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Wright's generalized hypergeometric functions associated with subclass of $k-$uniformly starlike functions with fixed two points [1]

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Abstract

In this paper we consider the class of functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \ \ (a_n \geq 0),$$

that are satisfying the condition

$$(1-\mu)\frac{f(z_0)}{z_0} + \mu f'(z_0) = 1, \ \ (0 \leq \mu \leq 1; \ \ -1 < z_0 < 1 \ \text{and} \ z_0 \neq 0).$$

We obtain coefficient bounds, distortion theorem and extreme points of the subclass of starlike functions associated with Wright's generalized hypergeometric functions. Furthermore, we discuss radius of convexity and closure properties for functions in this generalized class.

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1 Introduction

Let S be the class of functions $f(z)$ that are analytic in the unit disc $U = \{z : |z| < 1\}$ with $f(0) = 0$. Denote by T, the subclass of S consists of functions of the form

(1) $$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)$$

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and also denote T_1 ,the subclass of S consisting of functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, \ \ a_1 > 0). \tag{2}$$

where, either $f(z_0) = z_0\,(-1 < z_0 < 1; z_0 \neq 0)$ or, $f'(z_0) = 1\ (-1 < z_0 < 1)$.

Let T_μ be the subclass of T_1 satisfying

$$(1-\mu)\frac{f(z_0)}{z_0} + \mu f'(z_0) = 1 \ \text{ where } \ (-1 < z_0 < 1), \ \ 0 \leq \mu \leq 1. \tag{3}$$

Following Goodman [9, 10] , Rønning[15] defined two subclasses of S,
(i) for the functions f in S is said to be $k-$ uniformly starlike functions of order γ if

$$\text{Re}\ \left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in U, \ -1 < \gamma \leq 1, \ \text{and } k \geq 0$$

and (ii)for the functions f in S is said to be $k-$ uniformly convex functions of order γ if

$$\text{Re}\ \left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in U, \ -1 < \gamma \leq 1, \ \text{and } k \geq 0.$$

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_p, A_p$ and $\beta_1, B_1 \ldots, \beta_q, B_q$ $(p, q \in N = 1, 2, 3, ...$ satisfying the condition that

$$1 + \sum_{n=1}^{q} B_n - \sum_{n=1}^{p} A_n \geq 0. \ \ z \in U. \tag{4}$$

The Wright generalized hypergeometric function[19]

$${}_p\Psi_q[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z] =_p \Psi_q[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}; z]$$

is defined by

$${}_p\Psi_q[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z] = \sum_{n=0}^{\infty}\{\prod_{t=0}^{p}\Gamma(\alpha_t + tA_t\}\{\prod_{t=0}^{q}\Gamma(\beta_t + tB_t\}^{-1}\frac{z^n}{n!}, z \in U.$$

If $A_t = 1(t = 1, 2, ..., p)$ and $B_t = 1(t = 1, 2, ..., q)$ we have the relationship:

$$\begin{aligned} \Omega_p\Psi_q[(\alpha_t, 1)_{1,p}(\beta_t, 1)_{1,q}; z] &\equiv {}_pF_q(\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q; z) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n}\ \frac{z^n}{n!} \end{aligned} \tag{5}$$

($p \leq q+1;\ p,q \in N_0 = N \cup \{0\}; z \in U$) is the generalized hypergeometric function(see for details[8]) where N denotes the set of all positive integers and $(\alpha)_n$ is the Pochhammer symbol and

$$\Omega = \left(\prod_{t=0}^{p} \Gamma(\alpha_t)\right)^{-1} \left(\prod_{t=0}^{q} \Gamma(\beta_t)\right). \tag{6}$$

By using the generalized hypergeometric function Dziok and Srivastava [8] introduced the linear operator. In[4] Dziok and Raina extended the linear operator by using Wright generalized hypergeometric function . First we define a function

$${}_p\phi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] = \Omega\ z_p\Psi_q[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z].$$

Let $\mathcal{W}[(\alpha_n, A_n)_{1,p}; (\beta_n, B_n)_{1,q}] : S \to S$ be a linear operator defined by

$$\mathcal{W}[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}]f(z) \quad := \quad z\ {}_p\phi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] * f(z)$$

