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A generalization of Ostrowski-Grüss type inequality for twice differentiable mappings in Euclidean norm[1]

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Abstract

In this paper, we improve and further generalize Ostrowski-Grüss type inequality involving twice differentiable functions. Some applications for probability density function and generalized beta random variable are also given.

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1 Introduction

In 1938, Ostrowski [12] presented Ostrowski inequality for differentiable mappings with bounded derivatives. Since then there is an upsurge of obtaining sharp bounds of this inequality in terms of variety of Lebesgue spaces involving, at most, the first derivative which results in obtaining some new inequalities of Ostrowski type, for example, Ostrowski-Grüss type, Ostrowski-Čebyšev type, etc. The key role in obtaining these inequalities

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has been played by Grüss inequality, Čebyšev's functional and pre- Grüss inequality.

In 1997, Dragomir and Wang [5], by the use of the Grüss inequality proved the following Ostrowski- Grüss type integral inequality.

Theorem 1.1. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I^0 of I, and let $a, b \in I^0$ with $a < b$. If $\gamma \leq f' \leq \Gamma$, $x \in [a, b]$ for some constants $\gamma, \Gamma \in \mathbb{R}$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \leq \frac{1}{4}(b-a)(\Gamma-\gamma), \tag{1.1}$$

for all $x \in [a, b]$.

This inequality provides a connection between Ostrowski inequality [12] and the Grüss inequality [6].

The inequality (1.1) has been further extended by P. Cerone, S. S. Dragomir and J. Roumeliotis [2] for twice differentiable mappings as follows:

Theorem 1.2. *Let $f : I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval. Suppose that f is twice differentiable in the interior I^0 of I, and let $a, b \in I^0$ with $a < b$. If*

$$\gamma \leq f''(x) \leq \Gamma,$$

for some constants $\gamma, \Gamma \in \mathbb{R}$, then

$$\begin{aligned} &\left| f(x) + \left(\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right)\frac{f'(b)-f'(a)}{b-a} \right. \\ &\left. - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq\ & \frac{1}{8}(\Gamma-\gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2, \end{aligned} \tag{1.2}$$

for all $x \in [a,b]$.

For a generalization of (1.2), see [13] by A. Rafiq, N. A. Mir and Fiza Zafar.

In 2000, M. Matić, J. Pečarić and N. Ujević [9], by the use of pre-Grüss inequality improved Theorem 1.2 as follows:

Theorem 1.3. *Let the assumptions of Theorem 1.2 hold, then for all* $x \in [a,b]$, *we have*

$$\begin{aligned} &\left| f(x) + \left(\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right) \frac{f'(b) - f'(a)}{b-a} \right. \\ &\left. - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\leq \frac{(\Gamma - \gamma)\, l}{6\sqrt{5}} \sqrt{l^2 + 15\xi^2}, \end{aligned} \tag{1.3}$$

where

$$l = \frac{b-a}{2} \text{ and } \xi = x - \frac{a+b}{2}.$$

This result has been further improved by X. L. Cheng in [3] as follows:

Theorem 1.4. *Let the assumptions of Theorem 1.3 hold. Then for all* $x \in [a,b]$, *we have*

$$\begin{aligned} &\left| f(x) + \left(\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right) \frac{f'(b) - f'(a)}{b-a} \right. \\ &\left. - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\leq (\Gamma_2 - \gamma_2)\, G(a,b,x), \end{aligned} \tag{1.4}$$

where

$$G(a,b,x)=\begin{cases}\frac{1}{3(b-a)}\left(\left|(x-a)\left(x-\frac{a+b}{2}\right)(b-x)\right|\right.\\ \left.+\left(\frac{1}{12}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right)^{\frac{3}{2}}\right), & a\le x\le\frac{1}{3}(2a+b),\\ & \frac{1}{3}(a+2b)\le x\le b,\\ \frac{2}{3(b-a)}\left(\frac{1}{12}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right)^{\frac{3}{2}}, & \frac{1}{3}(2a+b)\le x\le\frac{1}{3}(a+2b).\end{cases}$$

