1} m_2^{k_2}\ldots m_r^{k_r})} \frac{t^n}{n!}$$

By comparing the coefficient of $t^n/n!$, we obtain the desired explicit formula.

Definition 4 *(Poly-Euler polynomials with a, b, c parameters):The Poly-Euler polynomials with a, b, c parameters may be defined by using the following generating function,*

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t} c^{xt} = \sum_{n=0}^{\infty} \mathbf{E}_n^{(k)}(x; a, b, c) \frac{t^n}{n!} \quad . \tag{25}$$

Now, in next theorem, we give an explicit formula for Poly-Euler polynomials with a, b, c parameters.

Theorem 4 *The generalized Poly-Euler polynomials with a, b, c parameters have the following explicit formula*

$$\mathbf{E}_n^{(k)}(x; a, b, c) = \sum_{m=0}^{n}\sum_{j=0}^{m}\sum_{i=0}^{j} \frac{2(-1)^{m-j+i}}{j^k} \binom{j}{i} (x\ln c - (m-j+i+1)\ln a - (m-j+i+1)\ln b)^n. \tag{26}$$

Proof. We can write

$$\sum_{n\geq 0}\mathbf{E}_n^{(k)}(x;a,b,c)\frac{t^n}{n!}=\frac{2Li_k(1-(ab)^{-t})}{a^{-t}((ab)^{-t}+1)}c^{xt}$$

$$=2a^{-t}\left(\sum_{n\geq 0}(-1)^n(ab)^{-nt}\right)\left(\sum_{n\geq 0}\frac{(1-(ab)^{-t})^m}{m^k}\right)c^{xt}$$

$$=a^{-t}\sum_{m\geq 0}\sum_{j=0}^{m}\sum_{i=0}^{j}\frac{2(-1)^{m-j+i}}{j^k}\binom{j}{i}(ab)^{-t(x+m-j+i)}c^{xt}$$

$$=\sum_{m\geq 0}\sum_{j=0}^{m}\sum_{i=0}^{j}\frac{2(-1)^{m-j+i}}{j^k}\binom{j}{i}e^{-t(x+m-j+i)\ln(ab)}e^{-t\ln a}e^{xt\ln c}=$$

$$\sum_{n\geq 0}\sum_{m\geq 0}\sum_{j=0}^{m}\sum_{i=0}^{j}\frac{2(-1)^{m-j+i}}{j^k}\binom{j}{i}\sum_{n\geq 0}(x\ln c-(m-j+i+1)\ln a-(m-j+i)\ln b)^n\frac{t^n}{n!}$$

$$=\sum_{n\geq 0}\sum_{m=0}^{n}\sum_{j=0}^{m}\sum_{i=0}^{j}\frac{2(-1)^{m-j+i}}{j^k}\binom{j}{i}(x\ln c-(m-j+i+1)\ln a-(m-j+i)\ln b)^n\frac{t^n}{n!}.$$

By comparing the coefficient of $t^n/n!$, we obtain the desired explicit formula.

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Sandwich results for certain subclasses of analytic functions defined by convolution [1]

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Abstract

In this paper, we obtain some applications of first order differential subordination and superordination results for analytic functions in the open unit disc. The results, which are presented in this paper, have relevant connections with various previous results.

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1 Introduction

Let H be the class of analytic functions in the unit disc $U = \{z \in C : |z| < 1\}$ and let $H[a, k]$ be the subclass of H consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} ... \ (a \in C). \tag{1.1}$$

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Also, let A_1 be the subclass of H consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

If f, $g \in H$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g.,[5] and [16]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $p, h \in H$ and let $\varphi(r, s, t; z) : C^3 \times U \to C$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is a solution of the differential superordination (1.3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant if $q(z) \prec p(z)$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants of (1.3) is called the best subordinant. Recently Miller and Mocanu [17] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [17], Bulboaca [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [6]. Ali et al. [1], have used the results of Bulboaca [4] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [25] obtained a sufficient condition for starlikeness of f in terms of the quantity $\dfrac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [23] obtained sufficient conditions for the normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