We observe that , for $f(z)$ of the form(1),we have

$$\mathcal{W}[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}]f(z) = z + \sum_{n=2}^{\infty} \Omega\sigma_n(\alpha_1)\ a_n z^n \tag{7}$$

where Ω is given by (6)and $\sigma_n(\alpha_1)$is defined by

$$\sigma_n(\alpha_1) = \frac{\Omega\Gamma(\alpha_1 + A_1(n-1)) \ldots \Gamma(\alpha_p + A_p(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_q + B_q(n-1))}. \tag{8}$$

If, for convenience, we write

$$\mathcal{W}[\alpha_1]f(z) = \mathcal{W}[(\alpha_1, A_1), \ldots, (\alpha_l, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q)]f(z) \tag{9}$$

introduced by Dziok and Raina[4]. In view of the relationship (5) the linear operator (7) and by setting $A_t = 1(t = 1, ..., q)$ and $B_t = 1(t = 1, ..., s)$, we are led immediately to the aforementioned Dziok- Srivastava operator which contains, as its further special cases, such other linear operators of Geometric Function Theory as the Hohlov operator, the Carlson-Shaffer operator[3], the Ruscheweyh derivative operator[16], the generalized Bernardi-Libera-Livingston operator[2], the fractional derivative operator[7], and so on (see, for the precise relationships, Dziok and Srivastava [4, 5, 6, 8]).

For $-1 \leq \gamma < 1$, we let $W_q^p([\alpha_1], \gamma, k, z_0)$ denote the subclass of starlike functions corresponding to the family UCV for functions $f(z)$ of the form (1) such that

$$\begin{aligned} &\text{Re}\ \left\{\frac{z(\mathcal{W}[\alpha_1]f(z))'}{(1-\lambda)\mathcal{W}[\alpha_1]f(z) + \lambda z(\mathcal{W}[\alpha_1]f(z))'} - \gamma\right\} \\ &\quad \geq k\left|\frac{z(\mathcal{W}[\alpha_1]f(z))'}{(1-\lambda)\mathcal{W}[\alpha_1]f(z) + \lambda z(\mathcal{W}[\alpha_1]f(z))'} - 1\right| \end{aligned} \tag{10}$$

For $k \geq 0$ and $-1 \leq \gamma < 1$, we let

$$\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0) = W_q^p([\alpha_1], \gamma, k, z_0) \cap T_\mu$$

the subclass of T_μ consisting of functions of the form (3) and satisfying the analytic criterion (10).

Using the techniques of Silverman [17] and motivated by the earlier works [11, 12, 13, 14] and [18], in this paper we obtain the coefficient bounds, distortion bounds, extreme points, radius of starlikeness and closure theorems for the functions belong to the class $\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$.

2 Main Results

Theorem 1 *A function $f(z)$ of the form (2) is in the class $W_q^p([\alpha_1], \gamma, k, z_0)$ if and only if*

$$\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)]\sigma_n(\alpha_1)a_n \leq a_1(1-\gamma), \tag{11}$$

$-1 \leq \gamma < 1, k \geq 0.$

The proof of the Theorem 1 is similar to that of Theorem 2.2, in [1], hence we omit the details.

Theorem 2 *Let $f(z)$ be defined by (2). Then $f(z) \in TS(\mu, \alpha, \beta, z_0)$ if and only if*

$$\sum_{n=2}^{\infty} \left\{ \frac{[n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)]}{1-\gamma} \sigma_n(\alpha_1) - [(1-\mu) + n\mu] z_0^{n-1} \right\} a_n \leq 1, \tag{12}$$

$-1 \leq \gamma < 1, k \geq 0.$

Proof. Since $(1-\mu)\frac{f(z_0)}{z_0} + \mu f'(z_0) = 1,\ \ (0 \leq \mu \leq 1;\ \ -1 < z_0 < 1$ and $z_0 \neq 0)$, we get

$$a_1 = 1 + \sum_{n=2}^{\infty} [(1-\mu) + n\mu] a_n z_0^{n-1}.$$

Substituting for a_1 in (11) we get (12).