Further, in [9] we can find the special cases of (1.3) i.e., midpoint and trapezoid inequalities in the form of following corollary:

Corollary 1.1. *Let the assumptions of Theorem 1.3 hold. Then*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{24}(b-a)\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\le\frac{1}{24\sqrt{5}}(\Gamma-\gamma)(b-a)^2. \tag{1.5}$$

Also,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{12}(b-a)\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\le\frac{1}{6\sqrt{5}}(\Gamma-\gamma)(b-a)^2. \tag{1.6}$$

Moreover, in [14], a sharp Simpson's inequality for absolutely continuous functions with derivatives, which belong to $L_2(a,b)$ was given as follows:

Theorem 1.5. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function, whose derivative $f'\in L_2(a,b)$. Then*

$$\left|\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(t)\,dt\right|$$

$$\le\frac{(b-a)^{\frac{3}{2}}}{6}\left[\left\|f'\right\|_2^2-\frac{(f(b)-f(a))^2}{b-a}\right]^{\frac{1}{2}}. \tag{1.7}$$

The inequality is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.

We know that for two mappings $f, g : [a, b] \to \mathbb{R}$, the Čebyšev functional is defined as

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt,$$

provided that f, g and fg are integrable on $[a, b]$.

Also in [9], we can find the pre-Grüss inequality as

$$T^2(f,g) \le T(f,f)\,T(g,g),$$

where $f, g \in L_2[a, b]$ and $T(f, g)$ is the Čebyšev's functional as defined above.

Moreover, we will use the Korkine's identity (see [8] & [10, p. 296]) which is defined as

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt \\ = \;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dtds, \end{aligned}$$

provided that $f, g : [a, b] \to \mathbb{R}$ are measurable and all the involved integrals exists.

In this paper, we improve and further generalize, by the use of Čebyšev's functional, the Matić et al. [9] results by providing first membership of the right side of (1.3) in terms of Euclidean norm. The bound in (1.3) is given in terms of functions whose derivatives are bounded whereas the right membership of the new inequality is in terms of larger class of absolutely continuous functions whose second derivative $f'' \in L_2(a, b)$ which enlarges the applicability of the underlying quadrature rules. Some applications for probability density function and generalized beta random variable are also given.

2 Main Results

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping whose first derivative is absolutely continuous and the second derivative $f'' \in L_2(a, b)$. Then we have the inequality*

$$\begin{aligned}
&\Bigg|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right]+h\frac{f(a)+f(b)}{2} \\
&-\left[\frac{1}{24}(3h-1)(b-a)^2-\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2\right]\frac{f'(b)-f'(a)}{b-a} \\
&-\frac{1}{b-a}\int_a^b f(t)\,dt\Bigg| \\
\leq\ & (b-a)^2\left[\frac{1}{2880}\left(4-15h+15h^2\right)+\frac{1}{24}(2-3h)(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right. \\
&\left.+\frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\left\|f''\right\|_2^2-\left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{2}(\Gamma-\gamma)(b-a)^2\left[\frac{1}{2880}\left(4-15h+15h^2\right)\right. \\
&\left.+\frac{1}{24}(2-3h)(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}},
\end{aligned}$$

(2.1) $\quad$ *if* $\gamma \leq f''(t) \leq \Gamma$, *almost everywhere* t *on* (a, b),

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ *and* $h \in [0, 1]$.

