Uniqueness of Meromorphic Function and its Differential Polynomial Concerning Weakly Weighted-Sharing

Hong-Yan Xu, Yi Hu

Abstract

In this paper, we introduce the definition of weakly weighted-sharing which is between "CM" and "IM". Using the notion of weakly weighted-sharing, we investigate problems of meromorphic functions that share a small function with its differential polynomial, and give some results and also answer some questions of Kit-Wing Yu, which were also studied by L.P.Liu and Y.X. Gu, S.H. Lin and W.C. Lin. Let S(f) be the set of meromorphic functions in the complex plane C which are small functions with respect to f.

2000 Mathematics Subject Classification: 30D30, 30D35.
Key words and phrases: Meromorphic function, Uniqueness theorem, Small function, Weakly-sharing.

1 Introduction and results

In this paper a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notation in Nevanlinna’s value distribution theory of meromorphic functions such as T(r, f), N(r, f) and m(r, f) (see [4] or [10]). We denote by S(r, f) any quantity satisfying S(r, f) = o(T(r, f)) as r → ∞ except possibly for a set of r of finite linear measure. A meromorphic function a(z) is called a small function with respect to f(z) if T(r, a) = S(r, f). Let S(f) be the set of meromorphic functions in the complex plane C which are small functions with respect to f.

1 Received 04 March, 2010
Accepted for publication (in revised form) 30 August, 2011
Let $a \in S(f)$, we say that two meromorphic functions $f$ and $g$ share a IM (CM) provided that $f - a$ and $g - a$ have the same zeros ignoring (counting) multiplicities.

Mues and Steinmetz [8], Gundersen [3], Yang [12] and Yi [11], and many other authors have obtained elegant results on the uniqueness problems of entire functions that share values CM or IM with their first or $n$-th derivatives.

In 2003, Yu [9] considered the uniqueness problem of an entire function or meromorphic function when it shares one small function with its derivative and proved the following results.

**Theorem A** Let $n \geq 1$, let $f$ be a non-constant entire function, $a \in S(f)$ and $a \neq 0, \infty$. If $f, f^{(n)}$ share a CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(n)}$.

**Theorem B** Let $n \geq 1$, let $f$ be a non-constant non-entire meromorphic function, $a \in S(f)$ and $a \neq 0, \infty$, $f$ and $a$ do not have any common pole. If $f, f^{(n)}$ share a CM and $4\delta(0, f) + 2(8 + n)\Theta(\infty, f) > 19 + 2n$, then $f \equiv f^{(n)}$.

In the same paper, Yu [9] posed the following open question:

**Question A** Can a CM shared value be replaced by an IM shared value in Theorem A?

**Question B** Is the condition $\delta(0, f) > 3/4$ sharp in Theorem A?

**Question C** Is the condition $4\delta(0, f) + 2(8 + n)\Theta(\infty, f) > 19 + 2n$ sharp in Theorem B?

**Question D** Can the condition $f$ and $a$ do not have any common pole” be deleted in Theorem B?

In 2004, Liu and Gu [7] applied a different method and obtained the following results.

**Theorem C** Let $f$ be a non-constant meromorphic function, $a \in S(f)$ and $a \neq 0, \infty$. If $f, f^{(n)}$ share a CM, $f$ and $a$ do not have any common pole of same multiplicity and $2\delta(0, f) + 4\Theta(\infty, f) > 5$, then $f \equiv f^{(n)}$.

**Theorem D** Let $n \geq 1$, let $f$ be a non-constant entire function, $a \in S(f)$ and $a \neq 0, \infty$. If $f, f^{(n)}$ share a CM and $\delta(0, f) > \frac{1}{2}$, then $f \equiv f^{(n)}$.

Let

$$L(f) = f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_0f,$$

be a differential polynomial on $f$, where $a_j (j = 0, 1, \cdots, n - 1) \in S(f)$.

**Question 1**: what happens if $f^{(n)}$ is replaced by $L(f)$ in Theorem C and D?

In order to state our results, we first introduce the definition of weakly weighted-sharing as followed.

