A Note on a Generalized Integral Operator

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Abstract
In this note, we introduce the new subclass \( SP_\lambda^k(\beta) \) of analytic functions and certain properties of a generalized integral operator are studied.

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1 Introduction
Let \( P_\lambda^k(\beta) \) denote the class of analytic functions \( p(z) \) defined in the unit disc \( U = \{ z : |z| < 1 \} \) with the following properties:

i. \( p(0) = 1 \)

ii. \( \int_0^{2\pi} \left| \Re \{ e^{i\lambda} p(z) - \beta \cos \lambda \} \right| \frac{1}{1 - \beta} d\theta \leq k\pi \cos \lambda \)

where, \( k \geq 2, \lambda \) real, \( |\lambda| \leq \frac{\pi}{2} \), \( 0 \leq \beta < 1 \) and \( z = re^{i\theta} \) for \( 0 \leq r < 1 \).

Let \( V_\lambda^k(\beta) \) \[5\] denote the class of analytic functions \( f \) defined in \( U \) satisfying the normalization properties \( f(0) = f'(0) - 1 = 0 \) and

\[ 1 + \frac{zf''(z)}{f'(z)} \in P_\lambda^k(\beta), \quad (z \in U) \]

where, \( k, \lambda \) and \( \beta \) are as above. For \( \beta = 0 \) we get the class \( V_\lambda^k \) of functions with bounded boundary rotation studied by Moulis \[4\]. Any function \( f(z) \in V_\lambda^k(\beta) \) if and only if

\[ \Re \left\{ e^{i\lambda} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \beta \cos \lambda, \quad \text{for} \quad |z| < \frac{k - \sqrt{k^2 - 4}}{2}. \]

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A function $f$ defined in $\mathcal{U}$ satisfying the normalization properties $f(0) = f'(0) - 1 = 0$ is said to be in the class $\mathcal{U}^\lambda_k(\beta)$ if

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}^\lambda_k(\beta), \quad (z \in \mathcal{U}).$$

From the definition of the above classes it follows that $f(z) \in \mathcal{V}^\lambda_k(\beta)$ if and only if $zf'(z) \in \mathcal{U}^\lambda_k(\beta)$.

Let $\mathcal{S}\mathcal{P}^\lambda_k(\beta)$ be the class of normalized functions $f$ such that

$$(1) \quad \frac{zf'(z)}{f(z)} - \left| \frac{zf'(z)}{f(z)} - 1 \right| \in \mathcal{P}^\lambda_k(\beta), \quad (z \in \mathcal{U})$$

where, $\beta$ is a real number with $0 \leq \beta < 1$. Now we consider the integral operator $F_n(z)$ [2], defined by

$$(2) \quad F_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} ... \left( \frac{f_n(t)}{t} \right)^{\alpha_n} dt$$

and we study its property.

**Remark 1** We observe that for $n = 1$ and $\alpha_1 = 1$, we obtain the integral operator of Alexander [1], $F(z) = \int_0^z \frac{f(t)}{t} dt$.

### 2 Main Result

**Theorem 1** Let $\alpha_i > 0 \ i \in \{1, 2, ..., n\}$ and $\beta_i$ be real numbers with the property $-1 \leq \beta_i \leq 1$ and $f_i \in \mathcal{S}\mathcal{P}^\lambda_k(\beta_i)$ for $i \in \{1, 2, ..., n\}$. If

$$0 < \sum_{i=1}^{n} \alpha_i(1 - \beta_i) \leq 1,$$

then the integral operator $F_n \in \mathcal{V}^\lambda_k(\gamma)$, where

$$\gamma = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1).$$

**Proof.** From (2), we have,

$$F'_n(z) = \left( \frac{f_1(z)}{z} \right)^{\alpha_1} ... \left( \frac{f_n(z)}{z} \right)^{\alpha_n}$$

$$F''_n(z) = \sum_{i=1}^{n} \alpha_i \left( \frac{f_i(z)}{z} \right)^{\alpha_i-1} \left( \frac{zf'_i(z) - f_i(z)}{z^2} \right) \prod_{j=1, j \neq i}^{n} \left( \frac{f_j(z)}{z} \right)^{\alpha_j}$$

$$\frac{F''_n(z)}{F'_n(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{zf'_i(z) - f_i(z)}{zf_i(z)} \right) = \sum_{i=1}^{n} \alpha_i \left( \frac{f'_i(z) - 1}{f_i(z) - \frac{1}{z}} \right)$$
so that
\[ \frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{zf'_i(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^{n} \alpha_i \left( \sum_{i=1}^{n} \frac{zf'_i(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i \]
which is equivalent to
\[ 1 + \frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^{n} \alpha_i \left( \sum_{i=1}^{n} \frac{zf'_i(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i + 1. \]
That is,
\[ \Re \left\{ 1 + \frac{zF''_n(z)}{F'_n(z)} \right\} = \sum_{i=1}^{n} \alpha_i \Re \left\{ \frac{zf'_i(z)}{f_i(z)} - \beta_i \right\} + \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{i=1}^{n} \alpha_i + 1. \]
Since, \( f_i \in SP^\lambda_k(\beta_i) \) for \( i \in \{1, 2, \ldots, n\} \)
\[ \Re \left\{ 1 + \frac{zF''_n(z)}{F'_n(z)} \right\} > \sum_{i=1}^{n} \alpha_i \left| \frac{zf'_i(z)}{f_i(z)} - i \right| + \sum_{i=1}^{n} \alpha_i (\beta_i - 1) + 1 > \sum_{i=1}^{n} \alpha_i (\beta_i - 1) + 1. \]
Therefore, \( F_n \in V_k^\lambda(\gamma) \), where \( \gamma = 1 + \sum_{i=1}^{n} \alpha_i (\beta_i - 1) \).