Corollary 1 *Let the function $f(z)$ defined by (2) belongs $\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$. Then*

$$a_n \leq \left\{ \frac{[n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)]}{1-\gamma} \sigma_n(\alpha_1) - [(1-\mu) + n\mu] z_0^{n-1} \right\}^{-1},$$

$n \geq 2,\ -1 \leq \alpha < 1,\ \ \beta \geq 0$ *with equality for*

$$f(z) = \frac{[n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)]\sigma_n(\alpha_1)z - (1-\gamma)z^n}{[n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)]\sigma_n(\alpha_1) - (1-\gamma)[(1-\mu) + n\mu] z_0^{n-1}}. \tag{13}$$

Theorem 3 *Let*

$$f_1(z) = z \text{ and} \tag{14}$$

$$f_n(z) = \frac{[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\sigma_n(\alpha_1)z-(1-\gamma)z^n}{[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\sigma_n(\alpha_1)-(1-\gamma)[(1-\mu)+n\mu]z_0^{n-1}}, \quad n \geq 2.$$

Then $f(z) \in \mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \quad \text{where } \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1.$$

Proof. The proof of the Theorem 3, follows on line similar to the proof of the theorem on extreme points given in Silverman [17].

3 A Distortion Theorem

Theorem 4 *Let the function* $f(z)$ *defined by (2) belong to* $\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$. *Then*

$$|f(z)| \geq a_1|z| \left\{1 - \frac{1-\gamma}{[2+k-\gamma-\lambda(\gamma+k)]\sigma_2(\alpha_1)}|z|\right\} \tag{15}$$

and

$$|f(z)| \leq a_1|z| \left\{1 + \frac{1-\gamma}{[2-k-\gamma-\lambda(\gamma+k)]\sigma_2(\alpha_1)}|z|\right\} \tag{16}$$

for $z \in U$.

Proof. In the view of (11) and the fact that $\sigma_n(\alpha_1)$ is non-decreasing for $n \geq 2$, we have

$$\begin{aligned} & [2+k-\gamma-\lambda(\gamma+k)]\sigma_2(\alpha_1) \sum_{n=2}^{\infty} a_n \\ \leq & \sum_{n=2}^{\infty} [n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\sigma_n(\alpha_1) a_n \\ \leq & \; a_1(1-\gamma) \end{aligned} \tag{17}$$

which is equivalent to,

$$\sum_{n=2}^{\infty} a_n \leq \frac{a_1(1-\gamma)}{[2+k-\gamma-\lambda(\gamma+k)]\sigma_2(\alpha_1)}\,. \tag{18}$$

Using (2) and (18), we obtain

$$\begin{aligned} |f(z)| &\geq a_1|z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq a_1|z| - |z|^2 \frac{a_1(1-\gamma)}{[2+k-\gamma-\lambda(\gamma+k)]\sigma_2(\alpha_1)} \\ &\geq a_1|z| \left\{1 - \frac{1-\gamma}{[2+k-\gamma-\lambda(\gamma+k)]\sigma_2(\alpha_1)}|z|\right\} \end{aligned}$$

and

$$|f(z)| \leq a_1|z| \left\{1 + \frac{1-\gamma}{[2+k-\gamma-\lambda(\gamma+k)]\sigma_2(\alpha_1)}|z|\right\}.$$

Hence the proof is complete.

Corollary 2 *Let $f(z)$ defined by (2) be in the class $\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$. Then $f(z)$ is included in a disk with its center at origin and radius r given by*

$$r = a_1 \left\{1 + \frac{1-\gamma}{[2+k-\gamma-\lambda(\gamma+k)]\sigma_2(\alpha_1)}\right\} \tag{19}$$