Proof. We defined in [13], the following kernel $K : [a, b]^2 \to \mathbb{R}$

$$K(x,t)=\begin{cases}\frac{1}{2}\left(t-\left(a+h\frac{b-a}{2}\right)\right)^2, & \text{if } t\in[a,x] \\ \frac{1}{2}\left(t-\left(b-h\frac{b-a}{2}\right)\right)^2, & \text{if } t\in(x,b].\end{cases}$$

Using Korkine's identity for K and f'', we obtain

$$\frac{1}{b-a}\int_a^b K(x,t)\, f''(t)\, dt - \frac{1}{b-a}\int_a^b K(x,t)\, dt \frac{1}{b-a}\int_a^b f''(t)\, dt$$

$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))\left(f''(t)-f''(s)\right) dtds, \tag{2.2}$$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ and $h \in [0,1]$.Further in [13], we have developed the following identities:

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K(x,t)\, f''(t)\, dt \\
&= \frac{1}{b-a}\int_a^b f(t)dt - \frac{h}{2}(f(a)+f(b)) - (1-h)\left[f(x) - \left(x-\frac{a+b}{2}\right) f'(x)\right] \\
&\quad + \frac{1}{8}h^2(b-a)\left(f'(b)-f'(a)\right),
\end{aligned}$$

$$\frac{1}{b-a}\int_a^b K(x,t)dt = \frac{1}{24}\left(3h^2-3h+1\right)(b-a)^2 + \frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2,$$

and

$$\frac{1}{b-a}\int_a^b f''(t)\, dt = \frac{f'(b)-f'(a)}{b-a}.$$

Then, by (2.2) we get

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)dt - (1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right) f'(x)\right] - \frac{h}{2}(f(a)+f(b)) \\
&- \left[\frac{1}{24}(3h-1)(b-a)^2 - \frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2\right]\frac{f'(b)-f'(a)}{b-a} \\
&= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))\left(f''(t)-f''(s)\right) dtds, \qquad (2.3)
\end{aligned}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0,1]$.

Using the Cauchy-Bunyakowski-Schwartz inequality for double integrals, we may write

$$\left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))\left(f''(t)-f''(s)\right)dtds\right| \tag{2.4}$$
$$\leq \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))^2\,dtds\right)^{\frac{1}{2}}$$
$$\times\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f''(t)-f''(s)\right)^2 dtds\right)^{\frac{1}{2}}.$$

However,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (K(x,t)-K(x,s))^2\,dtds \tag{2.5}$$
$$= \frac{1}{b-a}\int_a^b K^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b K(x,t)\,dt\right)^2,$$

$$\left(\frac{1}{b-a}\int_a^b K(x,t)\,dt\right)^2 \tag{2.6}$$
$$= (b-a)^4\left[\frac{1}{576}\left(1-6h+15h^2-18h^3+9h^4\right)\right.$$
$$\left.+\frac{1}{24}\left(1-4h+6h^2-3h^3\right)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{4}(1-h)^2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right],$$

and

$$\begin{aligned} & \frac{1}{b-a}\int_a^b K^2(x,t)\,dt \\ = & \frac{1}{20(b-a)}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^5+\left(b-h\frac{b-a}{2}-x\right)^5\right. \\ & \left.+\frac{1}{16}h^5(b-a)^5\right]. \end{aligned}$$

Taking $t = x - \frac{a+b}{2}$, we have

$$\begin{aligned} x-\left(a+h\frac{b-a}{2}\right) &= t+\frac{1}{2}(1-h)(b-a), \\ b-h\frac{b-a}{2}-x &= \frac{1}{2}(1-h)(b-a)-t. \end{aligned}$$

Thus,

$$\begin{aligned} & \left(x-\left(a+h\frac{b-a}{2}\right)\right)^5+\left(b-h\frac{b-a}{2}-x\right)^5 \\ = & \left(t+\frac{1}{2}(1-h)(b-a)\right)^5+\left(\frac{1}{2}(1-h)(b-a)-t\right)^5. \end{aligned}$$

For real numbers A and B, we have

$$A^5+B^5=(A+B)\left[\left(A^2+B^2\right)^2-(AB)^2-AB\left(A^2+B^2\right)\right].$$