For functions f given by (1.2) and $g \in A_1$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For functions $f, g \in A_1$, we define the linear operator $D_\lambda^m : A_1 \to A_1$ ($\lambda \geqslant 0, m \in N_0 = N \cup \{0\}, N = \{1, 2, ...\}$) by:

$$D_\lambda^0 (f * g)(z) = (f * g)(z)\,,$$

$$D_\lambda^1 (f * g)(z) = D_\lambda (f * g)(z) = (1 - \lambda)(f * g)(z) + z\,\lambda((f * g)(z))'\,,$$

and (in general)

$$D_\lambda^m (f * g)(z) = D_\lambda (D_\lambda^{m-1} (f * g)(z))$$

$$= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^m a_k b_k z^k \quad , \lambda \geqslant 0\,. \tag{1.5}$$

From (1.5), we can easily deduce that

$$\lambda z \,(D_\lambda^m (f * g)(z))' = D_\lambda^{m+1} (f * g)(z) - (1 - \lambda) D_\lambda^m (f * g)(z) \; (\lambda > 0). \tag{1.6}$$

We observe that the function $(f * g)(z)$ reduces to several interesting functions for different choices of the function g.

(i) For $\lambda = 1$ and $b_k = 1$ (or $g(z) = \frac{z}{1-z}$), we have $D_1^m(f * g)(z) = D^m f(z)$, where D^m is the Sălăgean operator introduced and studied by Sălăgean [21];

(ii) For $b_k = 1$ (or $g(z) = \frac{z}{1-z}$), we have $D_\lambda^m(f * g)(z) = D_\lambda^m f(z)$, where D_λ^m is the generalized Sălăgean operator introduced and studied by Al-Oboudi [2];

(iii) For $m = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \ (c \neq 0, -1, -2, ...), \tag{1.7}$$

where

$$(d)_k = \begin{cases} 1 & (k = 0; d \in C^* = C \backslash \{0\}) \\ d(d+1)...(d+k-1) & (k \in N; d \in C), \end{cases}$$

we have $D_\lambda^0(f * g)(z) = (f * g)(z) = L(a, c)f(z)$, where the operator $L(a, c)$ was introduced by Carlson and Shaffer [7];

(iv) For $m = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1 + l + \lambda(k-1)}{1 + l}\right]^s z^k \ (\lambda > 0; l, s \in N_0), \tag{1.8}$$

we see that $D_\lambda^0(f * g)(z) = (f * g)(z) = I(s, \lambda, l)f(z)$, where $I(s, \lambda, l)$ is the generalized multiplier transformation which was introduced and studied by Cătaş et al. [8]. The operator $I(s, \lambda, l)$, contains as special cases, the multiplier transformation (see [9]), the generalized Sălăgean operator introduced and studied by Al-Oboudi [2] which in turn contains as special case the Sălăgean operator (see [21]);

(v) For $m = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1}...(\alpha_l)_{k-1}}{(\beta_1)_{k-1}...(\beta_s)_{k-1}(1)_{k-1}} z^k, \tag{1.9}$$

where, $\alpha_i, \beta_j \in C^*$, $(i = 1, 2, ...l)$, $(j = 1, 2, ...s), l \leq s + 1, l, s \in N_0$, we see that, $D^0_\lambda(f * g)(z) = (f * g)(z) = H_{l,s}(\alpha_1)f(z)$, where $H_{l,s}(\alpha_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [10] (see also [11] and [12]). The operator $H_{l,s}(\alpha_1)$, contains in turn many interesting operators such as, Hohlov linear operator (see [13]), the Carlson-Shaffer linear operator (see [7] and [20]), the Ruscheweyh derivative operator (see [19]), the Bernardi-Libera-Livingston operator (see [3], [14] and [15]) and Owa-Srivastava fractional derivative operator (see [18]);

(vi) For $g(z)$ of the form (1.9), the operator $D^m_\lambda(f*g)(z) = D^m_\lambda(\alpha_1, \beta_1)f(z)$, introduced and studied by Selvaraj and Karthikeyan [22].