**Corollary 1** For parametric values \( k = 2, \lambda = 0 \), we get Theorem 1 in [3] which reads as:
Let \( \alpha_i > 0 \), for \( i \in \{1, 2, \ldots, n\} \), \( \beta_i \) be real numbers with the properties \( 0 \leq \beta_i < 1 \)
and \( f_i \in Sp(\beta_i) \) for \( i \in \{1, 2, \ldots, n\} \). If \( 0 < \sum_{i=1}^{n} \alpha_i \leq 1 \), then the integral operator \( F_n \)
is convex by the order \( 1 + \sum_{i=1}^{n} \alpha_i (\beta_i - 1) \).

**Corollary 2** For parametric values \( k = 2, \lambda = 0 = \beta \), we get Theorem 2.5 in [2], stated as:
Let \( \alpha_i, i \in \{1, 2, \ldots, n\} \) be real numbers with the properties \( \alpha_i > 0 \) for \( i \in \{1, 2, \ldots, n\} \),
\( \sum_{i=1}^{n} \alpha_i \leq 1 \) and \( 1 - \sum_{i=1}^{n} \alpha_i \in [0, 1) \). We consider the functions \( f_i \in SP_k \), for
\( i \in \{1, 2, \ldots, n\} \). In these conditions, the integral operator defined in \( F_n \) is convex by
\( 1 - \sum_{i=1}^{n} \alpha_i \) order.

**Theorem 2** Let \( \alpha_i, i \in \{1, 2, \ldots, n\} \) be real and positive and \( f_i \in SP^\lambda_k(\beta) \) for \( i \in \{1, 2, \ldots, n\} \). If \( 0 < \sum_{i=1}^{n} \alpha_i \leq \frac{1}{1-\beta} \), then the integral operator \( F_n \in V_k^\lambda(\gamma_1) \), where
\[ \gamma_i = (\beta - 1) \sum_{i=1}^{n} \alpha_i + 1. \]

**Proof.** Using (2), we have,

\[ F'_n(z) = \left( \frac{f_1(z)}{z} \right)^{\alpha_1} \cdots \left( \frac{f_n(z)}{z} \right)^{\alpha_n} \]

so that

\[ F''_n(z) = \sum_{i=1}^{n} \alpha_i \left( \frac{f_i(z)}{z} \right)^{\alpha_i-1} \left( \frac{zf'_i(z) - f_i(z)}{z^2} \right) \prod_{j=1, j \neq i}^{n} \left( \frac{f_j(z)}{z} \right)^{\alpha_j} \]

\[ 1 + \frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{z f'_i(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i + 1. \]

For \( f_i \in \mathcal{SP}_k^{\lambda}(\beta), \ i \in \{1, 2, \ldots, n\} \) we obtain,

\[ \Re \left\{ 1 + \frac{zF''_n(z)}{F'_n(z)} \right\} > (\beta - 1) \sum_{i=1}^{n} \alpha_i + 1. \]

Since, \( \beta \leq 1 \) the above inequality implies

\[ 0 < \sum_{i=1}^{n} \alpha_i \leq \frac{1}{1 - \beta}. \]

Therefore, \( F_n \in V_k^{\lambda}(\gamma_1) \) where \( \gamma_1 = (\beta - 1) \sum_{i=1}^{n} \alpha_i + 1. \)

**Corollary 3** For parametric values \( k = 2, \lambda = 0, \) we get Theorem 2 in [3], stated as.

Let \( \alpha_i, \) for \( i \in \{1, 2, \ldots, n\}, \ \beta_i \) be real positive numbers and \( f_i \in \mathcal{S}_p(\beta) \)

\( 0 \leq \beta < 1 \) and for \( i \in \{1, 2, \ldots, n\}. \) If \( 0 < \sum_{i=1}^{n} \alpha_i \leq \frac{1}{1 - \beta}, \) then the integral operator \( F_n \) is convex by the order \( (\beta - 1) \sum_{i=1}^{n} \alpha_i + 1. \)

By taking \( n = 1, \) we obtain the following Corollary.

**Corollary 4** Let \( \gamma \) be the real number, \( \gamma > 0. \) We suppose that the functions \( f \in \mathcal{SP}_k^{\lambda}(\beta) \) and \( 0 < \gamma \leq \frac{1}{1 - \beta}. \) With these conditions, the integral operator \( F_1(z) \in V_k^{\lambda}((\beta - 1)\gamma + 1). \)
Corollary 5 Let \( f \in S^p_k(\beta) \) and consisting the integral operator of Alexander,
\[
F(z) = \int_0^z \frac{f(t)}{t} \, dt.
\]
Then, \( F \in V^\lambda_k(\beta) \).

Since \( F(z) = \int_0^z \frac{f(t)}{t} \, dt \) we have,
\[
1 + \frac{zF''(z)}{F'(z)} = \frac{zf'(z)}{f(z)}.
\]
That is,
\[
\Re \left\{ 1 + \frac{zF''(z)}{F'(z)} \right\} = \Re \left\{ \frac{zf'(z)}{f(z)} - \beta \right\} + \beta > \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta > \beta.
\]
This implies that the Alexander operator belongs to \( V^\lambda_k(\beta) \).

Corollary 6 By taking \( k = 2, \lambda = 0 = \beta \), we get the Theorem 2.8 in [2], which reads as:
We suppose that \( f \in S^p \). In this condition, the integral operator of Alexander, defined by \( F_1(z) = \int_0^z \frac{f(t)}{t} \, dt \) is convex.

References


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