Regularity and Normality on L-Topological Spaces: (I)  

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Abstract

In the present paper, we define regularity and normality like strong $S_1$ regularity, $S_1$ regularity, weak $S_1$ regularity, strong $S_1$ normality, $S_1$ normality as well as weak $S_1$ normality on L-topological spaces. Also we investigate some of their properties and the relations between them.

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1 Introduction

The concept of fuzzy topology was first defined in 1968 by Chang [2] and redefined by Hutton and Reilly [4] and others. A new definition of fuzzy topology introduced by Badard [1] under the name of "smooth topology". The smooth topological space was rediscovered by Ramadan [5].

In the present work, it has been studied the concepts of separation axioms like strong $S_1$ regularity, $S_1$ regularity, weak $S_1$ regularity, strong $S_1$ normality, $S_1$ normality as well as weak $S_1$ normality on L-topological spaces. Also it has been investigated some of their properties and the relations between them.

2 Preliminaries

Throughout this paper, $L, L'$ represent two completely distributive lattice with the smallest element $0$ (or $\bot$) and the greatest element $1$ (or $\top$), where $0 \neq 1$. We define $M(L)$ to be the set of all non-zero $\lor$-irreducible (or coprime) elements in $L$ such
that \( a \in M(L) \) iff \( a \leq b \lor c \) implies \( a \leq b \) or \( a \leq c \). Let \( X \) be a non-empty usual set, and \( L^X \) be the set of all \( L \)-fuzzy sets on \( X \).

For each \( a \in L \), let \( \underline{a} \) denote the constant-valued \( L \)-fuzzy set with \( a \) as its value. Let \( \emptyset \) and \( 1 \) be the smallest element and the greatest element in \( L^X \), respectively. For the empty set \( \emptyset \subset L \), we define \( \land \emptyset = 1 \) and \( \lor \emptyset = 0 \).

For every \( L \)-fuzzy subset \( A \in L^X \), define its support set by \( \{ x \in X : A(x) > 0 \} \), denoted by \( \text{supp}(A) \).

**Definition 1** A \( L \)-fuzzy topology on \( X \) is a map \( \tau : L^X \to L \) satisfying the following three axioms:

1) \( \tau(\top) = \top \);
2) \( \tau(A \land B) \geq \tau(A) \land \tau(B) \) for every \( A, B \in L^X \);
3) \( \tau(\lor_{i \in \Delta} A_i) \geq \lor_{i \in \Delta} \tau(A_i) \) for every family \( \{ A_i | i \in \vartriangle \} \subseteq L^X \).

The pair \((X, \tau)\) is called an \( L \)-fuzzy topological space. For every \( A \in L^X \), \( \tau(A) \) is called the degree of openness of the fuzzy subset \( A \). For \( a \in L \) and a map \( \tau : L^X \to L \), we define
\[
\tau[a] = \{ A \in L^X | \tau(A) \geq a \}.
\]

**Definition 2** A smooth topological space (sts) [3] is an ordered pair \((X, \tau)\), where \( X \) is a non-empty set and \( \tau : L^X \to L' \) is a mapping satisfying the following properties:

\((O1)\) \( \tau(\emptyset) = \tau(1) = 1_L \),
\((O2)\) \( \forall A_1, A_2 \in L^X, \tau(A_1 \cap A_2) \geq \tau(A_1) \land \tau(A_2) \),
\((O3)\) \( \forall I, \tau(\bigcup_{i \in I} A_i) \geq \bigwedge_{i \in I} \tau(A_i) \).

**Definition 3** A smooth cotopology is defined as a mapping \( \Im : L^X \to L' \) which satisfies

\((C1)\) \( \Im(\emptyset) = \Im(1) = 1_L \),
\((C2)\) \( \forall B_1, B_2 \in L^X, \Im(B_1 \cup B_2) \geq \Im(B_1) \lor \Im(B_2) \),
\((C3)\) \( \forall I, \Im(\bigcap_{i \in I} B_i) \geq \bigwedge_{i \in I} \Im(B_i) \).

In this paper we suppose \( L' = L \).

