Existence Results for some Integral Equation with Modified Argument

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Abstract

One of the most important tool in establishing existence or uniqueness theorems for integral equations in the classical Banach’s contraction mapping principle or some of its generalizations.

In this paper we shall illustrate how one can establish existence results and approximate the solutions of certain integral equations in the case when the contraction mapping principle does not apply, but we are able to use instead the technique of nonexpansive mappings.

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1 Introduction

Most existence or existence and uniqueness theorems for integral equations are usually obtained by means of fixed point technique, e.g. by Schauder’s fixed point theorem or by the contractions mapping principle. One of the most important tool in establishing existence or uniqueness theorems for integral equations in the classical Banach’s contraction mapping principle or some of its generalizations.

In this paper we shall illustrate how one can approximate the solutions of certain integral equations in the case when the contraction mapping principle does not apply, but we are able to use instead the technique of nonexpansive mappings.
2 Preliminaries

The content of this section is taken from [2] and [5].

Let \((X, d)\) be a metric space. A mapping \(T : X \to X\) is said to be an \(\alpha\)-contraction if there exists \(\alpha \in [0, 1)\) such that

\[
d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X.
\]

The strict contraction condition ensures the existence and uniqueness of fixed point for a \(\alpha\)-contraction in a complete metric space and also the convergence of Picard iteration, defined by \(x_0 \in X\) and

\[
x_{n+1} = Tx_n, \quad n = 0, 1, 2, ...
\]

to that fixed point. In the case when \(\alpha = 1\), \(T\) is said to be nonexpansive.

As a complete study of nonexpansive mappings with respect to their fixed points could be better done in a normed space setting, we shall present some concepts and results that will be used later in the paper. Let \(K\) be a nonempty subset of a real normed linear space \(E\) and let \(T : K \to K\) be a map. A point \(x \in K\) is called a fixed point of \(T\) if \(Tx = x\). In this setting, \(T\) is nonexpansive if

\[
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K.
\]

We can now formulate one of the most important fixed point theorems for nonexpansive mappings, due to Browder, Ghode and Kirk, see for example [1]

**Theorem 2.1** Let \(K\) be a nonempty closed convex and bounded subset of a uniformly banach space \(E\). Then any nonexpansive mapping \(T : K \to K\) has at least a fixed point.

Let \(K\) be a convex subset of a normed linear space \(E\) and let \(T : K \to K\) be a self mapping. For \(x_0 \in K\) and \(\lambda \in [0, 1]\) the sequence \(x_n\) defined by

\[
x_{n+1} = (1 - \lambda) \cdot x_n + \lambda \cdot Tx_n, n = 0, 1, 2, ...
\]

is usually called Krasnoselskij iteration.

For \(x_0 \in K\) the sequence \(x_n\) defined by

\[
x_{n+1} = (1 - \lambda_n) \cdot x_n + \lambda_n \cdot Tx_n, n = 0, 1, 2, ...
\]

where \(\lambda_n \subset [0, 1]\) is a sequence of real number satisfying some appropriate condition, is called Mann iteration.

**Theorem 2.2** ([4]) Let \(K\) be a subset of a Banach space \(E\) and let \(T : K \to K\) be a nonexpansive mapping. For arbitrary \(x_0 \in K\), consider the Mann iteration process \(x_n\) under the following assumptions:

(a) \(x_n \in K\) for all positive integers;
(b) $0 \leq \lambda_n \leq b < 1$ for all positive integers;
(c) $\sum_{n=0}^{\infty} \lambda_n = \infty$. If $x_n$ is bounded, then $x_n - Tx_n \to 0$ as $n \to \infty$.

The following corollaries will be particularly important for application part of our paper.

**Corollary 2.1** ([5]) Let $K$ be a convex and compact subset of a Banach space $E$ and let $T : K \to K$ be a nonexpansive mapping. If the Mann iteration process $x_n$ satisfies assumptions (a)-(c) in Theorem 2.2, then $x_n$ converges strongly to a fixed point of $T$.

