A Generalized Integral Operator Associated with Functions of Bounded Boundary Rotation

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Abstract

In this paper, we define the subclass $V^{\lambda}_{k} (\beta, \delta, n)$ of analytic functions by using the generalized Al-Oboudi differential operator. We determine certain properties of the integral operator $I_{n}(f_{1}, \cdots, f_{m})$ for the functions belonging to the class $V^{\lambda}_{k} (\beta, \delta, n)$.

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1 Introduction

Let $A$ denote the class of all analytic functions of the form

\begin{equation}
    f(z) = z + \sum_{j=2}^{\infty} a_{j}z^{j}
\end{equation}

defined in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $S$ be the subclass of $A$ containing univalent functions defined in $U$. Let $P^{\lambda}_{k}(\beta)$ denote the class of analytic functions $p(z)$ defined in $U$ satisfying the following properties

i. $p(0) = 1$.

ii. $\int_{0}^{2\pi} \left| \Re^{i\lambda} p(z) - \beta \cos \lambda \right| \frac{d\theta}{1 - \beta} \leq k \pi \cos \lambda$

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where \( k \geq 2 \), \( \lambda \) is real, \(|\lambda| < \frac{\pi}{2}\), \( 0 \leq \beta < 1 \), \( z = re^{i\theta}, \ 0 \leq r < 1 \).

Let \( V_k^\lambda(\beta) \ [7] \) denote the class of functions \( f(z) \) analytic in \( \mathcal{U} \) satisfying the normalization conditions \( f(0) = f'(0) - 1 = 0 \) and

\[
1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_k^\lambda(\beta)
\]

where \( k, \lambda \) and \( \beta \) are as above.

For \( \beta = 0 \) we get the class \( V_k^\lambda \) of functions with bounded boundary rotation studied by Moulis \[6\].

Any function \( f(z) \in \mathcal{V}_k^\lambda(\beta) \) if and only if

\[
\Re \left\{ e^{i\lambda} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \beta \cos \lambda, \quad |z| < \frac{k - \sqrt{k^2 - 4}}{2}.
\]

A function \( f \in \mathcal{U} \) with the normalization properties \( f(0) = f'(0) - 1 = 0 \) is said to be in the class \( \mathcal{U}_k^\lambda(\beta) \) if \( \frac{zf'(z)}{f(z)} \in \mathcal{P}_k^\lambda(\beta) \).

For \( f \in \mathcal{A}, \mathcal{S} \), Sălăgean \[10\] introduced the differential operator

\[
D^n : \mathcal{A} \longrightarrow \mathcal{A}, \quad n \in \mathbb{N}
\]

defined as

\[
D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z)
\]

\[
D^n f(z) = D(D^{n-1} f(z)).
\]

Al-Oboudi \[2\] generalized this operator by considering \( D_\delta^n : \mathcal{A} \longrightarrow \mathcal{A}, \ n \in \mathbb{N}, \ \delta > 0 \) defined by

\[
D_\delta^0 f(z) = f(z)
\]

\[
D_\delta^1 = (1 - \delta)f(z) + \delta zf'(z) = D_\delta f(z)
\]

\[
D_\delta^n = D(D_\delta^{n-1} f(z)).
\]

From the above definition, if \( f \) is of the form (1), we have

\[
D_\delta^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j - 1)\delta]a_j z^j, \quad n \in \mathbb{N}_0,
\]

with \( D_\delta^0 f(0) = 0 \).

Let \( \mathcal{V}_k^\lambda(\delta, n) \) denote the class of functions \( f(z) \) analytic in \( \mathcal{U} \) with the normalization properties \( f(0) = f'(0) - 1 = 0 \) and

\[
\frac{z(D_\delta^n f(z))'}{D_\delta^n f(z)} \in \mathcal{P}_k^\lambda(\beta)
\]

where \( k \geq 2 \), \( \lambda \) is real, \(|\lambda| < \frac{\pi}{2}\), \( 0 \leq \beta < 1 \), \( z = re^{i\theta}, \ 0 \leq r < 1 \).