**Definition 1** Let $k$ be a positive integer, and let $f$ be a meromorphic function and $a \in S(f)$.

(i) $\overline{N}(r, a; f \geq k)$ denotes the counting function of zeros of $f - a$ whose multiplicities are not greater than $k$, where each zero is counted only once.

(ii) $\overline{N}(r, a; f \leq k)$ denotes the counting function of zeros of $f - a$ whose multiplicities are not less than $k$, where each zero is counted only once.

(iii) $N_p(r, a; f) = \overline{N}(r, a; f) + \sum_{k=2}^{p} \overline{N}(r, a; |f| \geq k)$. 

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Definition 2 \[5\] For any complex number \( c \in \mathbb{C} \cup \{\infty\} \), We denote by \( \delta_p(c, f) \) the quantity
\[
\delta_p(c, f) = 1 - \limsup_{r \to \infty} \frac{N_0(r, c; f)}{T(r, f)},
\]
where \( p \) is a positive integer. Clearly \( \delta_p(c, f) \geq \delta(c, f) \).

Let \( N_E(r, a) \) be the counting function of all common zeros of \( f - a \) and \( g - a \) with the same multiplicities, and \( N_0(r, a) \) be the counting functions of all common zeros of \( f - a \) and \( g - a \) ignoring multiplicities. Denotes by \( \overline{N}_E(r, a) \) and \( \overline{N}_0(r, a) \) the reduced counting functions of \( f \) and \( g \) corresponding to the counting functions \( N_E(r, a) \) and \( N_0(r, a) \), respectively. If
\[
\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_E(r, a) = S(r, f) + S(r, g),
\]
then we say that \( f \) and \( g \) share a "CM". If
\[
\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_0(r, a) = S(r, f) + S(r, g),
\]
then we say that \( f \) and \( g \) share a "IM".

Definition 3 Let \( f \) and \( g \) be two nonconstant meromorphic functions sharing a "IM", for \( a \in S(f) \cap S(g) \), and a positive integer \( k \) or \( \infty \).

(i) \( \overline{N}^k_E(r, a) \) denotes the counting function of zeros of \( f - a \) whose multiplicities are equal to the corresponding zeros of \( g - a \), both of their multiplicities are not greater than \( k \), where each zero is counted only once.

(ii) \( \overline{N}_0^k(r, a) \) denotes the reduced counting function of zeros of \( f - a \) which are zeros of \( g - a \), both of their multiplicities are not less than \( k \), where each zero is counted only once.

(iii) Let \( z_0 \) be the zeros of \( f - a \) with multiplicity \( p \) and zeros of \( g - a \) with multiplicity \( q \). Denote by \( \overline{N}_{f \geq k}(r, a; g) \) the reduced counting function of those zeros of \( f - a \) and \( g - a \) such that \( p > q = k \). \( \overline{N}_{g \geq k}(r, a; f) \) is defined analogously.

(iv) \( \overline{N}_*(r, a; f, g) \) denotes the reduce counting function of zeros of \( f - a \) whose multiplicities differ from the multiplicities of the corresponding zeros of \( g - a \).

Clearly
\[
\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f) \quad \text{and} \quad \overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g).
\]

Definition 4 For \( a \in S(f) \cap S(g) \), if \( k \) is a positive integer or \( \infty \), and
\[
\overline{N}(r, a; f) \leq k - \overline{N}_E^k(r, a) = S(r, f), \quad \overline{N}(r, a; f) \geq k + 1 - \overline{N}_0^{k+1}(r, a) = S(r, f);
\]
\[
\overline{N}(r, a; g) \leq k - \overline{N}_E^k(r, a) = S(r, g), \quad \overline{N}(r, a; g) \geq k + 1 - \overline{N}_0^{k+1}(r, a) = S(r, g).
\]
Or if \( k = 0 \) and
\[
\overline{N}(r, a; f) - \overline{N}_0(r, a) = S(r, f), \quad \overline{N}(r, a; g) - \overline{N}_0(r, a) = S(r, g),
\]
where \( \overline{N}_0(r, a) \) is the reduce counting functions of all common zeros of \( f - a \) and \( g - a \) ignoring multiplicities, then we say \( f \) and \( g \) weakly share \( a \) with weight \( k \). Here, we write \( f, g \) share \( "(a, k)" \) to mean that \( f, g \) weakly share \( a \) with weight \( k \).