**Corollary 2.2** ([5]) Let $K$ be a closed bounded convex subset of a real normed space $E$ and $T : K \to K$ be a nonexpansive mapping. If $I - T$ maps closed bounded subset of $E$ into closed subset of $E$ and $x_n$ is the Mann iteration, with $\lambda_n$ satisfying assumptions (a)-(c) in Theorem 2.2, then $x_n$ converges strongly to a fixed point of $T$ in $K$.

## 3 Existence and uniqueness theorems for some integral equations

Consider the following integral equation:

$$y(x) = f(x) + \int_{x_0}^{t} K(x, y(y(x)))dx,$$

where $x_0, t \in [a, b]$, $y, f \in C[a, b]$, $K \in C([a, b] \times [a, b])$.

Denote $c_x = \max\{x - a, b - x\}$, $\forall x \in [a, b]$,

and

$$(*)C_L = \{y \in C([a, b] \times [a, b]) : |y(t_1) - y(t_2)| \leq L \cdot |t_1 - t_2|, \forall t_1, t_2 \in [a, b]; L > 0\}$$

**Theorem 3.1** Assume that the following condition are satisfied:

(i) $K \in C([a, b] \times [a, b])$;

(ii) $\exists L_1 > 0, L_2 > 0$ such that

$$|K(s, u) - K(s, v)| \leq L_1 \cdot |u - v|, \forall s, u, v \in [a, b],$$

and

$$|f(t_1) - f(t_2)| \leq L_2 \cdot |t_1 - t_2|, \forall t_1, t_2 \in [a, b].$$

(iii) If $L$ is the Lipschitz constant involved in (*), then

$$M = \max\{|K(s, u)| : s, u \in [a, b]\}, \text{ and } M + L_2 \leq L.$$
(iv) If \( \exists x_0 \in [a, b] \) such that \( |f(x)| \leq f(x_0) \) then, either:

(a) \( M \cdot c_{x_0} \leq c_{y_0}, \quad y_0 = f(x_0); \) or

(b) \( x_0 = a, \quad M \cdot (b - a) \leq b - y_0, \quad K(s, u) \geq 0, \forall s, u \in [a, b]; \) or

(c) \( x_0 = b, \quad M \cdot (b - a) \leq y_0 - a, \quad K(s, u) \geq 0, \forall s, u \in [a, b]; \)

(v) \( L_1 \cdot (L + 1) \cdot c_{x_0} \leq 1 \)

Then there exists at least solutions of (1) in \( C_L \) which can be approximate by the Krasnoselskij iteration

\[
y_{n+1} = (1 - \lambda) \cdot y_n + \lambda f(x) + \lambda \int_{x_0}^{t} K(x, y_n(y_n(x)))dx, \quad t \in [a, b], n \geq 1
\]

where \( \lambda \in (0, 1) \) and \( y_1 \in C_L \).

**Proof.** The technique of proof is basically the one introduce in [2]. Consider the integral operator \( A : C_L \rightarrow C[a, b] \)

\[
(Ay)(t) = f(t) + \int_{x_0}^{t} K(s, y(s))ds, \quad x_0 \in [a, b].
\]

It is clear that \( y \in C_L \) is a solution of (1) if only if \( y \) is a fixed point of \( A \), that is \( y = Ay \). We prove that \( C_L \) is an invariant set with respect to \( A \), i.e., we have \( A(C_L) \subset C_L \).

If first condition (a) holds, then for any \( y \in C_L \) and \( t \in [a, b] \) we have:

\[
|(Ay)(t)| = \left| f(t) + \int_{x_0}^{t} K(s, y(y(s)))ds \right| \leq |f(t)| + \int_{x_0}^{t} |K(s, y(y(s)))|ds
\]

\[
\leq f(x_0) + M \cdot |t - x_0| = f(x_0) + M \cdot c_{x_0} \leq f(x_0) + c_{y_0} \leq b
\]

\[
|(Ay)(t)| \geq |f(t)| - \int_{x_0}^{t} K(s, y(y(s)))ds \geq -f(x_0) - M \cdot |t - x_0|
\]

\[
\geq -f(x_0) - c_{y_0} \geq a
\]

So, \( \forall y \in C_L \) we obtain (\( Ay)(t) \) \( \in C[a, b] \). We show that \( Ay \in C_L, \forall y \in C_L \).