In this paper, we obtain sufficient conditions for the normalized analytic function f defined by using the operator $D^m_\lambda(f * g)(z)$ to satisfy:

$$q_1(z) \prec \left(\frac{z}{D^m_\lambda(f * g)(z)}\right)^\delta \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U.

2 Definitions and Preliminaries

In order to prove our results, we shall make use of the following known results.

Definition 1 [17]. *Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\xi \in \partial U : \lim_{z\to\xi} f(z) = \infty\},$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

Lemma 1 [23]. *Let q be convex univalent function in U and let $\delta \in C, \gamma \in C^*$ with $Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\delta}{\gamma}\} > 0$. If p is analytic in U and*

$$\delta p(z) + \gamma zp'(z) \prec \delta q(z) + \gamma zq'(z),$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 2 [17]. *Let* q *be convex univalent in* U *and* $\gamma \in C$. *Further assume that* $Re\ \{\overline{\gamma}\} > 0$. *If* $p(z) \in H[q(0), 1] \cap Q$, $p(z) + \gamma z p'(z)$ *is univalent in* U, *then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z),$$

implies $p(z) \prec q(z)$ *and* q *is the best dominant.*

3 Applications to $D_\lambda^m(f * g)$ Operator

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda > 0; \ell \geq 0; p \in N; m \in N_0, z \in U$ and *The powers are the principle ones*.

Theorem 1 *Let* q *be univalent in* U *with* $q(0) = 1$, $\eta \in C^*$, $0 < \delta < 1$ *and* q *satisfy*

$$Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\delta}{\eta}\} > 0. \tag{3.1}$$

If $f \in A_1$ *satisfies the subordination:*

$$\chi(f, g, \lambda, m, \delta, \eta) \prec q(z) + \frac{\eta}{\delta} z q'(z), \tag{3.2}$$

where $\chi(f, g, \lambda, m, \delta, \eta)$ *is given by*

$$\chi(f, g, \lambda, m, \delta, \eta) =$$

$$(1 + \frac{\eta}{\lambda}) \left(\frac{z}{D_\lambda^m(f * g)(z)} \right)^\delta - \frac{\eta}{\lambda} \left(\frac{z}{D_\lambda^m(f * g)(z)} \right)^\delta \frac{D_\lambda^{m+1}(f * g)(z)}{D_\lambda^m(f * g)(z)}, \tag{3.3}$$

then

$$(\frac{z}{D_\lambda^m(f * g)(z)})^\delta \prec q(z) \tag{3.4}$$

and q *is the best dominant.*

Proof. Define a function p by

$$p(z) = (\frac{z}{D_\lambda^m(f*g)(z)})^\delta \quad (z \in U). \tag{3.5}$$

Then the function p is analytic in U and $p(0) = 1$. Therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

(3.6)
$$p(z) + \frac{\eta}{\delta} zp'(z) = (1 + \frac{\eta}{\lambda}) \left(\frac{z}{D_\lambda^m(f*g)(z)}\right)^\delta - \frac{\eta}{\lambda} \left(\frac{z}{D_\lambda^m(f*g)(z)}\right)^\delta \frac{D_\lambda^{m+1}(f*g)(z)}{D_\lambda^m(f*g)(z)}.$$

Using (3.2) and (3.6), we have

$$p(z) + \frac{\eta}{\delta} zp'(z) \prec q(z) + \frac{\eta}{\delta} zq'(z). \tag{3.7}$$

Hence, the assertion (3.4) now follows by using Lemma 1 with $\gamma = \frac{\eta}{\delta}$ $(\eta \in C, 0 < \delta < 1)$ and $\psi = 1$.