Obviously, if \( f \) and \( g \) share \( "(a, k)" \), then \( f \) and \( g \) share \( "(a, p)" \) for any \( p \) \( (0 \leq p \leq k) \). Also, we note that \( f \) and \( g \) share \( "IM" \) or \( "CM" \) if and only if \( f \) and \( g \) share \( "(a, 0)" \) or \( "(a, \infty)" \), respectively.

**Question 2:** Can a CM shared value be replaced by weakly weighted-sharing in Theorem C and Theorem D?

In this paper, we obtain some uniqueness theorems which answer **Question 1** and **Question 2** as followed.

**Theorem 1** Let \( n \geq 1 \) and \( 2 \leq k \leq \infty \), let \( f \) be a non-constant meromorphic function, \( a \in S(f) \) and \( a \neq 0, \infty \). Suppose that \( L(f) \) is defined by \((*)\), If \( f, L(f) \) share \( "(a, k)" \) and
\[
\begin{align*}
4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5, \\
\end{align*}
\]
then \( f \equiv L(f) \).

**Theorem 2** Let \( n \geq 1 \), let \( f \) be a non-constant meromorphic function, \( a \in S(f) \) and \( a \neq 0, \infty \). Suppose that \( L(f) \) is defined by \((*)\), If \( f, L(f) \) share \( "(a, 1)" \) and
\[
\begin{align*}
\left(\frac{7}{2} + n\right)\Theta(\infty, f) + \frac{3}{2}\delta_{2}(0, f) + \delta_{2+n}(0, f) > n + 5, \\
\end{align*}
\]
then \( f \equiv L(f) \).

**Theorem 3** Let \( n \geq 1 \), let \( f \) be a non-constant meromorphic function, \( a \in S(f) \) and \( a \neq 0, \infty \). Suppose that \( L(f) \) is defined by \((*)\), If \( f, L(f) \) share \( "(a, 0)" \) and
\[
\begin{align*}
(6 + 2n)\Theta(\infty, f) + \delta_{2}(0, f) + 2\Theta(0, f) + 2\delta_{2+n}(0, f) > (2n + 10), \\
\end{align*}
\]
then \( f \equiv L(f) \).

From Theorem 1.5-1.7 we have

**Corollary 1** Let \( f \) be a non-constant entire function and \( a \equiv a(z)(\not= 0, \infty) \) be a meromorphic function such that \( T(r, a) = S(r, f) \). If \( f, L(f) \) share \( "(a, k)" \), \( k \geq 2 \) and \( \delta_{2+n}(0, f) > \frac{1}{2} \), or if \( f, L(f) \) share \( "(a, 1)" \) and \( \delta_{2+n}(0, f) > \frac{1}{2} \), or if \( f, L(f) \) share \( "(a, 0)" \) and \( \delta_{2+n}(0, f) > 2 - \frac{1}{2}(\delta_{2}(0, f) + 2\Theta(0, f)) \), then \( f \equiv L(f) \), where \( L(f) \) is defined by \((*)\).
2 Some lemmas

Next, we introduce some notations for the following lemmas.