Now, for \( \forall t_1, t_2 \in [a, b] \) we have

\[
|(Ay)(t_1) - (Ay)(t_2)| \leq |f(t_1) - f(t_2)| + \int_{x_0}^{t_1} K(s, y(y(s)))ds - \int_{x_0}^{t_2} K(s, y(y(s)))ds
\]

\[
\leq |f(t_1) - f(t_2)| + \int_{t_1}^{t_2} K(s, y(y(s)))ds \leq L_2 \cdot |t_1 - t_2| + \int_{t_1}^{t_2} |K(s, y(y(s)))|ds
\]
Theorem 3.2
Assume that for second part of conclusion is given by Corollary 2.1 and Corollary 2.2.
Now, apply the Schauder’s fixed point theorem, a has at least a fixed point. The
which, in view of hypothesis (vi), proves that A is nonexpansive, hence continuous.

\[ |(Ay)(t) - (Ay)(z)| = \left| \int_{x_0}^{t} K(s, y(y(s))) ds - \int_{x_0}^{t} K(s, z(z(s))) ds \right| \leq \]

\[ \leq \int_{x_0}^{t} |K(s, y(y(s))) - K(s, z(z(s)))| ds \leq \int_{x_0}^{t} L_1 |y(y(s)) - z(z(s))| ds = \]

\[ = L_1 \cdot \int_{x_0}^{t} |y(y(s)) - y(z(s)) + y(z(s)) - z(z(s))| ds \leq \]

\[ L_1 \cdot \int_{x_0}^{t} (|y(y(s)) - y(z(s))| + |y(z(s)) - z(z(s))|) ds \]

\[ \leq L_1 \left[ L \int_{x_0}^{t} |y(s) - z(s)| ds + \max_{t \in [a, b]} |y(t) - z(t)| \cdot (t - x_0) \right] \]

Now, by applying the norm in last inequality, we get

\[ ||Ay - Az||_{C[a,b]} \leq L_1 \cdot (L + 1) \cdot c_{x_0} \cdot ||y - z||_{C[a,b]} \]

which, in view of hypothesis (vi), proves that A is nonexpansive, hence continuous.
Now, apply the Schauder’s fixed point theorem, a has at least a fixed point. The second part of conclusion is given by Corollary 2.1 and Corollary 2.2.

We consider the equation

\[ y(x) = f(x) + \int_{x_0}^{t} K(x, y(x), y(y(x))) dx, \]

where \( y, f \in C[a, b] \) and \( K \in C([a, b] \times [a, b] \times [a, b]) \) are given.
Denote \( c_x = \max\{x - a, b - x\}, \forall x \in [a, b] \) and

\[ (+)CL = \{ y \in C([a, b] \times [a, b]) : |y(t_1) - y(t_2)| \leq L \cdot |t_1 - t_2|, \forall t_1, t_2 \in [a, b] \}; L > 0 \]

**Theorem 3.2** Assume that for (2), the following conditions are satisfied:

(i) \( K \in C([a, b] \times [a, b] \times [a, b]) \);
(ii) \( \exists L_1 > 0, L_2 > 0 \) such that
\[
|K(s, u_1, u_2) - K(s, v_1, v_2)| \leq L_1 \cdot (|u_1 - v_1| + |v_2 - v_2|), \ \forall s, u_i, v_i \in [a, b], i = 1, 2
\]
and
\[
|f(t_1) - f(t_2)| \leq L_2 \cdot |t_1 - t_2|, \ \forall t_1, t_2 \in [a, b].
\]

(iii) If \( L \) is the Lipschitz constant involved in (*), then
\[
M = \max\{|K(s, u, v)| : s, u, v \in [a, b]\}, \text{ and } M + L_2 \leq L.
\]

(iv) If \( \exists x_0 \in [a, b] \) such that \( |f(x)| \leq f(x_0) \) then, either:
(a) \( M \cdot c_{x_0} \leq c_{g_0} \), \( y_0 = f(x_0) \); or
(b) \( x_0 = a, M \cdot (b - a) \leq b - y_0, K(s, u) \geq 0, \ \forall s, u \in [a, b] \); or
(c) \( x_0 = b, M \cdot (b - a) \leq y_0 - a, K(s, u) \geq 0, \ \forall s, u \in [a, b] \);

(v) \( L_1 \cdot (L + 2) \cdot c_{x_0} \leq 1. \)

Then there exists at least one solutions in \( C_L \) of (2).

