Fixed Point Theorems For 1-Set Contractions

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Abstract

Amann [2] and Nussbaum [5] have introduced the fixed point indices of k-set operators (0 ≤ k < 1) and condensing operators to derive some fixed point theorems. As a compliment, Li Guozhen [4] has defined the fixed point index of 1-set contractive operators and obtained some fixed point theorems of 1-set contractive operators. The main purpose of this paper is to generalize the result [4, Theorem 3] of Li Guozhen by relaxing the condition 'bounded' on the subset $D$ of the Banach space $E$. In addition, our method is also different from that of Guozhen.

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1 Introduction and Preliminary definitions

Fixed point theory for $k$-set contraction mappings has its origin in 1955, when G. Darbo took an essential step in extending the Schauder Fixed Point Theorem using the idea of the measure of noncompactness of Kuratowskii defined in 1939. Darbo defined a new class of mappings, the so called $k$-set contractions, using this number, called now the Kuratowskii measure of noncompactness. In 1967 Sadovskii used the idea of measure of noncompactness in proving a generalization of the Schauder Fixed Point Theorem for mappings which are now called condensing or densifying. Using this measure as well as other measure of noncompactness several classes of mappings were defined and at the same time various extensions of the Schauder’s Fixed Point Theorem were obtained. In 1973 Petryshyn [6] studied the fixed point theorem of 1-set contraction mappings.

Let $E$ be a real Banach space, $C$ be a closed convex set in $E$, $U$ be a bounded open...
set in $C$. Suppose that $T : \overline{U} \to C$ is a semiclosed 1-set contraction mapping, and $x_0 \in U$. Then the following boundary condition

$$x \neq tT(x) + (1 - t)x_0 \text{ for } x \in \partial U \text{ and } t \in (0, 1)$$

($\partial U$ denotes the boundary of $U$) guarantees the existence of a fixed point. Indeed this result is due to Li Guozhen who proved it using the fixed point index of 1-set contractive operators. Note that the above mentioned boundary condition is equivalent to the Leray-Schauder condition imposed by Browder for mappings of semicontractive type.

In this paper we generalize the above result [4, Theorem 3] of Li Guozhen by relaxing the condition 'bounded' on the subset $U$ of the Banach space $E$. Our method is also different from that of Guozhen.

We need the following preliminary definitions.

**Definition 1** Let $D$ be a bounded subset of a metric space $X$. Define the measure of noncompactness $\alpha(D)$ of $D$ by

$$\alpha(D) = \inf \{ \epsilon > 0 : D \text{ admits a finite covering of subsets of diameter } \leq \epsilon. \}$$

It follows that $\alpha(A) \leq \alpha(B)$ whenever $A \subset B, \alpha(A \cup B) = \max \{ \alpha(A), \alpha(B) \}, \alpha(\overline{A}(A)) = \alpha(A), \alpha(\lambda A) = |\lambda| \alpha(A), \alpha(\overline{A}) = \alpha(A), \alpha(A + B) \leq \alpha(A) + \alpha(B)$ and $\alpha(A) = 0$ if and only if $\overline{A}$ is compact.

**Definition 2** Let $T : X \to X$ be a continuous mapping of a Banach space $X$. Then $T$ is called a $k$-set contraction if, for all $A \subset X$ with $A$ bounded, $T(A)$ is bounded and $\alpha(TA) \leq k \alpha(A), 0 \leq k < 1$. If $\alpha(TA) < \alpha(A)$, for all $\alpha(A) > 0$, then $T$ is called densifying (or Condensing).

A k-set contraction, with $k \in [0, 1)$ is densifying, but the converse is not true. If $\alpha(TA) \leq \alpha(A)$, then $T$ is called a 1-set contraction.

**Definition 3** A continuous map $F : X \subseteq E \to E$ where $E$ is a Banach space is said to be completely continuous if $F(Y)$ is relatively compact for all bounded sets $Y \subseteq X$.

**Definition 4** A map $f : X \to X$ where $X$ is a Banach space, is said to be demiclosed if for every sequence $x_n \in X$ which converges weakly to $x$ in $X$ (denote by $x_n \rightharpoonup x$) and $f(x_n)$ converges strongly to $y$, we have $y = f(x)$.