4 Closure Theorems

Let the functions $f_j(z)$ be defined for $j = 1, 2, \ldots m$ by

$$f_j(z) = a_{1,j}\ z - \sum_{n=2}^{\infty} a_{n,j}\ z^n \tag{20}$$

$a_{1,j} > 0,\ a_n, j \geq 0,\ z \in U.$

Theorem 5 *Let $f_j(z)$ defined by (20) be in the class $\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{j=1}^{m} d_j f_j(z), \quad d_j \geq 0 \tag{21}$$

also in the same class $\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$, where

$$\sum_{j=1}^{m} d_j = 1. \tag{22}$$

Proof. From (21) we have

$$h(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n \tag{23}$$

where $b_1 = \sum\limits_{j=1}^{m} d_j a_{1,j}$ and $b_n = \sum\limits_{j=1}^{m} d_j a_{n,j}$ $(n = 2, 3, \dots)$.
Since $f_j(z) \in \mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$ $(j = 1, 2, \dots m)$ and by applying Theorem 2, we get

$$\sum_{n=2}^{\infty} \left\{ \frac{[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]}{1-\gamma} \sigma_n(\alpha_1) - [(1-\mu) + n\mu] z_0^{n-1} \right\} a_{n,j} \leq 1,$$

$$j = 1, 2, \dots m.$$

Therefore, we have

$$\sum_{n=2}^{\infty} \left\{ \frac{[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]}{1-\gamma} \sigma_n(\alpha_1) - [(1-\mu) + n\mu] z_0^{n-1} \right\} \Big(\sum_{j=1}^{m} d_j a_{n,j} \Big)$$

$$= \sum_{j=1}^{m} d_j \left[\sum_{n=2}^{\infty} \left\{ \frac{[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]}{1-\gamma} \sigma_n(\alpha_1) - [(1-\mu) + n\mu] z_0^{n-1} \right\} a_{n,j} \right]$$

$$\leq \sum_{j=1}^{m} d_j = 1 \quad \text{(by Theorem 2 and by (22))}.$$

Which implies that $h(z) \in \mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$ and so the proof is complete.

5 Radius of Convexity and Starlikeness

In this section we obtain the radius of starlikeness of order δ $(0 \leq \delta < 1)$, radius of convexity of order δ $(0 \leq \delta < 1)$, for the class $\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$.

Theorem 6 *Let $f \in TS(\mu, \alpha, \beta, z_0)$. Then*

1. *f is starlike of order $\delta (0 \leq \delta < 1)$ in the disc $|z| < r_1$; that is, $Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta$, where*

$$r_1 = \inf_{n \geq 2} \left\{ \sigma_n(\alpha_1) \frac{1-\delta}{n-\delta} \frac{[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]}{1-\alpha} \right\}^{\frac{1}{n-1}}.$$

2. *f is convex of order δ $(0 \leq \delta < 1)$ in the unit disc $|z| < r_2$, that is $Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta$, where*

$$r_2 = \inf_{n \geq 2} \left\{ \sigma_n(\alpha_1) \frac{(1-\delta)[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]}{n(n-\delta)(1-\gamma)} \right\}^{\frac{1}{n-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (14).

Proof. Given $f \in T_1$, and f is starlike of order δ,we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\delta. \tag{24}$$

For the left hand side of (24) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\ |z|^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty}a_n\ |z|^{n-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n\ |z|^{n-1} < a_1. \tag{25}$$

Substituting $a_1 = 1 + \sum\limits_{n=2}^{\infty}[(1-\mu)+n\mu]a_n z_0^{n-1}$ in (25), we have

$$\sum_{n=2}^{\infty}\left\{\frac{n-\delta}{1-\delta}|z|^{n-1} - [(1-\mu)+n\mu]z_0^{n-1}\right\}a_n \leq 1 \tag{26}$$

Using the fact, that $f \in \mathcal{TW}_q^p([\alpha_1],\gamma,k,z_0)$ if and only if

$$\sum_{n=2}^{\infty}\left\{\frac{[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]}{1-\gamma}\sigma_n(\alpha_1) - [(1-\mu)+n\mu]z_0^{n-1}\right\}a_n < 1.$$

We can say (24) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} - [(1-\mu)+n\mu]z_0^{n-1}$$
$$\leq \frac{[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]}{1-\gamma}\sigma_n(\alpha_1) - [(1-\mu)+n\mu]z_0^{n-1}.$$

Or, equivalently,

$$|z|^{n-1} < \frac{(1-\delta)[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]}{(n-\delta)(1-\gamma)}\sigma_n(\alpha_1)$$

which yields the starlikeness of the family.