The mapping \( \Im_t : L^X \to L' \), defined by \( \Im_t(A) = \tau(A^c) \) where \( \tau \) is a smooth topology on \( X \), is smooth cotopology on \( X \). Also \( \tau_\Im : L^X \to L' \), defined by \( \tau_\Im(A) = \Im(A^c) \) where \( \Im \) is a smooth cotopology on \( X \), is a smooth topology on \( X \) where \( A^c \) denotes the complement of \( A \) [3].

**Definition 4** Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be a mapping ; then [3], \( f \) is smooth continuous iff \( \Im_{\tau_2}(A) \leq \Im_{\tau_1}(f^{-1}(A)) \), \( \forall A \in L^Y \).
A map \( f : X \rightarrow Y \) is called smooth open (resp. closed) with respect to the smooth topologies \( \tau_1 \) and \( \tau_2 \) (resp. cotopologies \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \)), respectively, iff for each \( A \in L^X \) we have \( \tau_1(A) \leq \tau_2(f(A)) \) (resp. \( \mathcal{S}_1(A) \leq \mathcal{S}_2(f(A)) \)), where

\[
f(C)(y) = \sup \{ C(x) : x \in f^{-1}(\{y\}) \}, \text{ if } f^{-1}(\{y\}) \neq \emptyset,
\]
and \( f(C)(y) = 0 \) otherwise.

Let \( \tau : L^X \rightarrow L \) be a sts, and \( A \in L^X \), then the \( \tau \)-smooth closure of \( A \), denoted by \( \overline{A} \), is defined by

\[
\overline{A} = A, \text{ if } \mathcal{S}_\tau(A) = 1_L, \text{ and } \overline{A} = \bigcap \{ F : F \in L^X, F \supseteq A, \mathcal{S}_\tau(F) > \mathcal{S}_\tau(A) \}, \text{ if } \mathcal{S}_\tau(A) \neq 1_L.
\]

A map \( f : X \rightarrow Y \) is called a smooth homeomorphism with respect to the smooth topologies \( \tau_1 \) and \( \tau_2 \) iff \( f \) is bijective and \( f \) and \( f^{-1} \) are smooth continuous.

Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be two smooth topological spaces and \( f : X \rightarrow Y \) a bijective map. The following statements are equivalent [3]:
1. \( f \) is a smooth homeomorphism,
2. \( f \) is a smooth open and smooth continuous,
3. \( f \) is a smooth closed and smooth continuous.
A property which is preserved under smooth homeomorphism is said to be a smooth topological property.

A map \( f : X \rightarrow Y \) is called L-preserving (resp. strictly L-preserving) with respect to the L-topologies \( \tau_{1[a]} \) and \( \tau_{2[a]} \), for each \( a \in M(L) \) respectively, iff for every \( A, B \in L^Y \) with \( \tau_2(A), \tau_2(B) \geq a \), we have

\[
\tau_2(A) \geq \tau_2(B) \Rightarrow \tau_1(f^{-1}(A)) \geq \tau_1(f^{-1}(B))
\]
(resp. \( \tau_2(A) > \tau_2(B) \Rightarrow \tau_1(f^{-1}(A)) > \tau_1(f^{-1}(B)) \) [3].

Let \( f : X \rightarrow Y \) be a strictly L-preserving and continuous map with respect to the L-topologies \( \tau_{1[a]} \) and \( \tau_{2[a]} \), respectively, then for every \( A \in L^Y \) with \( \tau_2(A) \geq a \), \( f^{-1}(A) \supseteq \overline{f^{-1}(A)} \).