Lemma 1 Let \( f \) be a transcendental meromorphic function, \( L(f) \) be defined by (*), If \( L(f) \neq 0 \), we have

(i) \( N_2(r, 0; L) \leq N_{2+n}(r, 0; f) + n\overline{N}(r, \infty; f) + S(r, f); \)

(ii) \( N_2(r, 0; L) \leq T(r, L) - T(r, f) + N_{2+n}(r, 0; f) + S(r, f). \)

Proof: By the first fundamental theorem and the lemma of logarithmic derivatives, we get:

\[
N_2(r, 0; L) \leq N(r, 0; L) - \sum_{p=3}^{\infty} \overline{N}(r, 0; L \geq p) \\
= T(r, L) - m(r, \frac{1}{L}) - \sum_{p=3}^{\infty} \overline{N}(r, 0; L \geq p) + O(1) \\
\leq T(r, L) - m(r, \frac{1}{L}) - m(r, \frac{1}{f}) - \sum_{p=3}^{\infty} \overline{N}(r, 0; L \geq p) + O(1) \\
\leq T(r, L) - T(r, f) + N(r, 0; f) - \sum_{p=3}^{\infty} \overline{N}(r, 0; L \geq p) + S(r, f) \\
\leq T(r, L) - T(r, f) + N_{2+n}(r, 0; f) + \sum_{p=3+n}^{\infty} \overline{N}(f, r, 0; f) \\
- \sum_{p=3}^{\infty} \overline{N}(r, 0; L \geq p) + S(r, f) \\
\leq T(r, L) - T(r, f) + N_{2+n}(r, 0; f) + S(r, f).
\]

So this proves Lemma (ii).

Since

\[
T(r, L) = m(r, L) + N(r, \infty; L) \\
\leq m(r, f) + m(r, \frac{1}{f}) + N(r, \infty; f) + n\overline{N}(r, \infty; f) \\
= T(r, f) + n\overline{N}(r, \infty; f) + S(r, f).
\]

From this and Lemma (ii), we can prove Lemma (i).

Lemma 2 [11] Let \( k \) be a nonnegative integer or \( \infty \), \( F \) and \( G \) be two nonconstant meromorphic functions, \( F \) and \( G \) share “\((1, k)\)”. Let

\[
H = \left( \frac{F''}{F'} - 2 \frac{F''}{F - 1} \right) - \left( \frac{G''}{G'} - 2 \frac{G''}{G - 1} \right).
\]

If \( H \neq 0 \), \( 2 \leq k \leq \infty \), then

\[
T(r, F) \leq N_2(r, \infty; F) + N_2(r, 0; F) + N_2(r, \infty; G) + N_2(r, 0; G) + S(r, F) + S(r, G).
\]

The same inequalities holds for \( T(r, G) \).

When \( f \) and \( g \) share 1 “IM”, \( \overline{N}_L(r, 1; f) \) denotes the counting function of the 1-points of \( f \) whose multiplicities are greater than 1-points of \( g \), where each zero is counted only once. Similarly, \( \overline{N}_L(r, 1; g) \) denotes the counting function of those simple 1-points of \( f \) and \( g \), and \( \overline{N}_E(r, 1; f) \) denotes the counting function of those multiplicity 1-points of \( f \) and \( g \), each point in these counting functions is counted only once. In the same way, one can define \( \overline{N}_E^{1}(r, 1; g) \), \( \overline{N}_E^{2}(r, 1; g) \).
Lemma 3 If \( f, g \) be two nonconstant meromorphic functions such that they share 
\'\( (1, 1) \)'', then
\[
2N_L(r, 1; f) + 2N_L(r, 1; g) + N_E^2(r, 1; f) - N_{f>2}(r, 1; g) \leq N(r, 1; g) - N(r, 1; g).
\]

Lemma 4 Let \( f, g \) share 
\'\( (1, 1) \)''. Then
\[
N_{f>2}(r, 1; g) \leq \frac{1}{2}N(r, 0; f) + \frac{1}{2}N(r, \infty; f) - \frac{1}{2}N_0(r, 0; f') + S(r, f).
\]

Lemma 5 Let \( f \) and \( g \) be two nonconstant meromorphic functions sharing 
\'\( (1, 0) \)''. Then
\[
N_L(r, 1; f) + 2N_L(r, 1; g) + N_E^2(r, 1; f) - N_{f>1}(r, 1; f) - N_{g>1}(r, 1; f) 
\leq N(r, 1; g) - N(r, 1; g).
\]

