General Mathematics Vol. 19, No. 2 (2011), 41–48

On some Common Fixed Point Theorems in Uniform Spaces [1]

Alfred Olufemi Bosede

Abstract

In this paper, we establish some common fixed point theorems in uniform spaces by using a more general contractive condition than those of Aamri and El Moutawakil [1]. Our results not only improve a multitude of common fixed point results in literature but also generalize some of the results of Aamri and El Moutawakil [1].

2000 Mathematics Subject Classification: 47H10, 54H25.
Key words and phrases: Uniform space, $A-$distance, $E-$distance, φ-contractions.

1 Introduction

Let X be a nonempty set and let Φ be a nonempty family of subsets of $X \times X$. The pair (X, Φ) is called a uniform space if it satisfies the following properties:
(i) if G is in Φ, then G contains the diagonal $\{(x, x)|x \in X\}$;
(ii) if G is in Φ and H is a subset of $X \times X$ which contains G, then H is in Φ;
(iii) if G and H are in Φ, then $G \bigcap H$ is in Φ;
(iv) if G is in Φ, then there exists H in Φ, such that, whenever (x, y) and (y, z) are in H, then (x, z) is in H;
(v) if G is in Φ, then $\{(y, x)|(x, y) \in G\}$ is also in Φ.
Φ is called the uniform structure of X and its elements are called entourages or neighbourhoods or surroundings.
In Bourbaki [4] and Zeidler [14], (X, Φ) is called a quasiuniform space if property (v) is omitted.

Some authors such as Berinde [3], Jachymski [5], Kada et al [6], Rhoades [9], Rus

[1] *Received 13 August, 2009*
Accepted for publication (in revised form) 25 May, 2011

[11], Wang [13] and Zeidler [14] studied the theory of fixed point or common fixed point for contractive selfmappings in complete metric spaces or Banach spaces in general.

Within the last two decades, Kang [7], Montes and Charris [10] established some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform space.

Later, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A-distance and an E-distance.

Aamri and El Moutawakil [1] introduced and employed the following contractive definition: Let $f, g : X \longrightarrow X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \qquad \forall x, y \in X, \tag{1}$$

where $\psi : \Re^+ \longrightarrow \Re^+$ is a nondecreasing function satisfying
(i) for each $t \in (0, +\infty), \quad 0 < \psi(t)$,
(ii) $\lim_{n \longrightarrow \infty} \psi^n(t) = 0, \quad \forall\, t \in (0, +\infty)$.
ψ satisfies also the condition $\psi(t) < t$, for each $t > 0, \quad t \in \Re^+$.

In this paper, we shall establish some common fixed point theorems by using a more general contractive condition than (1).

We shall also employ the concepts of an A-distance, an E-distance as well as the notion of a nondecreasing function in this paper.

2 Preliminaries

The following definitions shall be required in the sequel.

Let (X, Φ) be a uniform space. Without loss of generality, $(X, \tau(\Phi))$ denotes a topological space whenever topological concepts are mentioned in the context of a uniform space (X, Φ). Definitions $1 - 6$ are contained in Aamri and El Moutawakil [1].

Definition 1 *If $H \in \Phi$ and $(x, y) \in H, (y, x) \in H$, x and y are said to be H-close. A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a Cauchy sequence for Φ if for any $H \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are H-close for $n, m \geq N$.*

Definition 2 *A function $p : X \times X \longrightarrow \Re^+$ is said to be an A-distance if for any $H \in \Phi$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in H$.*

Definition 3 *A function $p : X \times X \longrightarrow \Re^+$ is said to be an E-distance if*
(p_1) p is an A-distance,
(p_2) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall \; x, y \in X$.

Definition 4 *A uniform space (X, Φ) is said to be $Hausdorff$ if and only if the intersection of all $H \in \Phi$ reduces to the diagonal $\{(x, x)|x \in X\}$, i.e. if $(x, y) \in H$ for all $H \in \Phi$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $H \in \Phi$ is said to be symmetrical if $H = H^{-1} = \{(y, x)|(x, y) \in H\}$.*

Definition 5 *Let (X, Φ) be a uniform space and p be an A-distance on X.*
(i) X is said to be S-complete if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n \longrightarrow \infty} p(x_n, x) = 0$.
(ii) X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n \longrightarrow \infty} x_n = x$ with respect to $\tau(\Phi)$.
(iii) $f : X \longrightarrow X$ is said to be p-continuous if $\lim_{n \longrightarrow \infty} p(x_n, x) = 0$ implies that $\lim_{n \longrightarrow \infty} p(f(x_n), f(x)) = 0$.
(iv) $f : X \longrightarrow X$ is $\tau(\Phi)$-continuous if $\lim_{n \longrightarrow \infty} x_n = x$ with respect to $\tau(\Phi)$ implies $\lim_{n \longrightarrow \infty} f(x_n) = f(x)$ with respect to $\tau(\Phi)$.
(v) X is said to be p-bounded if $\delta_p = sup\{p(x, y)|x, y \in X\} < \infty$.