Putting $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$ in Theorem 1, we have the following corollary.

Corollary 1 *Let* $-1 \le B < A \le 1$ *and*

$$Re\left\{1 - \frac{2Bz}{1 + Bz} + \frac{\delta}{\eta}\right\} > 0.$$

If $f(z) \in A_1, \eta \in C^*, 0 < \delta < 1$ *and*

$$\chi(f, g, \lambda, m, \delta, \eta) \prec \frac{1 + Az}{1 + Bz} + \frac{\eta}{\delta} \frac{(A - B)z}{(1 + Bz)^2},$$

where $\chi(f, g, \lambda, m, \delta, \eta)$ *is given by* (3.3), *then*

$$\left(\frac{z}{D_\lambda^m(f*g)(z)}\right)^\delta \prec \frac{1 + Az}{1 + Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

And in particular, if $A = 1$, $B = -1$, $q(z) = \frac{1+z}{1-z}$ *and if*

$$\chi(f, g, \lambda, m, \delta, \eta) \prec \frac{1+z}{1-z} + \frac{\eta}{\delta}\frac{2z}{(1-z)^2},$$

where $\chi(f, g, \lambda, m, \delta, \eta)$ is given by (3.3), *then*

$$\left(\frac{z}{D_\lambda^m(f*g)(z)}\right)^\delta \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant.* Taking $m = 0, g(z)$ of the form (1.8) in Theorem 1 and using the identity (see [8])

(3.8)
$$\lambda z(I_p(m, \lambda, l)f(z))' = (1+l)I((m+1, \lambda, l)f(z) - [(1-\lambda)+l]I(m, \lambda, l)f(z) \ (\lambda > 0),$$

we have the following result.

Corollary 2 *Let* q *be univalent in* U, $\eta \in C^*$, $0 < \delta < 1$ *and* q *satisfy* (3.1). *If* $f(z) \in A_1$ *satisfies*

$$\left(1+\frac{(1+l)\eta}{\lambda}\right)\left(\frac{z}{I(m, \lambda, l)f(z)}\right)^\delta - \frac{(1+l)\eta}{\lambda}\left(\frac{z}{I(m, \lambda, l)f(z)}\right)^\delta \left(\frac{I(m+1, \lambda, l)f(z)}{I(m, \lambda, l)f(z)}\right)$$
$$\prec q(z) + \frac{\eta}{\delta}zq'(z),$$

then

$$\left(\frac{z}{I(m, \lambda, l)f(z)}\right)^\delta \prec q(z)$$

and q *is the best dominant.*

Taking $m = 0, g(z)$ of the form (1.9) in Theorem 1 and using the identity (see [10])

$$z(H_{l,s}(\alpha_1)f(z))' = \alpha_1 H_{l,s}(\alpha_1+1)f(z) - (\alpha_1 - 1)H_{l,s}(\alpha_1)f(z),$$

we have the following corollary.

Corollary 3 *Let q be univalent in U, $\eta \in C^*$, $0 < \delta < 1$ and q satisfy* (3.1). *If $f(z) \in A_1$ satisfies*

$$(1+\alpha_1\eta)\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta} - \alpha_1\eta\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta}\left(\frac{H_{l,s}(\alpha_1+1)f(z)}{H_{l,s}(\alpha_1)f(z)}\right)$$

$$\prec q(z) + \frac{\eta}{\delta} zq'(z),$$

then

$$\left(\frac{z}{H_{l,s}(\alpha_1)f(z)}\right)^{\delta} \prec q(z)$$

and q is the best dominant. (The power is the principle one.)

Remark 1 (i) *Taking $m = 0, g(z)$ of the form* (1.7) *and using the identity (see [7])*

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z),$$

in Theorem 1, we have the result obtained by Shanmugam et al. [24 , Theorem 3.1];

(ii) *Taking $m = 0$* and $g(z) = \frac{z}{1-z}$ *in Theorem 1, we have the result obtained by Shanmugam et al. [24 , Corollary 3.2].*

Now, by appealing to Lemma 2 we can prove the following theorem.