### 3 Main results

**Definition 7** A L-topology space \((X, \tau_{[a]})\) for each \( a \in M(L) \) is called
(a) strong \( s_1 \) regular (resp. strong \( S_2 \) regular) space iff for each \( C \in L^X \), satisfying \( \mathcal{S}_\tau(C) > 0 \), and each \( x \in X \) satisfying \( x \not\in \text{supp}C \), there exist \( A, B \in L^X \) with \( \tau(A), \tau(B) \geq a \) such that \( x \in \text{supp}A \) (resp. \( x \in \text{supp}(A \setminus B) \), \( \tau(A) \geq A(x) \), \( C \subseteq B, \tau(B) \geq \mathcal{S}_\tau(C) \) and \( A \cap B = \emptyset \) (resp. \( A \subseteq (B)^C \)),
(b) \( s_1 \) regular (resp. \( S_2 \) regular) space iff for each \( C \in L^X \), satisfying \( \mathcal{S}_\tau(C) > 0 \),
and each $x \in X$ satisfying $x \notin \text{supp} C$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $x \in \text{supp} A$ (resp. $x \in \text{supp} (A \setminus B), \tau(A) \geq A(x), C \subseteq B, \tau(B) \geq \exists_\tau(C)$ and $A \cap B = \emptyset$ (resp. $A \subseteq B^c$).

(c) weak $S_1$ regular (resp. weak $S_2$ regular) space iff for each $C \in L^X$, satisfying $\exists_\tau(C) > 0$, and each $x \in X$ satisfying $x \notin \text{supp} C$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $x \in \text{supp} A \setminus \text{supp} B^o$ (resp. $x \in \text{supp} (A \setminus B^o), \tau(A) \geq A(x), C \subseteq B, \tau(B) \geq \exists_\tau(C)$ and $A^o \cap B^o = \emptyset$ (resp. $A^o \subseteq (B^o)^c$).

**Definition 8** An $L$-topology space $(X, \tau_a)$ for each $a \in M(L)$ is called
(a) strong $S_1$ normal (resp. strong $S_2$ normal) space iff for each $C, D \in L^X$ such that $C \subseteq (D^c)$ (resp. $C \cap D = \emptyset), \exists_\tau(C) > 0$ and $\exists_\tau(D) > 0$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $C \subseteq A, \tau(A) \geq \exists_\tau(C), D \subseteq B, \tau(B) \geq \exists_\tau(D)$ and $A \cap B = \emptyset$ (resp. $A \subseteq (B^c)^c$),
(b) $S_1$ normal (resp. $S_2$ normal) space iff for each $C, D \in L^X$ such that $C \subseteq (D^c)$ (resp. $C \cap D = \emptyset), \exists_\tau(C) > 0$ and $\exists_\tau(D) > 0$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $C \subseteq A, \tau(A) \geq \exists_\tau(C), D \subseteq B, \tau(B) \geq \exists_\tau(D)$ and $A \cap B = \emptyset$ (resp. $A \subseteq (B^c)^c$),
(c) weak $S_1$ normal (resp. weak $S_2$ normal) space iff for each $C, D \in L^X$ such that $C \subseteq (D^c)$ (resp. $C \cap D = \emptyset), \exists_\tau(C) > 0$ and $\exists_\tau(D) > 0$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $C \subseteq A, \tau(A) \geq \exists_\tau(C), D \subseteq B, \tau(B) \geq \exists_\tau(D)$ and $A^o \cap B^o = \emptyset$ (resp. $A^o \subseteq (B^o)^c$).

**Lemma 1** Let $(X, \tau_a)$ be an $L$-topology space for each $a \in M(L), A, B \in L^X$ and $\tau(A), \tau(B) \geq a$. Then the following properties hold:
(i) $\text{supp} A \setminus \text{supp} B \subseteq \text{supp} (A \setminus B)$,
(ii) $\text{supp} A \setminus \text{supp} B \subseteq \text{supp} A \setminus \text{supp} B^o$,
(iii) $A \setminus B \subseteq A \setminus B^o$,
(iv) $A \cap B = \emptyset$ implies $A \subseteq B^c$.

**Proof.** (i) Consider $x \in \text{supp} A \setminus \text{supp} B$. Then we obtain $A(x) > 0$ and $B(x) = 0$. Hence, $\min(A(x), 1 - B(x)) = A(x) > 0$, i.e., $x \in \text{supp} (A \setminus B)$. The reverse inclusion in (i) is not true as can be seen from the following counterexample. Let $X = \{x_1, x_2\}, A(x_1) = 0.5, B(x_1) = 0.3$. Then we have $x_1 \in \text{supp} (A \setminus B)$ and $x_1 \notin \text{supp} A \setminus \text{supp} B$.
(ii) and (iii) easily follow from $B^o \subseteq B \subseteq \overline{B}$.
(iv) See [4].