Definition 6 *Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Two selfmappings f and g on X are said to be p-compatible if, for each sequence $\{x_n\}_{n=0}^{\infty}$ of X such that $\lim_{n \longrightarrow \infty} p(f(x_n), u) = \lim_{n \longrightarrow \infty} p(g(x_n), u) = 0$ for some $u \in X$, then we have $\lim_{n \longrightarrow \infty} p(f(g(x_n)), g(f(x_n))) = 0$.*

Our aim in this paper is to establish some common fixed point theorems by using a more general contractive condition than (1). We shall employ the following contractive definition: Let $f, g : X \longrightarrow X$ be selfmappings of X. There exist $L \geq 0$ and a nondecreasing function $\psi : \Re^+ \longrightarrow \Re^+$ satisfying conditions (i) and (ii) of (1) such that $\forall\ x, y \in X$, we have

$$p(f(x), f(y)) \leq e^{Lp(x,g(x))}\psi(p(g(x), g(y))), \qquad \forall x, y \in X, \tag{2}$$

where e^x denotes the exponential of x.

Remark 1 *The contractive condition* (2) *is more general than* (1) *in the sense that if $L = 0$ in the above inequality, then we obtain* (1)*, which was employed by Aamri and El Moutawakil [1].*

The following Lemma contained in Aamri and El Moutawakil [1], Kang [7] and Montes and Charris [10] shall be required in the sequel.

Lemma 1 *Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be arbitrary sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $\Re^+$ converging to 0. Then, for $x, y, z \in X$, the following hold:*
(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$, $\forall\ n \in N$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.
(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$, $\forall\ n \in N$, then $\{y_n\}_{n=0}^{\infty}$ converges to z.
(c) If $p(x_n, x_m) \leq \alpha_n$, $\forall\ m > n$, then $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, Φ).

The following remark is contained in Aamri and El Moutawakil [1].

Remark 2 *A sequence in X is p-Cauchy if it satisfies the usual metric property.*

3 Main Results

Theorem 1 *Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X such that X is p-bounded and S-complete. For arbitrary $x_0 \in X$, define a sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by*

$$x_n = f(x_{n-1}), \qquad n = 1, 2, ...$$

Suppose that f and g are commuting p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that
(i) $f(X) \subseteq g(X)$,
(ii) $p(f(x_i), f(x_i)) = 0, \quad \forall\ x_i \in X, \quad i = 0, 1, 2, ...,$
(iii) $f, g : X \longrightarrow X$ satisfy the contractive condition (2).
Suppose also that $\psi : \Re^+ \longrightarrow \Re^+$ is a nondecreasing function satisfying conditions (i) and (ii) of inequality (1).
Then, f and g have a common fixed point.

Proof. For arbitrary $x_0 \in X$, select $x_1 \in X$ such that $f(x_0) = g(x_1)$. Similarly, for $x_1 \in X$, select $x_2 \in X$ such that $f(x_1) = g(x_2)$.
Continuing this process, we select $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$.

We now show that the sequence $\{f(x_n)\}_{n=0}^{\infty}$ so generated is a p-Cauchy sequence. Indeed, since $x_n = f(x_{n-1}),\ n = 1, 2, ...$, then by using the contractive condition (2) together with conditions (ii) and (iii) of the Theorem, we get

$$(3) \quad \begin{aligned} p(f(x_n), f(x_{n+m})) &\leq e^{Lp(x_n, g(x_n))}\psi(p(g(x_n), g(x_{n+m}))) \\ &= e^{Lp(f(x_{n-1}), f(x_{n-1}))}\psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\ &= e^{L(0)}\psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\ &= e^{0}\psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\ &= \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\ &\leq \psi[e^{Lp(x_{n-1}, g(x_{n-1}))}\psi(p(g(x_{n-1}), g(x_{n+m-1})))] \\ &= \psi[e^{L(0)}\psi(p(f(x_{n-2}), f(x_{n+m-2})))] \\ &= \psi[e^{0}\psi(p(f(x_{n-2}), f(x_{n+m-2})))] \\ &= \psi^2(p(f(x_{n-2}), f(x_{n+m-2}))) \\ &\leq ... \leq \psi^n(p(f(x_0), f(x_m))) \leq \psi^n(\delta_p(X)), \end{aligned}$$