Theorem 2 *Let q be convex univalent in U with $q(0) = 1$, $0 < \delta < 1, \eta \in C$ and:*

$$Re\{\eta\} > 0. \tag{3.9}$$

If $f \in A_1$, $\left(\frac{z}{D_{\lambda}^{m}(f*g)(z)}\right)^{\delta} \in H[q(0),1] \cap Q$, $\chi(f,g,\lambda,m,\delta,\eta)$ is univalent in U, and

$$q(z) + \frac{\lambda}{\delta} zq'(z) \prec \chi(f,g,\lambda,m,\delta,\eta),$$

where $\chi(f, g, \lambda, m, \delta, \eta)$ *is given by* (3.3)*, then*

$$q(z) \prec \left(\frac{z}{D_\lambda^m(f * g)(z)}\right)^\delta$$

and q *is the best subordinant.*

Proof. Define a function p by

$$p(z) = \left(\frac{z}{D_\lambda^m(f * g)(z)}\right)^\delta \quad (z \in U).$$

Then simple computations shows that

$$p(z) + \frac{\eta}{\delta} z p'(z) = \chi(f, g, \lambda, m, \delta, \eta),$$

where $\chi(f, g, \lambda, m, \delta, \eta)$ is given by (3.3). Theorem 2 follows as an application of Lemma 2.

Remark 2 *Taking* $m = 0$ *and* $g(z)$ *in the form* (1.7) *in Theorem 2, we have the result of Shanmugam et. al. [24, Theorem 4.1].*

4 Sandwich Theorems

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.

Theorem 3 *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1, \eta \in C$ *and* $0 < \delta < 1$. *Suppose that* q_2 *satisfies* (3.1). *If* $f \in A_1$, $\left(\frac{z}{D_\lambda^m(f * g)(z)}\right)^\delta \in H[q(0), 1] \cap Q$, $\chi(f, g, \lambda, m, \delta, \eta)$ *given by* (3.3) *is univalent in* U *and if*

$$q_1(z) + \frac{\eta}{\delta} z q_1'(z) \prec \chi(f, g, \lambda, m, \delta, \eta) \prec q_2(z) + \frac{\eta}{\delta} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{z}{D_\lambda^m(f * g)(z)}\right)^\delta \prec q_2(z)$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant.*

Remark 3 *Taking* $m = 0$ *and* $g(z)$ *in the form* (1.7) *in Theorem 3, we obtain the result obtained by Shanmugam et al. [24, Theorem 5.1].*

Specializing the parameter m and $g(z)$ in Theorems 1, 2 and 3, we obtain the sandwich theorems for the corresponding operators.

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Strong convergence of a modified implicit iteration process for the finite family of $\psi-$uniformly pseudocontractive mappings in Banach spaces [1]

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Abstract

The purpose of this paper is to establish the strong convergence of a implicit iteration process with errors to a common fixed point for a finite family of $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings in real Banach spaces. The results presented in this paper extend and improve the corresponding results of Refs. [3, 7, 12]. The remark at the end is important.

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1 Introduction

Form now onward, we assume that E is a real Banach space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \text{ and } ||f^*|| = ||x||\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

Let $\Psi := \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is a strictly increasing mapping such that $\psi(0) = 0\}$.