For \( \delta = 1, \ n = 1 \), we get the class \( \mathcal{V}_k^\lambda(\beta) \) studied by Moulis \[7\].
If $\delta = 1$, $n = 0$, we get the class $U^\lambda_\delta$ studied by Moulis [6].

Any function $f(z) \in V^\lambda_k(\beta, \delta, n)$ if and only if

$$\Re \left\{ e^{i \lambda} \left( 1 + \frac{z(D_\delta^n f(z))'}{D_\delta^n(f(z))} \right) \right\} > \beta \cos \lambda, \quad |z| < \frac{k - \sqrt{k^2 - 4}}{2}.$$ 

Let $n, m \in \mathbb{N}_0$ and $\alpha_i > 0$, $1 \leq i \leq m$. We define the integral operator $I_n : A^n \rightarrow A$

$$I_n(f_1, \cdots, f_m)(z) = \int_0^z \left( \frac{D_\delta^n f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{D_\delta^n f_m(t)}{t} \right)^{\alpha_m} dt, \quad z \in U,$$

where $f_i \in A$ and $D_\delta^n$ is the Al-Oboudi differential operator.

For parametric values of $n = 0$, $\delta = 1$ we have the integral operator

$$I_0(f_1, \cdots, f_m)(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_m(t)}{t} \right)^{\alpha_m} dt,$$

introduced in [4].

If $n = 0$, $\delta = 1$, $m = 1$, $\alpha_1 = \cdots = \alpha_m = 0$ and $D_0^0 f_1(z) = D_0^0 f(z) = f(z) \in A$, we have the integral operator of Alexander

$$I_0(f)(z) = \int_0^z \frac{f(t)}{t} dt \text{ introduced in } [1].$$

For $n = 0$, $\delta = 1$, $m = 1$, $\alpha_1 = \alpha \in [0, 1]$, $\alpha_2 = \cdots = \alpha_m = 0$ and $D_0^0 f_1(z) = D_0^0 f(z) = f(z) \in S$, we have the integral operator

$$I(f)(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha dt \text{ studied in } [9].$$

If $\alpha_i \in \mathbb{C}$ for $1 \leq i \leq m$, then we have the integral operator $I_n(f_1, \cdots, f_m)$ studied in [8].

2 Main Result

**Theorem 1** Let $f_i \in V^\lambda_k(\beta_i, \delta, n)$ for $1 \leq i \leq m$ with $0 \leq \beta_i < 1$, and

$n \in \mathbb{N}_0$, also let $\alpha_i > 0$, $1 \leq i \leq m$. If $\sum_{i=1}^{m} \alpha_i(1 - \beta_i) \leq 1$, then

$I_n(f_1, \cdots, f_m) \in V^\lambda_k(\gamma)$, with $\gamma = 1 + \sum_{i=1}^{m} \alpha_i(\beta_i - 1)$.

**Proof:** From (2), for $1 \leq i \leq m$, we have

$$\frac{D_\delta^n f_i(z)}{z} = 1 + \sum_{j=2}^{\infty} [1 + (j - 1)\delta]^n a_j z^{j-1}, \quad n \in \mathbb{N}_0$$
and
\[ \frac{D^n_{\delta} f_i(z)}{z} \neq 0, \quad \forall z \in U. \]

Consider,
\[ I_n(f_1, \ldots, f_m)(z) = \int_0^z \left( \frac{D^n_{\delta} f_1(t)}{t} \right)^{\alpha_1} \ldots \left( \frac{D^n_{\delta} f_m(t)}{t} \right)^{\alpha_m} dt. \]

On successive differentiation of \( I_n(f_1, \ldots, f_m) \), we get
\[ I_n(f_1, \ldots, f_m)'(z) = \left( \frac{D^n_{\delta} f_1(z)}{z} \right)^{\alpha_1} \ldots \left( \frac{D^n_{\delta} f_m(z)}{z} \right)^{\alpha_m} \]
\[ I_n(f_1, \ldots, f_m)''(z) = \sum_{i=1}^m \alpha_i \left( \frac{D^n_{\delta} f_i(z)}{z} \right)^{\alpha_i-1} \frac{z(D^n_{\delta} f_i(z))' - D^n_{\delta} f_i(z)}{z^2} \prod_{j=1, j \neq i}^m \left( \frac{D^n_{\delta} f_j(z)}{z} \right)^{\alpha_j} \]