**Proposition 1** Let $(X, \tau_a)$ be an $L$-topology space for each $a \in M(L)$. Then the relationships as shown in Fig. 2. hold.

**Proof.** All the implications in Fig. 2 are straightforward consequences of Lemma 1. As an example we prove that strong $S_1$ normal implies strong $S_2$ normal. Suppose that $(X, \tau_a)$ is a strong $S_1$ normal space and let $C, D \in L^X$ such that $C \cap D = \emptyset, \exists_\tau(C) > 0$ and $\exists_\tau(D) > 0$. 

\[\Box\]
From Lemma 1 (iv) it follows that $C \subseteq D^c$. Since $(X, \tau_{[a]})$ is strong $S_1$ normal there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $C \subseteq A, \tau(A) \geq \tau(C), D \subseteq B, \tau(B) \geq \tau(D)$ and $\overline{A} \cap \overline{B} = \emptyset$. From Lemma 1 it follows that $\overline{A} \subseteq (B)^c$ and hence $(X, \tau_{[a]})$ is strong $S_2$ normal.

strong $S_1$ regular $\Rightarrow$ $S_1$ regular $\Rightarrow$ weak $S_1$ regular

\[ \downarrow \quad \downarrow \quad \downarrow \]

strong $ST_2$ regular $\Rightarrow$ $ST_2$ regular $\Rightarrow$ weak $ST_2$ regular

strong $S_1$ normal $\Rightarrow$ $S_1$ normal $\Rightarrow$ weak $S_1$ normal

\[ \downarrow \quad \downarrow \quad \downarrow \]

strong $S_2$ normal $\Rightarrow$ $S_2$ normal $\Rightarrow$ weak $S_2$ normal

Fig. 2. Relationship between the different regularity and normality notions.

Proposition 2 The $S_i$ ($i = 1, 2$) regularity (resp. normality) property is a topological property. when $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be an smooth homeomorphism or $f : (X, \tau_{1[a]}) \rightarrow (Y, \tau_{2[a]})$ be an homeomorphism for each $a \in M(L)$.

Proof. As an example we give the proof for $S_2$ normality when $f : X \rightarrow Y$ be a homeomorphism from $S_2$ normal space $(X, \tau_{1[a]})$ onto a space $(Y, \tau_{2[a]})$ for each $a \in M(L)$. Let $C, D \in L^Y$ such that $C \cap D = \emptyset, \exists \tau_2(C) > 0$ and $\exists \tau_2(D) > 0$. Since $f$ is bijective and continuous, from $C' \in \tau_{2[a]}$ we have $f^{-1}(C') \in \tau_{1[a]}$. From here, $\tau_2(C') \geq a$ then $\tau_1(f^{-1}(C')) \geq a$. It follows that $\tau_1(f^{-1}(C')) \geq \tau_2(C')$, hence $\tau_1((f^{-1}(C'))') \geq \tau_2(C') > 0$.

Now we obtain that $\exists \tau_2(f^{-1}(C)) \geq \exists \tau_2(C) > 0$. Similarly, $\exists \tau_1(f^{-1}(D)) \geq \exists \tau_2(D) > 0$. We know that $f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D) = f^{-1}(\emptyset) = \emptyset$. Since $(X, \tau_{1[a]})$ is $S_2$ normal, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $f^{-1}(C) \subseteq A, \tau_1(A) \geq \tau_1(f^{-1}(C)), f^{-1}(D) \subseteq B, \tau_1(B) \geq \tau_1(f^{-1}(D))$ and $A \subseteq B^c$. Since $f$ is L-open and L-closed, it follows that $\tau_2(f(A)) \geq \tau_1(A), \tau_2(f(B)) \geq \tau_1(B), \exists \tau_2(C) \geq \exists \tau_1(f^{-1}(C))$ and $\exists \tau_2(D) \geq \exists \tau_1(f^{-1}(D))$, and hence, $\tau_2(f(A)) \geq \exists \tau_1(f^{-1}(C)) = \exists \tau_2(C), \tau_2(f(B)) \geq \exists \tau_1(f^{-1}(D)) = \exists \tau_2(D), C \subseteq f(A), D \subseteq f(B)$ and $f(A) \subseteq f(B^c) = (f(B))^c$. So $(Y, \tau_{2[a]})$ is $S_2$ normal.