Definition 1 *A mapping $T : K \to K$ is called $\psi-$uniformly pseudocontractive if there exist mapping $\psi \in \Psi$ and $j(x - y) \in J(x - y)$ such that*

$$\langle Tx - Ty, j(x - y) \rangle \le ||x - y||^2 - \psi(||x - y||), \ \forall x, \ y \in K. \tag{1.1}$$

Definition 2 *A mapping $S : D(S) \subset E \to E$ is called $\psi-$uniformly accretive if there exist mapping $\psi \in \Psi$ and $j(x - y) \in J(x - y)$ such that*

$$\langle Sx - Sy, j(x - y) \rangle \ge \psi(||x - y||), \ \forall x, \ y \in E. \tag{1.2}$$

Remark 1 *1. Taking $\psi(a) := \psi(a)a$, $\forall a \in [0, \infty)$, $(\psi \in \Psi)$, we get the usual definitions of $\psi-$ pseudocontractive and $\psi-$ accretive mappings.*

2. Taking $\psi(a) := \gamma a^2; \gamma \in (0, 1)$, $\forall a \in [0, \infty)$, $(\psi \in \Psi)$, we get the usual definitions of strongly pseudocontractive and strongly accretive mappings.

3. T is $\psi-$uniformly pseudocontractive iff $S = I - T$ is $\psi-$uniformly accretive.

It is known that T is strongly pseudocontractive if and only if $(I - T)$ is strongly accretive.

In 2001, Xu and Ori [12] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, ..., N\}$), with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in K$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

which can written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \ \forall n \geq 1, \tag{1.3}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$ (here the $modN$ function takes values in I). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [7], Oslilike proved the following theorem.

Theorem 1 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:*

(i) $0 < \alpha_n < 1$,
(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$,
(iii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$.

From arbitrary $x_0 \in K$, *define the sequence* $\{x_n\}$ *by the implicit iteration process (1.3). Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in I\}$ *if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

Definition 3 *A normed space* E *is said to satisfy Opial's condition if for any sequence* $\{x_n\}$ *in* E, $x_n \rightharpoonup x$ *implies that* $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ *for all* $y \in E$ *with* $y \neq x$.

In [3], Chen et al proved the following theorem.

Theorem 2 *Let* K *be a nonempty closed convex subset of a q-uniformly smooth and p-uniformly convex Banach space* E *that has the Opial property. Let* s *be any element in* $(0, 1)$ *and let* $\{T_i\}_{i=1}^N$ *be a finite family of strictly pseudocontractive self-maps of* K *such that* $\{T_i\}_{i=1}^N$, *have at least one common fixed point. For any point* x_0 *in* K *and any sequence* $\{\alpha_n\}_{n=1}^\infty$ *in* $(0, s)$, *define the sequence* $\{x_n\}$ *by the implicit iteration process (1.3). Then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^N$.

Inspired and motivated by the above said facts, we suggest the following implicit iteration process with errors and define the sequence $\{x_n\}$ as follows

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n + u_n, \ \forall n \geq 1, \tag{1.4}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{u_n\}$ is a summable sequence in K.

Clearly, this iteration process contains the process (1.3) as its special case.

The purpose of this paper is to study the strong convergence of the implicit iteration process (1.4) to a common fixed point for a finite family of $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings in real Banach spaces. The results presented in this paper extend and improve the corresponding results of Refs. [3, 7, 12].

2 Main Results

The following lemma is now well known.

Lemma 3 *Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 4 *[5] Let $\{\theta_n\}$ be a sequence of nonnegative real numbers, $\{\lambda_n\}$ be a real sequence satisfying*

$$0 \leq \lambda_n \leq 1, \ \sum_{n=0}^{\infty} \lambda_n = \infty$$

and let $\psi \in \Psi$. If there exists a positive integer n_0 such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \psi(\theta_{n+1}) + \sigma_n,$$

for all $n \geq n_0$, with $\sigma_n \geq 0, \forall n \in \mathbb{N}$, and $\sigma_n = 0(\lambda_n)$, then $\lim_{n\to\infty} \theta_n = 0$.