Lemma 6 Let \( f, g \) share 
\'\( (1, 0) \)''. Then
\[
N_L(r, 1; f) \leq N(r, 0; f) + N(r, \infty; f) + S(r, f).
\]

Lemma 7 Let \( f, g \) share 
\'\( (1, 0) \)''. Then
\begin{enumerate}
  \item \( N_{f>1}(r, 1; g) \leq N(r, 0; f) + N(r, \infty; f) - N_0(r, 0; f') + S(r, f) \);
  \item \( N_{g>1}(r, 1; f) \leq N(r, 0; g) + N(r, \infty; f) - N_0(r, 0; f') + S(r, g) \);
\end{enumerate}

Using the method of [1] and [2], we can prove Lemmas 2.3-2.7 easily. Here, we omit them.

3 Proof of main Theorems

Proof of Theorem 1.5

Let
\[
F = \frac{f}{a}, \quad G = \frac{L(f)}{a}.
\]

From the conditions of Theorem 1.5, we know that \( F \) and \( G \) share 
\'\( (1, k) \)'', and from (4), we have
\[
T(r, F) = O(T(r, f)) + S(r, f), \quad T(r, G) = O(T(r, f)) + S(r, f).
\]

\[
N(r, \infty; F) = N(r, \infty; G) + S(r, f).
\]

Let \( H \) be defined as in Lemma 2.2. Suppose that \( H \neq 0 \). It follows from Lemma 2.2 that
\[
T(r, G) \leq N_2(r, \infty; F) + N_2(r, 0; F) + N_2(r, \infty; G) + N_2(r, 0; G) + S(r, F) + S(r, G).
\]
Using Lemma 2.1, we have
\[ T(r, L) \leq N_2(r, \infty; f) + N_2(r, 0; f) + N_2(r, \infty; L) + N_2(r, 0; L) + S(r, f) \]
\[ \leq N_{2+n}(r, 0; f) + T(r, L) - T(r, f) + N_{2+n}(r, 0; f) \]
\[ + 4N(r, \infty; f) + S(r, f), \]

i.e.
\[ T(r, f) \leq 2N_{2+n}(r, 0; f) + 4N(r, \infty; f) + S(r, f). \]
Which contradicts the assumption (1) of Theorem 1.5. Thus, \( H \equiv 0. \) That is
\[ F'' \frac{F'}{F} - 2 F' \frac{G'}{G} \equiv \frac{G''}{G} - 2 \frac{G'}{G - 1}. \]

It follows that
\[ \frac{1}{F - 1} = \frac{A}{G - 1} + B, \]
where \( A(\neq 0) \) and \( B \) are constants. Thus
\[ F = \frac{(B + 1)G + (A - B - 1)}{BG + (A - B)} \]
and \( T(r, F) = T(r, G) + S(r, f). \)

Next, we consider three cases.

Case 1. \( B \neq 0, -1. \) If \( A - B - 1 \neq 0 \), then by (7) we know
\[ N \left( r, \frac{-A + B + 1}{B + 1}; G \right) = N(r, 0; F). \]
By the Nevanlinna second fundamental theorem and Lemma 2.1 we have
\[ N_2(r, \infty; G) + N_2(r, 0; G) + N\left( r, \frac{-A + B + 1}{B + 1}; G \right) + S(r, G) \]
\[ = N_2(r, \infty; G) + N(r, 0; G) + N(r, 0; F) + S(r, f), \]

i.e.
\[ T(r, L) < \frac{N(r, \infty; f) + N(r, 0; L) + N(r, 0; f) + S(r, f)}{N_2(r, \infty; f) + N(r, 0; f) + N_{2+n}(r, 0; f) + N(r, 0; f) + S(r, f)}. \]
Then
\[ T(r, f) < \frac{N(r, \infty; f) + 2N_{2+n}(r, 0; f) + S(r, f)}{N_2(r, \infty; f) + 2N_{2+n}(r, 0; f) + S(r, f)}. \]
Which contradicts the assumption (1).