Proposition 3 Let $f : X \rightarrow Y$ be an injective, L-closed, L-continuous map with respect to the L-topologies $\tau_{1[a]}$ and $\tau_{2[a]}$ respectively for each $a \in M(L)$. If $(Y, \tau_{2[a]})$ is $S_i$ ($i = 1, 2$) regular (resp. normality); then so is $(X, \tau_{1[a]})$. 
Proof. As an example we give the proof for $S_1$ regularity. Let $C \in L^X$, satisfy $\exists \tau_1(C) > 0$ and let $x \in X$ be such that $x \notin suppC$. Since $f$ is injective and L-closed we have $f(x) \notin supp(C)$ and $\exists \tau_2(f(C)) \geq \exists \tau_1(C) > 0$. Since $(Y, \tau_2)$ is $S_1$ regular, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $f(x) \in suppA, \tau_2(A) \geq A(f(x)), f(C) \subseteq B, \tau_2(B) \geq \exists \tau_2(f(C))$ and $A \cap B = 0$. Since $f$ is injective and L-continuous, if $A \in \tau_{[a]}$ then $f^{-1}(A) \in \tau_{[a]}$. Hence when $\tau_2(A) \geq a$ then $\tau_1(f^{-1}(A)) \geq a$. Thus $\tau_1(f^{-1}(A)) \geq \tau_2(A) \geq A(f(x)) = f^{-1}(A)(x)$. Similarly, $\tau_1(f^{-1}(B)) \geq \tau_2(B) \geq \exists \tau_1(C)$. We know that $C \subseteq (f^{-1}(B)), f^{-1}(A)(x) = A(f(x)) > 0$, i.e., $x \in supp f^{-1}(A)$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(0) = 0$ and hence $(X, \tau_{[a]})$ is $S_1$ regular.

Proposition 4 Let $f : X \rightarrow Y$ be a strictly L-preserving, injective, L-closed and L-continuous map with respect to the L-topologies $\tau_{[a]}$ and $\tau_{[a]}$ respectively for each $a \in M(L)$. If $(Y, \tau_{[a]})$ is strong $S_i$ ($i = 1, 2$) regular (resp. normal); then so is $(X, \tau_{[a]})$.

Proof. As an example we proof the strong $S_2$ regularity. Let $C \in L^X$, satisfying $\exists \tau_1(C) > 0$ and let $x \in X$ such that $x \notin suppC$. Since $f$ is injective and L-closed we have $f(x) \notin supp(C)$ and $\exists \tau_2(f(C)) \geq \exists \tau_1(C) > 0$. Since $(Y, \tau_{[a]})$ is $S_2$ regular, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $f(x) \in supp(A \setminus B), \tau_2(A) \geq A(f(x)), f(C) \subseteq B, \tau_2(B) \geq \exists \tau_2(f(C))$ and $A \subseteq (B)^c$. As $f$ is injective, L-continuous and strictly L-preserving it follows that $\tau_1(f^{-1}(A)) \geq \tau_2(A) \geq A(f(x)) = f^{-1}(A)(x), \tau_1(f^{-1}(B)) \geq \exists \tau_1(C), C \subseteq (f^{-1}(B)), [f^{-1}(A) \setminus f^{-1}(B)](x) = (f^{-1}(A) \cap (f^{-1}(B))^c)(x) \geq (A \cap f^{-1}(B)^c)(x) = f^{-1}(A \setminus B)(x) = (A \setminus B)f(x) > 0$, i.e., $x \in supp(f^{-1}(A) \setminus f^{-1}(B))$ and $f^{-1}(A) \subseteq f^{-1}(B)^c \subseteq (f^{-1}(B))^c$, and hence $(X, \tau_{[a]})$ is strong $S_2$ regular.

References


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