Thus we obtain,
\[ \frac{zI_n(f_1, \ldots, f_m)''(z)}{I_n(f_1, \ldots, f_m)'(z)} + 1 = \sum_{i=1}^m \alpha_i \left[ \frac{z(D^n_{\delta} f_i(z))'}{D^n_{\delta} f_i(z)} - \frac{1}{z} \right]. \]

This relation is equivalent to
\[ \Re \left\{ e^{i\lambda} \left( \frac{zI_n(f_1, \ldots, f_m)''(z)}{I_n(f_1, \ldots, f_m)'(z)} + 1 \right) \right\} = \sum_{i=1}^m \Re e^{i\lambda} \left\{ \alpha_i \frac{z(D^n_{\delta} f_i(z))'}{D^n_{\delta} f_i(z)} - \sum_{i=1}^m \alpha_i \right\} + 1. \]

Since \( f_i \in \mathcal{V}_{\alpha}^\lambda(\beta_i, \delta, n) \), we get
\[ \Re \left\{ e^{i\lambda} \left( \frac{zI_n(f_1, \ldots, f_m)''(z)}{I_n(f_1, \ldots, f_m)'(z)} + 1 \right) \right\} > \sum_{i=1}^m \Re e^{i\lambda} \alpha_i - \sum_{i=1}^m \alpha_i + 1 \]
\[ + \sum_{i=1}^m \alpha_i (\beta_i - 1). \]

Hence \( I_n(f_1, \ldots, f_m)(z) \in \mathcal{V}_{\alpha}^\lambda(\gamma) \), where \( \gamma = 1 + \sum_{i=1}^m \alpha_i (\beta_i - 1) \).
Corollary 1 For parametric values \( n = 0, \delta = 1, k = 2, \lambda = 0 \), we get the following result [3].

Let \( \alpha_i, i \in \{1, 2, ..., n\} \) be real numbers with the properties \( \alpha_i > 0 \) for \( i \in \{1, 2, ..., n\} \)
and \( \sum_{i=1}^{n} \alpha_i \leq n + 1 \). We suppose that the functions \( f_i \),
i \( i \in \{1, 2, ..., n\} \) are the starlike functions of order \( \frac{1}{\alpha_i} \), i.e., \( f_i \in \mathcal{S}^* \left( \frac{1}{\alpha_i} \right) \) for all
i \( i \in \{1, 2, ..., n\} \). Then the integral operator defined in (3) is convex.

For \( \beta_1 = \beta_2 = \cdots = \beta_m = \beta, \delta = 1, \) and \( n = 0 \), similarly we prove the following theorem.

Theorem 2 Let \( \alpha_i \) be real numbers with the properties \( \alpha_i > 0 \) for
i \( i \in \{1, 2, ..., n\} \) with \( \sum_{i=1}^{m} \alpha_i \leq 1 \). We suppose that the function
\( f_i \in \mathcal{V}_k^\lambda (\beta, 1, 0) \). Then the integral operator defined in (3) belongs to \( \mathcal{V}_k^\lambda (\gamma) \), where
\( \gamma = 1 - \sum_{i=1}^{m} \alpha_i \).

Corollary 2 For parametric values \( n = 0, \delta = 1, k = 2, \lambda = 0 \), we get the following result [3].

Let \( \alpha_i, i \in \{1, 2, ..., n\} \) be real numbers with the properties \( \alpha_i > 0 \) for \( i \in \{1, 2, ..., n\} \)
and \( \sum_{i=1}^{n} \alpha_i \leq 1 \). We suppose that the functions \( f_i \),
i \( i \in \{1, 2, ..., n\} \) are the starlike functions. Then the integral operator defined in (3) is convex by order \( 1 - \sum_{i=1}^{n} \alpha_i \).

References


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