Now, if $A = t + \frac{1}{2}(1-h)(b-a)$, $B = \frac{1}{2}(1-h)(b-a) - t$, then

$$\begin{aligned} A^2+B^2 &= \left(t+\frac{1}{2}(1-h)(b-a)\right)^2+\left(\frac{1}{2}(1-h)(b-a)-t\right)^2 \\ &= 2t^2+\frac{(1-h)^2(b-a)^2}{2}, \\ AB &= \frac{1}{4}(1-h)^2(b-a)^2-t^2, \\ A+B &= (1-h)(b-a). \end{aligned}$$

Thus,

$$\left(x-\left(a+h\frac{b-a}{2}\right)\right)^5+\left(b-h\frac{b-a}{2}-x\right)^5$$
$$= 5\left(1-h\right)\left(b-a\right)^5\left[\frac{1}{80}\left(1-h\right)^4+\frac{1}{2}\left(1-h\right)^2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right].$$

Therefore,

$$\frac{1}{b-a}\int_a^b K^2\left(x,t\right)dt \tag{2.7}$$
$$= \frac{1}{4}\left(b-a\right)^4\left[\frac{1}{80}\left(1-5h+10h^2-10h^3+5h^4\right)\right.$$
$$\left.+\frac{1}{2}\left(1-h\right)^3\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\left(1-h\right)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right].$$

Using (2.6) and (2.7) in (2.5), we get

$$\frac{1}{2\left(b-a\right)^2}\int_a^b\int_a^b\left(K\left(x,t\right)-K\left(x,s\right)\right)^2dtds \tag{2.8}$$
$$= \left(b-a\right)^4\left[\frac{1}{2880}\left(4-15h+15h^2\right)+\frac{1}{24}\left(2-5h+3h^2\right)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right.$$
$$\left.+\frac{1}{4}h\left(1-h\right)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right].$$

Moreover,

$$\frac{1}{2\left(b-a\right)^2}\int_a^b\int_a^b\left(f''\left(t\right)-f''\left(s\right)\right)^2dtds \tag{2.9}$$
$$= \frac{1}{b-a}\left\|f''\right\|_2^2-\left(\frac{f'\left(b\right)-f'\left(a\right)}{b-a}\right)^2.$$

Using $(2.3-2.5, 2.8-2.9)$,we deduce the first inequality.

Moreover, if $\gamma \leq f''(t) \leq \Gamma$ almost everywhere t on (a,b), then, by using Grüss inequality, we have

$$0 \leq \frac{1}{b-a}\int_a^b f''^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b f''(t)\,dt\right)^2 \leq \frac{1}{4}(\Gamma-\gamma)^2,$$

which proves the last inequality of (2.1).

Remark 2.1. *(i) We can get the best estimation from (2.1), only when* $x = \frac{a+b}{2}$ *i.e.,*

$$\left|(1-h) f\left(\frac{a+b}{2}\right) + h\frac{f(a)+f(b)}{2} - \frac{1}{24}(3h-1)(b-a)^2 \frac{f'(b)-f'(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{1}{24\sqrt{5}}(b-a)^2\left(4-15h+15h^2\right)^{\frac{1}{2}}\left[\frac{1}{b-a}\left\|f''\right\|_2^2 - \left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{48\sqrt{5}}(\Gamma-\gamma)(b-a)^2\left(4-15h+15h^2\right)^{\frac{1}{2}},$$

$$\text{(2.10)} \qquad if\ \gamma \leq f''(t) \leq \Gamma,\ \text{almost everywhere } t \text{ on } (a,b).$$

As

$$4 - 15h + 15h^2 \leq 4,\ \forall\ h \in [0,1].$$

and is minimum for $h = \frac{1}{2}$, *implies*

$$\frac{1}{48\sqrt{5}}\left(4-15h+15h^2\right)^{\frac{1}{2}} \leq \frac{1}{24\sqrt{5}}.$$

Thus (2.1) shows an overall improvement of the inequality obtained by Matić et al. [9].