(2) Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar the proof of (1).

Remark 1 *We note that the radius of starlikeness and convexity are independent of the fixed point z_0.*

6 Convex families

Suppose B is nonempty subset of the real interval $(0,1)$,we define $\mathcal{TW}_q^p([\alpha_1],\gamma,k,B)$ by

$$\mathcal{TW}_q^p([\alpha_1],\gamma,k,B)=\bigcup_{z_i\in B}\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_i).$$

If B consists of a single element say z_0 then $\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_0)$ is a convex family. Because if $f_1(z)$ and $f_2(z)$ are in $\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_0)$, then it can be seen that for $0\le\eta\le1$, $\eta f_1(z)+(1-\eta)f_2(z)$ is in $\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_0)$. To examine this class for other subsets of B, we prove the following lemma

Lemma 1 *If $f(z)\in\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_0)\cap\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_1)$, where z_0 and z_1 are distinct positive numbers then $f(z)=z$.*

Proof. For the functions of the form (2), we have

$$a_1=1+\sum_{n=2}^{\infty}a_n[(1-\mu)+n\mu]z_0^{n-1}$$

and

$$a_1=1+\sum_{n=2}^{\infty}a_n[(1-\mu)+n\mu]z_1^{n-1}.$$

That is, $a_n[(1-\mu)+n\mu][z_1^{n-1}-z_0^{n-1}]=0$.
Hence $a_n\equiv0$ for $n\ge2$, and so the results follows.

Theorem 7 *If B is contained in the interval $(0,1)$ and $0\le\alpha<1$, $\beta\ge0$, then $\mathcal{TW}_q^p([\alpha_1],\gamma,k,B)$ is a convex family if and only if B is connected.*

Proof. Let B be connected. Suppose $z_0,z_1\in B$ with $z_0\le z_1$. If $f(z)$ of the form (2) is in $\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_0)$ and $g(z)=b_1z-\sum\limits_{n=2}^{\infty}b_nz^n$ is in $\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_1)$ then for $0\le\eta\le1$, we shall prove that there exists a z_2 $(z_0\le z_2\le z_1)$ such that $h(z)=\eta f(z)+(1-\eta)g(z)$ is in $\mathcal{TW}_q^p([\alpha_1],\gamma,k,z_2)$. Set

$$\begin{aligned}t(z)&=[(1-\mu)\frac{h(z}{z}+\mu h'(z)]\\&=\eta\left[a_1-\sum_{n=2}^{\infty}a_nz^{n-1}((1-\mu)+n\mu)\right]\\&\quad+(1-\eta)\left[b_1-\sum_{n=2}^{\infty}b_nz^{n-1}((1-\mu)+n\mu)\right]\\t(z)&=1+\eta\sum_{n=2}^{\infty}\left[a_n\{(1-\mu)+n\mu\}(z_0^{n-1}-z^{n-1})\right]\\&\quad+(1-\eta)\sum_{n=2}^{\infty}\left[b_n\{(1-\mu)+n\mu\}(z_1^{n-1}-z^{n-1})\right]\end{aligned}\tag{27}$$

and we observe that $f(z)$ is real when z is real with $t(z_0) \geq 1$ and $t(z_1) \leq 1$.
Hence for some z_1, $z_0 \leq z_2 \leq z_1$, we have $t(z_2) = 1$. Since z_1, z_2 and η are arbitrary, the family $\mathcal{TW}_q^p([\alpha_1], \gamma, k, B)$ is convex.