**Definition 5** Let $T : X \to X$ be a continuous mapping of a Banach space $X$. Then $T$ is called a semiclosed 1-set contraction mapping, if $T$ is a 1-set contraction and $I - T$ is closed.

In what follows we will require the following fixed point theorems which we state here as lemma.
Lemma 1 [3, Theorem (c.8)] Let $E$ be a real Banach space, $C \subseteq E$ be closed and convex, $U \subseteq C$ an open subset containing 0, and $F : \overline{U} \to C$ a condensing map. Assume that $\|F(x)\| \leq \|x\|$ for all $x \in \partial U$. Then $F$ has a fixed point.

Lemma 2 [6, Proposition 1]. If $D$ is a bounded open subset of a Banach space $X$ with 0 in $D$ and $T$ a condensing mapping of $\overline{D}$ into $X$ which satisfies the boundary condition
\[
T(x) = \alpha x \quad \text{for some} \ x \in \partial D, \ \text{then} \ \alpha \leq 1.
\]
Then $F(T) \subseteq \overline{D}$ is nonempty, where $F(T)$ denotes the set of fixed points of $T$.

2 Fixed point results for 1-set contractions

Theorem 1 Let $C$ be a closed convex subset of a Banach space $E, U$ an open subset of $C$ with $0 \in U$ and let $F : \overline{U} \to C$ a semiclosed 1-set contraction map with $F(\overline{U})$ bounded. Assume that $\|F(x)\| \leq \|x\|$ for all $x \in \partial U$. Then $F$ has a fixed point.

Proof. For $n \geq 2$, define
\[
F_n := (1 - \frac{1}{n})F : \overline{U} \to C.
\]
It is easy to see that each $F_n$ is a $k$-set contraction with contraction constant $k = 1 - \frac{1}{n}$. Since every $k$-set contraction map is a condensing map, we have each $F_n : \overline{U} \to X$ condensing. We also notice that each $F_n$ satisfies the boundary condition
\[
\|F_n(x)\| \leq \|x\|, \ x \in \partial U.
\]
Therefore by Lemma 1.6 each $F_n$ has a fixed point $x_n \in \overline{U}$, that is $F_n(x_n) = x_n$. Now since $F(x_n) - x_n = \frac{1}{n}F(x_n)$, and $F(\overline{U})$ is bounded, it follows that $F(x_n) - x_n \to 0$ as $n \to \infty$. In view of this and the assumed closedness of $I - F$, we see that $0 \in (I - F)(\overline{U})$. Hence there exist an $x \in \overline{U}$ such that $(I - F)x = 0$, that is $x = F(x)$. When $0 \not\in U$, then the following generalizations of the above theorem holds.

Theorem 2 Let $C$ be a closed convex subset of a Banach space $E, U$ an open subset of $C$ and $F : \overline{U} \to C$ a semiclosed 1-set contraction with $F(\overline{U})$ bounded. If $F$ satisfies the following conditions
(i) There exists $x_0 \in U$ such that if $F(x) - x_0 = k(x - x_0)$ for some $x$ in $\partial U$, then $k \leq 1$. Then $F$ has a fixed point in $\overline{U}$.

Proof. Consider the set $V = U - x_0 = \{x - x_0 : x \in U\}$. It follows that $0 \in V, \partial V = \partial U - x_0, \overline{V} = \overline{U} - x_0$ and $V$ is open. Define $F' : \overline{V} \to C$ by $F'(x - x_0) = F(x) - x_0$ where $x \in \overline{U}$. Then $F'$ is a 1-set contraction. Moreover if $F'(y) = ky$ for some $y$ in $\partial V$, then we have $F(x) - x_0 = k(x - x_0)$ for some $x \in \partial U$ which gives $k \leq 1$ by condition(i). Therefore we have $\|F'(y)\| \leq \|y\|$, for every $y \in \partial V$. Furthermore,
\((I - F')(V)\) is closed, since \((I - F')V = (I - F)(U)\). We also have \(F'(V)\) bounded as \(F(U)\) is bounded. Thus \(F'\) and \(V\) satisfy all the conditions of Theorem 2.1. Hence, there exists a \(y \in V\) such that \(F'(y) = y\); that is \(F(x) - x_0 = x - x_0\) with \(x \in U\). Hence \(F(x) = x\).