(ii) *For* $h = 1$*, i.e.,* $x = \frac{a+b}{2}$*, (2.1) gives*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{12}(b-a)\left(f'(b) - f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

$$\leq \frac{1}{12\sqrt{5}}(b-a)^2\left[\frac{1}{b-a}\|f''\|_2^2 - \left(\frac{f'(b) - f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{24\sqrt{5}}(\Gamma - \gamma)(b-a)^2,$$

(2.11) *if* $\gamma \leq f''(t) \leq \Gamma$, *almost everywhere* t *on* (a, b),

which is perturbed trapezoid inequality (corrected trapezoid rule)and it is not difficult to see that it is better than the simple trapezoid inequality.

(iii) *For* $h = 0$ *and* $x = \frac{a+b}{2}$*, (2.1) gives*

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{24}(b-a)\left(f'(b) - f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

$$\leq \frac{1}{12\sqrt{5}}(b-a)^2\left[\frac{1}{b-a}\left\|f''\right\|_2^2 - \left(\frac{f'(b) - f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{24\sqrt{5}}(\Gamma - \gamma)(b-a)^2,$$

(2.12) *if* $\gamma \leq f''(t) \leq \Gamma$, *almost everywhere* t *on* (a, b).

which is perturbed mid-point inequality.

(iv) *For* $h = \frac{1}{2}$ *and* $x = \frac{a+b}{2}$*, (2.1) gives*

$$\left| \frac{f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)}{4} - \frac{1}{48}(b-a)\left(f'(b) - f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

$$\leq \frac{1}{48\sqrt{5}}(b-a)^2\left[\frac{1}{b-a}\left\|f''\right\|_2^2 - \left(\frac{f'(b) - f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

$$\leq \frac{1}{96\sqrt{5}}(\Gamma-\gamma)(b-a)^2,$$

$$\text{(2.13)} \qquad if \quad \gamma \leq f''(t) \leq \Gamma, \text{ almost everywhere } t \text{ on } (a,b).$$

which is a linear combination of Trapezoid and Mid-point rule.

(v) *For* $h=\frac{1}{3}$ *and* $x=\frac{a+b}{2}$, *(2.1) gives*

$$\left|\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{1}{24\sqrt{30}}(b-a)^2\left[\frac{1}{b-a}\left\|f''\right\|_2^2-\left(\frac{f'(b)-f'(a)}{b-a}\right)^2\right]^{\frac{1}{2}},$$

$$\leq \frac{1}{48\sqrt{30}}(\Gamma-\gamma)(b-a)^2,$$

$$\text{(2.14)} \qquad if \quad \gamma \leq f''(t) \leq \Gamma, \text{ almost everywhere } t \text{ on } (a,b).$$

which is a variant of Simpson's inequality for twice differentiable function f, f'' *is integrable and there exist constants* $\gamma,\Gamma\in\mathbb{R}$ *such that* $\gamma\leq f''(t)\leq\Gamma$, $t\in(a,b)$.

The estimations (2.10), (2.11), (2.12), (2.13) *and* (2.14) *are expressed in terms of second derivative of the integrand which are useful when the higher derivatives of* f *does not exist or are very large at some points in the domain. Moreover, the three point quadrature rule* (2.13) *which is a linear combination of Trapezoid and Mid-point rule, offers better estimations than the simple three point Simpson's rule* (2.14).

Remark 2.2. *In [9], the result corresponding to* (2.11) *was given, but with* $\frac{1}{6\sqrt{5}}$ *in place of our factor* $\frac{1}{24\sqrt{5}}$ *showing an improvement of factor* $\frac{1}{4}$ *as it can be seen from* (1.6). *Also in [3],* (2.11) *was given with a factor of* $\frac{1}{18\sqrt{3}}$ *which shows that* (2.11) *also offers better estimation than as given in [3]. Moreover, we have also been able to present bounds for three point quadrature rules as given in* (2.13) *and* (2.14) *where* (2.14) *is a extension of* (1.7) *for twice differentiable mappings.*

3 Applications

3.1 Application in Numerical integration

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$ and $h_i = x_{i+1} - x_i = h = \frac{(b-a)}{n}$, $i = 0, \cdots, n-1$, then we have the following quadrature formula.