Theorem 5 *Let $\{T_1, T_2, ..., T_N\} : K \to K$ be N, $\psi-$uniformly pseudo-contractive mappings with $\{T_n x_n\}$ bounded and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.4) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, $\lim_{n\to\infty}(1-\alpha_n) = 0$ and $\|u_n\| = 0(1-\alpha_n)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.*

Proof. Since each T_i is $\psi-$uniformly pseudocontractive, we have from (1.1)

(2.1) $\langle T_i x - T_i y, j(x-y)\rangle \leq ||x-y||^2 - \psi(||x-y||), \ i = 1, \ 2, \ \cdots, \ N.$

We know that the mappings $\{T_1, T_2, ..., T_N\}$ have a common fixed point in K, say w, then the fixed point set $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ is nonempty.

We will show that w is the unique fixed point of F. Suppose there exists $q \in F(T_1)$ such that $w \neq q$ i.e., $\|w-q\| > 0$. Then

$$\text{(AR)} \qquad \psi(\|w-q\|) > 0.$$

Since ψ is strictly increasing with $\psi(0) = 0$. Then, from the definition of $\psi-$uniformly pseudocontractive mapping,

$$\begin{aligned} ||w-q||^2 &= \langle w-q, j(w-q)\rangle = \langle T_1 w - T_1 q, j(w-q)\rangle \\ &\leq ||w-q||^2 - \psi(||w-q||), \end{aligned}$$

implies

$$\psi(||w-q||) \leq 0,$$

contracditing (AR), which implies the uniqueness. Hence $F(T_1) = \{w\}$. Similarly we can prove that $F(T_i) = \{w\}$; $i = 2, 3, ..., N$. Thus $F = \{w\}$.

We set

$$\begin{aligned} M_1 &= ||x_0 - w|| + \sup_{n\geq 0} ||T_n x_n - w||, \\ M_2 &= 1 + \sup_{n\geq 0} ||u_n||. \end{aligned}$$

Obviously M_1, $M_2 < \infty$. Let $M_3 = M_1 + M_2$.

It is clear that $||x_0 - w|| \leq M_1 < M_3$. Let $||x_{n-1} - w|| \leq M_1 < M_3$. Next we will prove that $||x_n - w|| \leq M_3$.

Consider

$$\begin{aligned} ||x_n - w|| &= ||\alpha_n x_{n-1} + (1-\alpha_n) T_n x_n + u_n - w|| \\ &= ||\alpha_n (x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w) + u_n|| \\ &\leq \alpha_n ||x_{n-1} - w|| + (1-\alpha_n)||T_n x_n - w|| + \|u_n\| \\ &\leq \alpha_n M_1 + (1-\alpha_n) M_1 + M_2 \\ &= M_1 + M_2 \\ &= M_3. \end{aligned}$$

So, from the above discussion, we conclude that the sequence $\{x_n - w\}$ is bounded. Let $M_4 = \sup_{n\geq 0} ||x_n - w||$.

Denote $M = M_3 + M_4$. Obviously $M < \infty$.

The real function $f : [0, \infty) \to [0, \infty)$, defined by $f(t) = t^2$ is increasing and convex. For all $\lambda \in [0, 1]$ and $t_1, t_2 > 0$ we have

$$((1-\lambda)t_1 + \lambda t_2)^2 \leq (1-\lambda)t_1^2 + \lambda t_2^2. \tag{2.2}$$

Consider

$$\begin{aligned} ||x_n - w||^2 &= ||\alpha_n x_{n-1} + (1-\alpha_n)T_n x_n + u_n - w||^2 \\ &= ||\alpha_n(x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w) + u_n||^2 \\ &\leq [\alpha_n \|x_{n-1} - w\| + (1-\alpha_n)\|T_n x_n - w\| + \|u_n\|]^2 \\ &\leq \alpha_n \|x_{n-1} - w\|^2 + (1-\alpha_n)\|T_n x_n - w\|^2 + \|u_n\|^2 + 2M\|u_n\| \\ &\leq \|x_{n-1} - w\|^2 + M^2(1-\alpha_n) + \|u_n\|^2 + 2M\|u_n\|. \qquad (2.3) \end{aligned}$$