If \( A - B - 1 = 0 \), then by (7) we know \( F = (B + 1)G/(BG + 1) \). Obviously,
\[ N \left( r, \frac{1}{B}; G \right) = N(r, \infty; F). \]
Similar to the arguments in the above, we also have a contradiction.

Case 2. \( B = -1. \) Then (7) becomes \( F = A/(A + 1 - G) \).
If $A + 1 \neq 0$, then $N(r, A + 1; G) = N(r, \infty; F)$. Similarly, we can deduce a contradiction as in Case 1.

If $A + 1 = 0$, then $FG \equiv 1$, that is

$$f \cdot L(f) \equiv a^2. \quad (8)$$

From (8), we have

$$N(r, 0; f) + N(r, \infty; f) = S(r, f). \quad (9)$$

Since $N(r, f^{(n)}/f) = S(r, f), m(r, f^{(n)}/f) = S(r, f)$, then $T(r, f^{(n)}/f) = S(r, f)$.

From (9), we obtain

$$2T\left(r, \frac{f}{a}\right) = T\left(r, \frac{f^2}{a^2}\right) + O(1) = T\left(r, \frac{L}{f}\right) + O(1) + S(r, f).$$

i.e.

$$T(r, f) = S(r, f),$$

we can get a contradiction.

Case 3. $B = 0$. Then (7) becomes $F = (G + A - 1)/A$.

If $A - 1 \neq 0$, then $N(r, 1/(G + A - 1)) = N(r, 1/F)$. Similarly, we can again deduce a contradiction as in Case 1.

If $A - 1 = 0$, then $F \equiv G$, that is

$$f \equiv L(f).$$

This completes the proof of the Theorem 1.5.

**Proof of Theorem 1.6:** Let $F, G$ be given by (4), from the assumption of Theorem 1.6, we know that $F$ and $G$ share ")((1, 1)"

Let $H$ be defined as in Lemma 2.2. Suppose that $H \neq 0$. Since $F, G$ share ")((1, 1)"

we can get

$$N(r, \infty; H) \leq N(r, \infty; F) + N(1; F \geq 2) + N(r, 0; F \geq 2) + N(r, ; G \geq 2) + N_0(r, 0; F') + N_0(r, 0; G') + S(r, f), \quad (10)$$

and

$$N(r, 1; F = 1) \leq N(r, 0; H) + S(r, f) \leq N(r, \infty; H) + S(r, f), \quad (11)$$

where $N_0(r, 0; F')$ is the reduced counting function of those zeros of $F'$ which are not the zeros of $F(F - 1)$, and $N_0(r, 0; G')$ is similarly defined.

By the second fundamental theorem, we see that

$$T(r, F) + T(r, G) \leq N(r, \infty; F) + N(r, 0; F) + N(r, \infty; G) + N(r, 0; G) + N(r, 1; F) + N(r, 1; G) - N_0(r, 0; F') - N_0(r, 0; G') + S(r, F) + S(r, G). \quad (12)$$
Using Lemmas 2.3 and 2.4, (10) and (11) we can get
\[
\begin{align*}
\overline{N}(r, 1; F) + \overline{N}(r, 1; G) \\
\leq N(r, 1; F) + N(r, 1; G) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) \\
+ \overline{N}_E(r, 1; F) + \overline{N}(r, 1; G) \\
\leq N(r, 1; F) + N(r, 1; G) - \overline{N}_L(r, 1; F) \\
- \overline{N}_L(r, 1; G) + \overline{N}_{F \geq 2}(r, 1; G) \\
\leq \overline{N}(r, 0; F) \geq 2 + \overline{N}(r, 0; G) \geq 2 + \overline{N}(r, \infty; F) \\
+ \overline{N}_s(r, 1; F, G) + T(r, G) - m(r, 1; G) + O(1) + \frac{1}{2} \overline{N}(r, \infty; F) \\
- \overline{N}_L(r, 1; F) - \overline{N}_L(r, 1; G) + \frac{1}{2} \overline{N}(r, 0; F) \\
+ N_0(r, 0; F') + N_0(r, 0; G') + S(r, F) + S(r, G).
\end{align*}
\]