Theorem 3.1. *Let I_n be the subdivision of the interval $[a, b]$ and let the assumptions of Theorem 2.1 hold. Then,*

$$\left| \int_a^b f(t)\,dt - (1-\delta) h \sum_{i=0}^{n-1} \left[f(\xi_i) - \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) \right] \right.$$
$$+ \sum_{i=0}^{n-1} \left[\frac{1}{24} h^2 (3\delta - 1) - \frac{1}{2}(1-\delta) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \left(f'(x_{i+1}) - f'(x_i) \right)$$
$$\left. -\delta \frac{h}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] \right|$$
$$\leq \left(\frac{b-a}{n} \right)^2 \left[\frac{1}{2880} \left(4 - 15\delta + 15\delta^2 \right) + \right.$$
$$+ \frac{1}{24} (2 - 3\delta)(1-\delta) \sum_{i=0}^{n-1} \left(\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h} \right)^2$$
$$\left. + \frac{1}{4} \delta (1-\delta) \sum_{i=0}^{n-1} \left(\frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h} \right)^4 \right]^{\frac{1}{2}}$$
$$\left[\frac{b-a}{n} \left\| f'' \right\|_2^2 - \sum_{i=0}^{n-1} \left(f'(x_{i+1}) - f'(x_i) \right)^2 \right]^{\frac{1}{2}}.$$

Proof. Apply inequality (2.1) on the interval $[x_i, x_{i+1}]$, $i = 0, \cdots, n-1$ to get,

$$\left| \int_{x_i}^{x_{i+1}} f(t)\,dt - (1-\delta) h \left[f(\xi_i) - \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) \right] \right.$$

$$+\left[\frac{1}{24}h^{2}\left(3\delta-1\right)-\frac{1}{2}\left(1-\delta\right)\left(\xi_{i}-\frac{x_{i}+x_{i+1}}{2}\right)^{2}\right]\left(f^{'}\left(x_{i+1}\right)-f^{'}\left(x_{i}\right)\right)$$

$$\left.-\delta\frac{h}{2}\left[f\left(x_{i}\right)+f\left(x_{i+1}\right)\right]\right|$$

$$\leq h^{\frac{5}{2}}\left[\frac{1}{2880}\left(4-15\delta+15\delta^{2}\right)+\frac{1}{24}\left(2-3\delta\right)\left(1-\delta\right)\left(\frac{\xi_{i}-\frac{x_{i}+x_{i+1}}{2}}{h}\right)^{2}\right.$$

$$\left.+\frac{1}{4}\delta\left(1-\delta\right)\left(\frac{\xi_{i}-\frac{x_{i}+x_{i+1}}{2}}{h}\right)^{4}\right]^{\frac{1}{2}}\left[\int_{x_{i}}^{x_{i+1}}\left(f^{''}\left(t\right)\right)^{2}dt-\frac{\left(f^{'}\left(x_{i+1}\right)-f^{'}\left(x_{i}\right)\right)^{2}}{h}\right]^{\frac{1}{2}},$$

for all $i=0,\cdots,n-1$.

Summing over i from 0 to $n-1$, using triangular inequality and Cauchy-Schwartz discrete inequality, we get,

$$|R(f,f',I_{n},\xi,\delta)|$$

$$\leq\sum_{i=0}^{n-1}\left|\int_{x_{i}}^{x_{i+1}}f\left(t\right)dt-\left(1-\delta\right)h\left[f\left(\xi_{i}\right)-\left(\xi_{i}-\frac{x_{i}+x_{i+1}}{2}\right)f^{'}\left(\xi_{i}\right)\right]\right.$$

$$+\left[\frac{1}{24}h^{2}\left(3\delta-1\right)-\frac{1}{2}\left(1-\delta\right)\left(\xi_{i}-\frac{x_{i}+x_{i+1}}{2}\right)^{2}\right]\left(f^{'}\left(x_{i+1}\right)-f^{'}\left(x_{i}\right)\right)$$