which implies that

$$(4) \qquad p(f(x_n), f(x_{n+m})) \leq \psi^n(\delta_p(X),$$

where $p(f(x_0), f(x_m))) \leq \delta_p(X)$.
But, X is p-bounded, therefore, $\delta_p(X) = \sup\{p(x, y)|x, y \in X\} < \infty$.
Since $\psi : \Re^+ \longrightarrow \Re^+$ is a nondecreasing function, then by using condition (ii) of inequality (1) in (4), we get

$$\lim_{n \longrightarrow \infty} \psi^n(\delta_p(X)) = 0$$

and hence,

$$p(f(x_n), f(x_{n+m})) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore, by using Lemma 1(c), we have that $\{f(x_n)\}_{n=0}^{\infty}$ is a p-Cauchy sequence. But X is S-complete. Hence, $\lim_{n\longrightarrow\infty} p(f(x_n),u)) = 0$, for some $u \in X$.
Since $x_n \in X$ implies that $f(x_{n-1}) = g(x_n)$, therefore, we have $\lim_{n\longrightarrow\infty} p(g(x_n),u)) = 0$.
Also, since f and g are p-continuous, then

$$\lim_{n\longrightarrow\infty} p(f(g(x_n)), f(u)) = \lim_{n\longrightarrow\infty} p(g(f(x_n)), g(u)) = 0.$$

But f and g are commuting, therefore $fg = gf$. Hence,

$$\lim_{n\longrightarrow\infty} p(f(g(x_n)), f(u)) = \lim_{n\longrightarrow\infty} p(f(g(x_n)), g(u)) = 0.$$

By applying Lemma 1(a), we have that $f(u) = g(u)$.
Since $f(u) = g(u)$ and $fg = gf$, then we have $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$.

We need to show that $p(f(u), f(f(u))) = 0$. Suppose on the contrary that $p(f(u), f(f(u))) \neq 0$. By using the contractive definition (2) and the condition that $\psi(t) < t, \ \ \forall\, t > 0$, we obtain

$$\begin{aligned} p(f(u), f(f(u))) &\leq e^{Lp(u,g(u))}\psi(p(g(u), g(f(u)))) \\ &= e^{Lp(f(u),f(u))}\psi(p(f(u), f(f(u)))) \\ &= e^{L(0)}\psi(p(f(u), f(f(u)))) \\ &= e^{0}\psi(p(f(u), f(f(u)))) \\ &= \psi(p(f(u), f(f(u)))) \\ &< p(f(u), f(f(u))), \end{aligned} \tag{5}$$

which is a contradiction. Hence, $p(f(u), f(f(u))) = 0$.
By using condition (ii) of the Theorem, we have $p(f(u), f(u)) = 0$.
Therefore, since $p(f(u), f(f(u))) = 0$ and $p(f(u), f(u)) = 0$, by using Lemma 1(a), we get $f(f(u)) = f(u)$, which implies that $f(u)$ is a fixed point of f.
But, $f(u) = f(f(u)) = f(g(u)) = g(f(u))$, which shows that $f(u)$ is also a fixed point of g. Thus, $f(u)$ is a common fixed point of f and g.
The proof of when f and g are $\tau(\Phi)$-continuous is similar since S-completeness implies p-Cauchy completeness.
This completes the proof.

Remark 3 *The existence result in Theorem 1 is a generalization of Theorem 3.1 of Aamri and El Moutawakil [1].*

The uniqueness of the common fixed point of f and g is established by the next two Theorems.

Theorem 2 *Let $(X, \Phi), f, g, \psi, \{x_n\}_{n=0}^{\infty}$ be as defined in Theorem 1 above and p an E-distance on X. Then, f and g have a unique common fixed point.*

Proof. Since an E-distance function p is also an A-distance, then by Theorem 1 above, we know that f and g have a common fixed point. Suppose that there exist $u, v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$.
We need to show that $u = v$. Suppose on the contrary that $u \neq v$, i.e. let $p(u, v) \neq 0$. Then, by using the contrative definition (2) and the condition that $\psi(t) < t, \ \forall\, t > 0$, we obtain

$$\begin{aligned} p(u,v) &= p(f(u), f(v)) \\ &\leq e^{Lp(u,g(u))}\psi(p(g(u), g(v))) \\ &= e^{Lp(u,u)}\psi(p(u,v)) \\ &= e^{L(0)}\psi(p(u,v)) \\ &= e^{0}\psi(p(u,v)) \\ &= \psi(p(u,v)) \\ &< p(u,v), \end{aligned} \tag{6}$$

which is a contradiction. Hence, we have $p(u, v) = 0$.
Similarly, we have $p(v, u) = 0$. By applying condition (p_2) of Definition 3, we obtain $p(u, u) \leq p(u, v) + p(v, u)$, and hence $p(u, u) = 0$.
Since $p(u, u) = 0$ and $p(u, v) = 0$, then by using Lemma 1(a), we get $u = v$.
This completes the proof.