Combining (12) and (13), we can obtain
\[
T(r, F) \leq \frac{7}{2} \overline{N}(r, \infty; F) + N_2(r, 0; F) + N_2(r, 0; G) + \frac{1}{2} \overline{N}(r, 0; F) \\
\leq \frac{7}{2} \overline{N}(r, \infty; F) + \frac{3}{2} N_2(r, 0; F) + N_2(r, 0; G) + S(r, f).
\]

By the definition of $F, G$ and Lemma 2.1(i), we have
\[
T(r, f) \leq \frac{7}{2} \overline{N}(r, \infty; f) + \frac{3}{2} N_2(r, 0; f) + N_2(r, 0; L) + S(r, f) \\
\leq \left(\frac{7}{2} + n\right) \overline{N}(r, \infty; f) + \frac{3}{2} N_2(r, 0; f) + N_{2+n}(r, 0; f) + S(r, f).
\]

By the Lemma 2.1, we have
\[
T(r, f) \leq \left(\frac{7}{2} + n\right) N(r, \infty; f) + \frac{3}{2} N_2(r, 0; f) + N_{2+n}(r, 0; f) + S(r, f).
\]

So
\[
\left(\frac{7}{2} + n\right) \Theta(\infty, f) + \frac{3}{2} \delta_2(0, f) + \delta_{2+n}(0, f) \leq n + 5.
\]

Which contradicts the assumption (2) of Theorem 1.6. Thus, $H \equiv 0$.

Similar to the arguments in Theorem 1.5, we can prove that the conclusions of Theorem 1.6 hold.

**Proof of Theorem 1.7:** Let $F, G$ be given by (4), from the assumption of Theorem 1.6, we know that $F$ and $G$ share "$(1, 0)$".

Let $H$ be defined as in Lemma 2.2. Suppose that $H \not\equiv 0$. Since $F, G$ share "$(1, 0)$", we can get
\[
N(r, \infty; H) \leq \overline{N}(r, \infty; F) + \overline{N}(1; F) \geq 2 + \overline{N}(r, 0; F) \geq 2 \\
+ \overline{N}(r, G) \geq 2 + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) \\
+ N_0(r, 0; F') + N_0(r, 0; G') + S(r, f),
\]
and
\[
N_E^{1}(r, 1; F) = N_E^{1}(r, 1; G) + S(r, f), \quad N_E^{1}(r, 1; F) = N_E^{1}(r, 1; G) + S(r, f),
\]
Using Lemmas 2.5–2.7 and (14) and (15), we get

\[ N_E^1(r, 1; F) \leq N(r, \infty; H) + S(r, f). \]

Combining (12) and (16) and by Lemma 2.1, we can obtain

\[ T(r, f) \leq 6N(r, \infty; f) + N_2(r, 0; f) + 2\overline{N}(r, 0; f) + 2\overline{N}_2(r, 0; L) + S(r, f) \]
\[ \leq (2n + 6)\overline{N}(r, \infty; f) + N_2(r, 0; f) + 2\overline{N}(r, 0; f) \]
\[ + 2\overline{N}_{2+n}(r, 0; f) + S(r, f) \]

So

\[ (6 + 2n)\Theta(\infty, f) + \delta_2(0, f) + 2\Theta(0, f) + 2\delta_{2+n}(0, f) \leq (2n + 10). \]

Which contradicts the assumption (3) of Theorem 1.7. Thus, \( H \equiv 0. \)

Similar to the arguments in Theorem 1.5, we can prove that the conclusions of Theorem 1.7 hold.

4 Acknowledgements

The authors want to thanks the referee for his/her thorough review. This work was supported by the Natural Science Foundation of Jiang-Xi Province in China(Grant No. 2010GQS0119) and the Youth Foundation of Education Bureau of Jiangxi Province in China(Grant No. GJJ10223).

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