$$\left.-\delta\frac{h}{2}\left[f\left(x_{i}\right)+f\left(x_{i+1}\right)\right]\right|$$

$$\leq h^{\frac{5}{2}}\left(\sum_{i=0}^{n-1}\left(\left[\frac{1}{2880}\left(4-15\delta+15\delta^{2}\right)+\frac{1}{24}\left(2-3\delta\right)\left(1-\delta\right)\left(\frac{\xi_{i}-\frac{x_{i}+x_{i+1}}{2}}{h}\right)^{2}\right.\right.\right.$$

$$\left.\left.\left.+\frac{1}{4}\delta\left(1-\delta\right)\left(\frac{\xi_{i}-\frac{x_{i}+x_{i+1}}{2}}{h}\right)^{4}\right]^{\frac{1}{2}}\right)^{2}\right)^{\frac{1}{2}}.$$

$$\cdot\left(\sum_{i=0}^{n-1}\left(\left[\int_{x_i}^{x_{i+1}}\left(f''(t)\right)^2dt-\frac{\left(f'(x_{i+1})-f'(x_i)\right)^2}{h}\right]^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}$$

$$\leq h^2\left[\frac{1}{2880}\left(4-15\delta+15\delta^2\right)+\frac{1}{24}(2-3\delta)(1-\delta)\sum_{i=0}^{n-1}\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^2\right.$$

$$\left.+\frac{1}{4}\delta(1-\delta)\sum_{i=0}^{n-1}\left(\frac{\xi_i-\frac{x_i+x_{i+1}}{2}}{h}\right)^4\right]^{\frac{1}{2}}\left[h\left\|f''\right\|_2^2-\sum_{i=0}^{n-1}\left(f'(x_{i+1})-f'(x_i)\right)^2\right]^{\frac{1}{2}}.$$

Thus, we get the required result.

Remark 3.1. *Note that if we choose $\delta=\frac{1}{2}$, $\xi_i=\frac{x_i+x_{i+1}}{2}$, then we get the quadrature rule which is a linear combination of midpoint rule and trapezoid rule and it offers the best estimate.*

3.2 Application for Probability Density Functions

Let X be a random variable having the p.d.f $f:[a,b]\to\mathbb{R}_+$ and the cumulative distribution function $F:[a,b]\to[0,1]$, i.e.,

$$F(x)=\int_a^x f(t)\,dt,\ x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]\subseteq[a,b].$$

Then, we may have the following:

Theorem 3.2. *Under the above assumptions and if the probability density* *function f belongs to $L_2[a,b]$, then we have the inequality*

$$\left|(1-h)\left[F(x)-\left(x-\frac{a+b}{2}\right)f(x)\right]+\frac{h}{2}-\frac{b-E(X)}{b-a}\right.$$
$$\left.-\left[\frac{1}{24}(3h-1)(b-a)^2-\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2\right]\frac{f(b)-f(a)}{b-a}\right|$$
$$\leq (b-a)^2\left[\frac{1}{2880}\left(4-15h+15h^2\right)+\frac{1}{24}(2-3h)(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right.$$
$$\left.+\frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{(b-a)^2(M-m)}{2}\left[\frac{1}{2880}\left(4-15h+15h^2\right)+\right.$$
$$\left.+\frac{1}{24}\left(2-5h+3h^2\right)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{4}h(1-h)\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^4\right]^{\frac{1}{2}},$$

(3.1) $\qquad$ *if* $m \leq f' \leq M$, *almost everywhere on* $[a,b]$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ *and* $h \in [0,1]$.

Proof. Put in (2.1), F instead of f to get (3.2) and the details are omitted.

Corollary 3.1. *Under the above assumptions, we have*

$$\left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{b-E(X)}{b-a}-\frac{(3h-1)}{24}(b-a)(f(b)-f(a))\right|$$
$$\leq \frac{1}{24\sqrt{5}}\left(4-15h+15h^2\right)^{\frac{1}{2}}(b-a)^2\left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{48\sqrt{5}}(M-m)\left(4-15h+15h^2\right)^{\frac{1}{2}}(b-a)^2,$$

(3.2) $\qquad$ *if* $m \leq f' \leq M$, *almost everywhere on* $[a,b]$.

