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On the equation $x^n + y^n = z^n$ [1]

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Abstract

We prove the non existence of non zero integral solution to the equation $x^n + y^n = z^n$ for few cases by categorizing the triplet (x, y, z).

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1 Introduction

Fermat's last theorem states that the equation:

$$x^n + y^n = z^n \tag{1}$$

(where n is a positive integer) has no non-zero integral solution x, y, z when n exceeds 2.

This theorem was coined by mathematician Fermat, he himself has not given any formal proof for this theorem, but he proved this result for the case $n = 4$ using the method of infinite descendant. Using a similar method, Euler proved the theorem for $n = 3$ (see [1]). Like wise many mathematicians have proved particular cases of this theorem (for recent one see [4]). However no correct proof was found for 357 years when Andrew wiles finally published a proof using very deep methods in 1995.(see [2], [3])

In this note we prove few cases of Fermat's last theorem by categorising the triplet (x, y, z) involved in equation(1)

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1.1 Alternate form of Fermat's last theorem

Equation (1) is equivalent to the following set of equations:

$$(4x+1)^n + (4y+1)^n = (2z)^n \tag{2}$$

$$(4x+3)^n + (4y+3)^n = (2z)^n \tag{3}$$

$$(4x+1)^n + (4y+3)^n = (2z)^n \tag{4}$$

$$(4x+1)^n + (2y)^n = (4z+1)^n \tag{5}$$

$$(4x+1)^n + (2y)^n = (4z+3)^n \tag{6}$$

$$(4x+3)^n + (2y)^n = (4z+1)^n \tag{7}$$

$$(4x+1)^n + (2y)^n = (4z+3)^n \tag{8}$$

where x, y, z are integer variables and n is a positive integer. Therefore proving Fermat's last theorem is equivalent to proving the non existence of non zero integral solution of the equations (2) to (8) when n exceeds 2.

2 Main results

2.1 Lemmas

Lemma 1 $\frac{1+3^{2k}}{2}$ *is always an odd integer.*

Proof. First we prove the relation $3^{2k} \equiv 1 \ mod\ 4$. This relation is true when $k = 1$. Assume that the relation is true for $k = 1, 2, ..., r$. Consider $3^{2(r+1)} = 3^2 3^{2r} \equiv 1 \ mod\ 4$. Hence the relation is true when $k = r + 1$, so by induction priciple this relation is true for any positive integer k. Now consider $3^{2k} + 1 = (3^{2k} - 1) + 2 = 4m + 2 = 2(2m + 1)$ for some positive integer m. This establishes the lemma.

Lemma 2 $\frac{1+3^{2k-1}}{4}$ *is always an odd integer.*

Proof. Proof is immediate from the expression

$$3^{2k-1} + 1 = (3 + 1)(1 - 3 + 3^2 - \cdots + 3^{2k-2})$$

Lemma 3 $\frac{3^{2k-1}-1}{2}$ *is always an odd integer.*

Proof. Proof is immediate from the expression

$$3^{2k-1} - 1 = (3 - 1)(1 + 3 + 3^2 + \cdots + 3^{2k-2})$$

2.2 Theorems

Theorem 1 *Equation(2) and (3) has no integer solution if $n \geq 2$*

Proof. Consider the following binomial expansion

$$(4n+1)^k + (4m+1)^k = \sum_{i=0}^{k-1} \binom{k}{i} \left((4n)^{k-i} + (4m)^{k-i}\right) + 2$$

which equals 2 times an odd integer for any integers m, n and positive integer k.This gives us the inferration that: 2 divides $(4n+1)^k + (4m+1)^k$, but no other higher powers of 2 divides $(4n+1)^k + (4m+1)^k$, in similar way we can show that 2 divides $(4n+3)^k + (4m+3)^k$, but no other higher powers of 2 divides $(4n+3)^k + (4m+3)^k$, this proves the theorem.

Theorem 2 *Equation(4) has no integer solution if n is even, and in case when $n \geq 3$ is odd it has no integer solution if the variables x, y is of the form $x = 2x',\ y = 2y'$ or $x = 2x' - 1,\ y = 2y' - 1$*

Proof. Consider the following binomial expansion

$$(4n+1)^k + (4m+3)^k = \sum_{i=0}^{k-1} \binom{k}{i} \left((4n)^{k-i} + (4m)^{k-i}3^i\right) + (1+3^k)$$

$$= 2(\textit{an odd integer})$$

for any integer m, n when k is even (by lemma1). From this we conclude that 2 divides $(4n+1)^k + (4m+3)^k$ and no other higher powers of 2 divides $(4n+1)^k + (4m+3)^k$. This proves the first part of the theorem. If n and m belongs to the same parity and $k \geq 3$ is odd, then from the above expansion we conclude that 2^2 divides $(4n+1)^k + (4m+3)^k$ (by lemma 2) and no other higher powers of 2 divides $(4n+1)^k + (4m+3)^k$. This proves the second part of the theorem.

Theorem 3 *Equation (5) and (8) has no integer solution if the following conditions are satisfied (i) $n \geq 3$ is odd and (ii) if the variables x, z is of the form $x = 2x' - 1, z = 2z'$ or $x = 2x', z = 2z' - 1$*

Proof. Consider the following binomial expansion,

$$(4n+1)^k - (4m+1)^k = \sum_{i=0}^{k-1} \binom{k}{i} \left((4n)^{k-i} - (4m)^{k-i}\right)$$

From this expansion, we conclude that: if n and m belongs to different parity and $k \geq 3$ is odd then 2^2 divides $(4n+1)^k - (4m+1)^k$ and no other higher powers of 2 divides $(4n+1)^k - (4m+1)^k$, this proves the first part of the theorem, analogous proof goes to the second part of the theorem.

Theorem 4 *Equation (6) and (7) has no integer solution if $n \geq 3$ is odd.*

Proof. Consider the following binomial expansion,

$$(4n+3)^k - (4m+1)^k = \sum_{i=0}^{k-1} \binom{k}{i} \left((4n)^{k-i} 3^i - (4m)^{k-i}\right) + (3^k - 1)$$

From this expansion we conclude that: if $k \geq 3$ is odd then 2 divides $(4n+3)^k - (4m+1)^k$ and no other higher powers of 2 divides $(4n+3)^k - (4m+1)^k$ (by lemma 3).Thus we got the first part of the theorem. An analogous proof goes to the second part of the theorem.

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