**Proof.** It is know that \( C_L \) is a nonempty convex and compact subset of the Banach space \((C[a, b], \|\cdot\|)\), where \( \|\cdot\| \) is the usual sup norm. Consider the integral operator
\[
A : C_L \to C[a, b]
\]
\[
(Ay)(t) = f(t) + \int_{x_0}^{t} K(s, y(s), y(y(s))) ds, \ x_0, t \in [a, b], y \in C_L
\]

The solutions of equation (2) is the fixed points of integral operator \( A \).

We will show that \( A(C_L) \subset C_L \).

In the case (a), we have
\[
|(Ay)(t)| = \left| f(t) + \int_{x_0}^{t} K(s, y(s), y(y(s))) ds \right| \leq |f(t)| + \int_{x_0}^{t} |K(s, y(s), y(y(s)))| ds \leq
\]
\[
\leq |f(x_0)| + M \cdot |t - x_0| \leq f(x_0) + M \cdot c_{x_0} \leq f(x_0) + c_{g_0} \leq b
\]
\[
|(Ay)(t)| \geq |f(t)| - \int_{x_0}^{t} |K(s, y(s), y(y(s)))| ds \geq |f(t)| - M \cdot |t - x_0| \geq -f(x_0) - c_{g_0} \geq a
\]

So, \( Ay \in [a, b] \). Let us prove that \( Ay \in C_L, \forall y \in C_L \).

Let \( t_1, t_2 \in [a, b] \).

\[
|(Ay)(t_1) - (Ay)(t_2)| \leq |f(t_1) - f(t_2)| + \int_{t_1}^{t_2} |K(s, y(s), y(y(s)))| ds \leq L_2 \cdot |t_1 - t_2| +
\]
\[ +M \cdot |t_1 - t_2| \leq L \cdot |t_1 - t_2|. \]

So, \( Ay \in C_L, \forall y \in C_L. \)

Let \( y, z \in C[a, b] \)

\[
|(Ay)(t) - (Az)(t)| \leq \int_{x_0}^{t} |K(s, y(s), y(y(s))) - K(s, z(s), z(z(s)))| \, ds \leq L \cdot \int_{x_0}^{t} |y(s) - z(s)| + |y(y(s)) - z(z(s))| \, ds =
\]

\[
= L_1 \cdot \int_{x_0}^{t} (|y(s) - z(s)| + |y(y(s)) - y(z(s)) + y(z(s)) - z(z(s))|) \, ds \leq
\]

\[
L_1 \left[ \int_{x_0}^{t} |y(s) - z(s)| \, ds + \int_{x_0}^{t} (|y(y(s)) - y(z(s)) + y(z(s)) - z(z(s))|) \, ds \right]
\]

\[
\leq L_1 \left[ \int_{x_0}^{t} |y(s) - z(s)| + L \cdot \int_{x_0}^{t} |y(s) - z(s)| \, ds + \int_{x_0}^{t} \max_{t \in [a, b]} |y(t) - z(t)| \, ds \right].
\]

Now, by letting maximum in last inequality, we get

\[
\|Ay - Az\|_{C[a, b]} \leq L_1 \cdot (L + 2) \cdot c_{x_0} \cdot \|y - z\|_{C[a, b]}
\]

which, in view of condition (v), proves that \( A \) is nonexpansive, hence continuous. Applying the Schauder’s fixed point theorem we obtain the conclusion of the theorem.

**Particular case**

If \( f(x) \equiv y_0(\text{const.}) \), then by Theorem 3.1 we find the results in [2].

**References**


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