Theorem 3 *Let $(X, \Phi), p, \psi$ and $\{x_n\}_{n=0}^{\infty}$ be as defined in Theorem 1 above. Suppose that f and g are p-compatible, p-continuous or $\tau(\Phi)$-continuous selfmappings of X satisfying conditions (i), (ii) and (iii) of Theorem 1 above. Then, f and g have a unique common fixed point.*

Proof. By Theorem 1 above, we know that f and g have a common fixed point. Hence, for some $u \in X$, we have $\lim_{n\longrightarrow\infty} p(f(x_n, u)) = \lim_{n\longrightarrow\infty} p(g(x_n, u)) = 0$. Since f and g are p-continuous, then

$$\lim_{n\longrightarrow\infty} p(f(g(x_n)), f(u)) = \lim_{n\longrightarrow\infty} p(g(f(x_n)), g(u)) = 0.$$

Also, since f and g are p-compatible, then

$$\lim_{n\longrightarrow\infty} p(f(g(x_n)), g(f(x_n))) = 0.$$

By applying condition (p_2) of Definition 3, we obtain

$$p(f(g(x_n)), g(u)) \leq p(f(g(x_n)), g(f(x_n))) + p(g(f(x_n)), g(u)).$$

Letting $n \longrightarrow \infty$ and using Lemma 1(a) yields

$$\lim_{n\longrightarrow\infty} p(f(g(x_n)), g(u)) = 0.$$

Since $\lim_{n\longrightarrow\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\longrightarrow\infty} p(f(g(x_n)), g(u)) = 0$, then by Lemma 1(a), we obtain $f(u) = g(u)$.
The rest of the proof is as in Theorem 1 and is therefore omitted.
This completes the proof.

Remark 4 *The uniqueness result in Theorem 3 is a generalization of Theorem 3.3 of Aamri and El Moutawakil [1].*

References

[1] M. Aamri, D. El Moutawakil, *Common Fixed Point Theorems for E-contractive or E-expansive Maps in Uniform Spaces*, Acta Math. Acad. Paedagog. Nyházi (N.S.), 20(1), 2004, 83-91.

[2] V. Berinde, *A priori and a posteriori Error Estimates for a Class of φ-contractions*, Bulletins for Applied and Computing Math., 1999, 183-192.

[3] V. Berinde, *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare, 2002.

[4] N. Bourbaki, *Éléments de mathématique. Fasc. II. Livre III: Topologie générale. Chapitre 1: Structures topologiques. Chapitre 2: Structures uniformes. Quatriéme édition.*, Actualités Scientifiques et Industrielles, No. 1142, Hermann, Paris, 1965.

[5] J. Jachymski, *Fixed Point Theorems for Expansive Mappings*, Math. Japon., 42(1), 1995, 131-136.

[6] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex Minimization Theorems and Fixed Point Theorems in Complete Metric Spaces*, Math. Japon., 44(2), 1996, 381-391.

[7] S. M. Kang, *Fixed Points for Expansion Mappings*, Math. Japon., 38(4), 1993, 713-717.

[8] M. O. Olatinwo, *Some Common Fixed Point Theorems for Selfmappings in Uniform Spaces*, Acta Math. Acad. Paedagog. Nyházi (N.S.), 23, 2007, 47-54.

[9] B. E. Rhoades, *A Comparison of Various Definitions of Contractive Mappings*, Trans. Amer. Math. Soc., 226, 1977, 257-290.

[10] J. Rodríguez-Montes, J. A. Charris, *Fixed Points for W-contractive or W-expansive Maps in Uniform Spaces: toward a unified approach*, Southwest J. Pure Appl. Math., 1, 2001, 93-101.

[11] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[12] I. A. Rus, A. Petrusel, G. Petrusel, *Fixed Point Theory: 1950-2000, Romanian Contributions*, House of the Book of Science, Cluj-Napoca, 2002.

[13] S. Z. Wang, B. Y. Li, Z. M. Gao, K. Iséki, *Some Fixed Point Theorems on Expansion Mappings*, Math. Japon., 29(4), 1984, 631-636.

[14] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. 1, Springer-Verlag, New York, 1986.

Alfred Olufemi Bosede
Lagos State University
Department of Mathematics
Ojo, Lagos State, Nigeria
e-mail: aolubosede@yahoo.co.uk