Conversely, suppose B is not connected. Then we can take $z_0, z_1 \in B$, $z_2 \notin B$ such that $z_0 < z_2 < z_1$. Let us assume $f(z)$ and $g(z)$ are not both identity function. Then using (27) fixing $z = z_2$ and allow η to vary,

$$\begin{aligned} t(z) &= t(z_2, \eta) \\ &= 1 + \eta \sum_{n=2}^{\infty} \left[a_n \{(1-\mu) + n\mu\} (z_0^{n-1} - z_2^{n-1}) \right] \\ &\quad + (1-\eta) \sum_{n=2}^{\infty} \left[b_n \{(1-\mu) + n\mu\} (z_1^{n-1} - z_2^{n-1}) \right]. \end{aligned}$$

Since $t(z_2, 0) > 1$ and $t(z_2, 1) < 1$, there must exists a λ_0, $0 < \eta_0 < 1$, for which $t(z_2, \eta_0) = 1$. Hence $h(z) \in \mathcal{TW}_q^p([\alpha_1], \gamma, k, z_2)$ for $\eta = \eta_0$. Since $z_2 \notin B$ from the Lemma 1, it follows that $h(z) \in \mathcal{TW}_q^p([\alpha_1], \gamma, k, B)$. Therefore $\mathcal{TW}_q^p([\alpha_1], \gamma, k, B)$ is not a convex family.

Concluding Remarks: Observe that, if $A_t = 1 (t = 1, 2, ..., p); B_t = 1 (t = 1, 2, ..., q)$ and $\lambda = 0$ specializing the parameters p, q, $\alpha_1, \alpha_2, \ldots, \alpha_p$, and $\beta_1, \beta_2, \ldots, \beta_q$, γ, k in the class $\mathcal{TW}_q^p([\alpha_1], \gamma, k, z_0)$, we obtain various classes introduced and studied in the literature (see [11, 12, 13, 14, 17, 18]).

References

[1] M. K. Aouf, G. Murugusundaramoorthy, *On a subclass of uniformly convex functions defined by the Dziok-Srivastava Operator*, Austral. J. Math. Anal. Appl., 1, 2008.

[2] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., 135, 1969, 429-446.

[3] B. C. Carlson, S. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., 15, 1984, 737-745.

[4] J. Dziok, R. K. Raina, *Families of analytic functions associated with the Wright generalized hypergeometric function*, Demonstratio Math., 37, no. 3, 2004, 533-542.

[5] J. Dziok, R. K. Raina, H. M. Srivastava, *Some classes of analytic functions associated with operators on Hilbert space involving Wrights generalized hypergeometric function*, Proc. Jangjeon Math. Soc., 7, 2004, 4355.

[6] J. Dziok, H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., 103, 1999, 113.

[7] J. Dziok, H. M. Srivastava, *Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function*, Adv. Stud. Contemp. Math., 5, 2002, 115-125.

[8] J. Dziok, H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform and Spec. Funct., 14, 2003, 7-18.

[9] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., 56, 1991, 87-92.

[10] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl., 155, 1991, 364-370.

[11] S. R. Kulkarni, U. H. Naik, *A new class of univalent functions with negative coefficients with two fixed points*, Indian J. Appl. Math., 27(7), 1996, 681-688.

[12] G. Murugusundaramoorthy, K. G. Subramanian, P. Balasubrahmanyam, *On uniformly convex functions with two fixed points*, Ganita, vol. 53, no. 2, 2002, 109-116.

[13] G. Murugusundaramoorthy, K. Vijaya, *On certain subclass of starlike functions with two fixed points*, Journal. Indian. Math. Soc., vol. 72, no. 1-4, 2004, 131-139.

[14] R. K. Raina, T. S. Nahar, *A certain subclass of analytic functions with negative coefficients and fixed points*, Kyungpook. Math. J., 40, 2000, 39-48.

[15] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., 118, 1993, 189-196.

[16] S. Ruscheweyh, *New criteria for Univalent Functions*, Proc. Amer. Math. Soc., 49, 1975, 109-115.

[17] H. Silverman, *Extreme points of univalent functions with fixed points*, Trans. Amer. Math. Soc., 219, 1976, 387-395.

[18] B. A. Uralegaddi, C. Somanatha, *Generalized class of univalent functions with two fixed points*, Tamkang. J. Math., 24, no. 1, 1993, 57-66.

[19] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function,* Proc. London. Math. Soc., 46, 1946, 389-408.

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