From lemma 1 and (1.4), we have

$$\begin{aligned} \|x_n - w\|^2 &= \|\alpha_n x_{n-1} + (1-\alpha_n) T_n x_n + u_n - w\|^2 \\ &= \|\alpha_n(x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w) + u_n\|^2 \\ &\leq {\alpha_n}^2 ||x_{n-1} - w||^2 + 2(1-\alpha_n)\langle T_n x_n - w, j(x_n - w)\rangle \\ &\quad +2\langle u_n, j(x_n - w)\rangle \\ &\leq {\alpha_n}^2 ||x_{n-1} - w||^2 + 2(1-\alpha_n)\|x_n - w\|^2 \\ &\quad -2(1-\alpha_n)\psi(\|x_n - w\|) + 2M\|u_n\|. \qquad (2.4) \end{aligned}$$

Substituting (2.3) in (2.4), and with the help of $\|u_n\| = 0(1-\alpha_n)$ (implies $\|u_n\| = (1-\alpha_n)t_n$; $t_n \to 0$ as n$\to \infty$) we get

$$\begin{aligned}\|x_n - w\|^2 \leq & [\alpha_n^2 + 2(1-\alpha_n)]||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) \\ & +2M^2(1-\alpha_n)^2 + 2(1-\alpha)\|u_n\|^2 + 4M(1-\alpha)\|u_n\| \\ & +2M\|u_n\| \\ = & \left[1 + (1-\alpha_n)^2\right]||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) \\ & +2M^2(1-\alpha_n)^2 + 2(1-\alpha)\|u_n\|^2 + 2M[1 + 2(1-\alpha)]\|u_n\| \\ \leq & ||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) + 3M^2(1-\alpha_n)^2 \\ & +2(1-\alpha)\|u_n\|^2 + 6M\|u_n\| \\ \leq & ||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) \\ & +(1-\alpha_n)[3M^2(1-\alpha_n) + 2(1+3M)t_n]. \end{aligned} \tag{2.5}$$

Denote

$$\begin{aligned}\theta_n &= ||x_{n-1} - w||, \\ \lambda_n &= 2(1-\alpha_n), \\ \sigma_n &= (1-\alpha_n)[3M^2(1-\alpha_n) + 2(1+3M)t_n].\end{aligned}$$

Condition $\lim_{n\to\infty}(1-\alpha_n) = 0$ assures the existence of a rank $n_0 \in \mathbb{N}$ such that $\lambda_n = 2(1-\alpha_n) \leq 1$, for all $n \geq n_0$. Now with the help of $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, $\lim_{n\to\infty}(1-\alpha_n) = 0$ and lemma 2, we obtain from (2.5) that

$$\lim_{n\to\infty} ||x_n - w|| = 0,$$

completing the proof.

Corollary 6 *Let $\{T_1, T_2, ..., T_N\} : K \to K$ be N, $\psi-$uniformly pseudo-contractive mappings with $\{T_n x_n\}$ bounded and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.3) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and $\lim_{n\to\infty}(1-\alpha_n) = 0$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.*

Remark 2 *Theorem 3 extend and improve the theorems 1-2 in the following directions:*

1) The strictly pseudocontractive mappings are replaced by the more general $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings;

2) Theorem 3 holds in real Banach space whereas the results of theorem 2 are in q-uniformly smooth and p-uniformly convex Banach space;

3) We do not need the assumption $\lim_{n\to\infty} d(x_n, F)$ as in theorem 1;

4) Weak convergence in theorem 2 is replaced by the strong convergence in theorem 3;

5) One can easily see that if we take $\alpha_n = 1-\frac{1}{\sqrt{n}}$, then $\sum(1-\alpha_n) = \infty$, but $\sum(1-\alpha_n)^2 = \infty$. Hence the conclusion of theorem 1 is not true in all cases.

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