3.3 Application for generalized beta random variable

If X is a beta random variable with parameters $\beta_3 > -1$, $\beta_4 > -1$ and for $\beta_2 > 0$ and any β_1, the generalized beta random variable

$$Y = \beta_1 + \beta_2 X,$$

is said to have a generalized beta distribution [7] and the probability density function of the generalized beta distribution of beta random variable is,

$$f(x) = \begin{cases} \frac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}}, & \text{for } \beta_1 < x < \beta_1+\beta_2 \\ 0, & \text{otherwise,} \end{cases},$$

where $\beta(l,m)$ is the beta function with $l,\ m > 0$ and is defined as

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

For p, $q > 0$ and $h \in [0,1)$, we choose,

$$\begin{aligned} \beta_1 &= \frac{h}{2}, \\ \beta_2 &= (1-h), \\ \beta_3 &= p-1, \\ \beta_4 &= q-1. \end{aligned}$$

Then, the probability density function associated with generalized beta random variable

$$Y = \frac{h}{2} + (1-h)X,$$

takes the form

$$f(x) = \begin{cases} \frac{\left(x-\frac{h}{2}\right)^{p-1}\left(1-\frac{h}{2}-x\right)^{q-1}}{\beta(p,q)(1-h)^{p+q-1}}, & \frac{h}{2} < x < 1-\frac{h}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Now,

$$E(Y) = \int_{\frac{h}{2}}^{1-\frac{h}{2}} x f(x)\, dx = (1-h)\frac{p}{p+q} + \frac{h}{2}.$$

$$\frac{df(x;p,q)}{dx} = \frac{\left(x-\frac{h}{2}\right)^{p-2}\left(1-\frac{h}{2}-x\right)^{q-2}}{(1-h)^{p+q-1}\beta(p,q)} \times \left[(p-1)-(p-q)\frac{h}{2}-(p+q-2)x\right],$$

and

$$\left\|f'(.;p,q)\right\|_2^2 = \frac{1}{(1-h)^3\beta^2(p,q)}\left[(p-1)^2\beta(2p-3,2q-1) + (q-1)^2\beta(2p-1,2q-3) - 2(p-1)(q-1)\beta(2p-2,2q-2)\right].$$

Then, by Theorem 3.2, we may state the following.

Proposition 3.1. *Let X be a beta random variable with parameters (p,q). Then, for generalized beta random variable*

$$Y = \frac{h}{2} + (1-h)X,$$

we have the inequality

$$\begin{aligned}
&\Bigg|(1-h)\left[\Pr(Y\le x)-\left(x-\frac{1}{2}\right)f(x)-\frac{q}{p+q}\right]\\
&-\left[\frac{1}{24}(3h-1)-\frac{1}{2}(1-h)\left(x-\frac{1}{2}\right)^2\right](f(1)-f(0))\Bigg|\\
\le\ &\frac{1}{(1-h)^{\frac{3}{2}}\beta(p,q)}\Bigg[\frac{1}{2880}\left(4-15h+15h^2\right)+\\
&\frac{1}{24}(2-3h)(1-h)\left(x-\frac{1}{2}\right)^2+\frac{1}{4}h(1-h)\left(x-\frac{1}{2}\right)^4\Bigg]^{\frac{1}{2}}\times\\
&\big[(p-1)^2\beta(2p-3,2q-1)+(q-1)^2\beta(2p-1,2q-3)\\
&-2(p-1)(q-1)\beta(2p-2,2q-2)\\
&-(1-h)^3\beta^2(p,q)(f(1)-f(0))^2\big]^{\frac{1}{2}},
\end{aligned}\tag{3.3}$$

for all $x\in\left[\frac{h}{2},1-\frac{h}{2}\right]$.

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