**Corollary 1** [4, Theorem 3] Let \(E\) be a real Banach space, \(X\) be a closed convex set in \(E\), \(D\) be a bounded open set in \(X\). Suppose that \(T : \overline{D} \to X\) is a semiclosed 1-set contraction mapping, and \(x_0 \in D\) is such that

\[x \neq tTx + (1 - t)x_0, \quad x \in \partial D, \quad t \in (0, 1].\]

Then \(T\) has a fixed point in \(\overline{D}\).

**Theorem 3** Let \(U\) be a bounded open convex subset of a reflexive Banach space \(X\) with \(0 \in U\). Suppose that \(F : \overline{U} \to X\) is a 1-set contraction map satisfying the boundary condition \(\|F(x)\| \leq \|x\|\) for all \(x \in \partial U\). Then if \(I - F\) is demiclosed, \(F\) has a fixed point.

**Proof.** Consider for each \(n \geq 2\), the mapping

\[(1)\] \(F_n := (1 - \frac{1}{n})F : \overline{U} \to X.\)

As in Theorem 2.1 it is easy to see that each \(F_n : \overline{U} \to X\) is condensing. We also notice that each \(F_n\) satisfies the boundary condition

\[\|F_n(x)\| \leq \|x\|, \quad x \in \partial U.\]

Therefore by [6, proposition 1] \(F_n\) has a fixed point \(u_n \in U\), that is,

\[(2)\] \(F_n(u_n) = u_n, \quad \forall n \geq 2.\)

Hence from (1) and (2) we obtain that

\[F_n(u_n) = u_n = (1 - \frac{1}{n})F(u_n)\]

Since \(X\) is reflexive, every bounded sequence of elements of \(X\) contains a weakly convergent subsequence. Again, a closed convex subset of a Banach space is weakly closed. Therefore since \(\overline{U}\) is bounded, there exists a subsequence \(S\) of integers and \(u \in \overline{U}\) with

\[u_n \rightharpoonup u \quad \text{as} \quad n \to \infty \quad \text{in} \quad S.\]

In addition since \(u_n = (1 - \frac{1}{n})F(u_n)\) we have

\[||(I - F)(u_n)|| = \frac{1}{n}||F(u_n)|| \to 0 \quad \text{as} \quad n \to \infty.\]

Thus \((I - F)(u_n)\) converges strongly to 0. Hence the demiclosedness of \(I - F\) implies that \((I - F)u = 0\), which implies that \(u = F(u)\).
Corollary 2 [1, Theorem 3.3] Let $U$ be a bounded open convex subset of a uniformly convex Banach space $X$, with $0 \in U$ and $F : \overline{U} \to X$ a nonexpansive map. Then either
(A1) $F$ has a fixed point in $\overline{U}$ or
(A2) there exist $\lambda \in (0,1)$ and $u \in U$ with $u = \lambda F(u)$

Corollary 3 Let $U$ be a bounded open convex subset of a reflexive Banach space $X$, with $0 \in U$. Suppose that $F : \overline{U} \to X$ is given by $F := F_1 + F_2$, where $F_1 : \overline{U} \to X$ is a completely continuous mapping, $F_2 : \overline{U} \to X$ is a 1-set contraction. Then $F$ has a fixed point if the following two conditions are satisfied.
i) $\|F(x)\| \leq \|x\|$ for all $x \in \partial U$.
ii) $I - F$ is demiclosed.

Proof. Let $A$ be a bounded subset of $\overline{U}$ with $\alpha(A) > 0$. Then
$$\alpha(F(A)) = \alpha((F_1 + F_2)A) \leq \alpha(F_1(A)) + \alpha(F_2(A)).$$

Since $F_1$ maps bounded sets to pre-compact sets we have
$$\alpha(F(A)) \leq \alpha(F_2(A)) \leq \alpha(A)$$

which implies that $F$ is a 1-set contraction. Hence by Theorem 2.4, $F$ has a fixed point.

Remark 1 The class of 1-set contractive operators include completely continuous operators, strict set-contractive operators, condensing operators, nonexpansive maps, semi-contractive maps, LANE maps and others (Acta. Math. Sinica, 17(2001), 103-112). So the results in this section remain valid for the above maps.

References


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