

About a class of linear positive operators¹

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Abstract

In this paper we construct a class of linear positive operators $(L_m)_{m \geq 1}$ with the help of some nodes. We study the convergence and we demonstrate the Voronovskaja-type theorem for them. By particularization, we obtain some known operators.

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1 Introduction

In this section, we recall some notions and operators which we will use in this article.

Let \mathbb{N} be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let $p_{m,k}(x)$ the fundamental polynomials of Bernstein, defined as follows

$$(1.1) \quad p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k},$$

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for any $x \in [0, 1]$ and any $k \in \{0, 1, \dots, m\}$ (see [5] or [21]). For the following construction see [15]. Define the natural number m_0 by

$$(1.2) \quad m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{if } \beta \in \mathbb{R} \setminus \mathbb{Z} \\ \max\{1, 1 - \beta\}, & \text{if } \beta \in \mathbb{Z}. \end{cases}$$

For the real number β , we have that

$$(1.3) \quad m + \beta \geq \gamma_\beta$$

for any natural number m , $m \geq m_0$, where

$$(1.4) \quad \gamma_\beta = m_0 + \beta = \begin{cases} \max\{1 + \beta, \{\beta\}\}, & \text{if } \beta \in \mathbb{R} \setminus \mathbb{Z} \\ \max\{1 + \beta, 1\}, & \text{if } \beta \in \mathbb{Z}. \end{cases}$$

For the real numbers α, β , $\alpha \geq 0$, we note

$$(1.5) \quad \mu^{(\alpha, \beta)} = \begin{cases} 1, & \text{if } \alpha \leq \beta \\ 1 + \frac{\alpha - \beta}{\gamma_\beta}, & \text{if } \alpha > \beta. \end{cases}$$

For the real numbers α and β , $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha, \beta)}$ and

$$(1.6) \quad 0 \leq \frac{k + \alpha}{m + \beta} \leq \mu^{(\alpha, \beta)}$$

for any natural number m , $m \geq m_0$ and for any $k \in \{0, 1, \dots, m\}$.

For the real numbers α and β , $\alpha \geq 0$, m_0 and $\mu^{(\alpha, \beta)}$ defined by (1.2)-(1.6), let the operators $P_m^{(\alpha, \beta)} : C([0, \mu^{(\alpha, \beta)}]) \rightarrow C([0, 1])$, defined for any function $f \in C([0, \mu^{(\alpha, \beta)}])$ by

$$(1.7) \quad (P_m^{(\alpha, \beta)} f)(x) = \sum_{k=0}^m p_{m,k}(x) f\left(\frac{k + \alpha}{m + \beta}\right),$$

for any natural number m , $m \geq m_0$ and for any $x \in [0, 1]$. These operators are named Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [20]. In [20], the domain of definition of the Stancu operators is $C([0, 1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

Remark 1.1. For $\alpha = \beta = 0$ we obtain the Bernstein operators.

Remark 1.2. For $\alpha = 0$, $p \in \mathbb{N}_0$ and choosing m by $m + p$ and p by $m - p$, we obtain the Schurer operators.

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [4] a sequence of linear positive operators $(L_m)_{m \geq 1}$, $L_m : C_B([0, \infty)) \rightarrow C_B([0, \infty))$, defined for any function $f \in C_B([0, \infty))$ by

$$(1.8) \quad (L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^m \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right),$$

for any $x \in [0, \infty)$ and any $m \in \mathbb{N}$, where $C_B([0, \infty)) = \{f \mid f : [0, \infty) \rightarrow \mathbb{R}, f \text{ bounded and continuous on } [0, \infty)\}$.

For $m \in \mathbb{N}$ consider the operators $S_m : C_2([0, \infty)) \rightarrow C([0, \infty))$ defined for any function $f \in C_2([0, \infty))$ by

$$(1.9) \quad (S_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right),$$

for any $x \in [0, \infty)$, where $C_2([0, \infty)) = \left\{f \in C([0, \infty)) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}$.

The operators $(S_m)_{m \geq 1}$ are named Mirakjan-Favard-Szász operators and were introduced in 1941 by G. M. Mirakjan in [11].

They were intensively studied by J. Favard in 1944 in [8] and O. Szász in 1950 in [22].

Let for $m \in \mathbb{N}$ the operators $V_m : C_2([0, \infty)) \rightarrow C([0, \infty))$ be defined for any function $f \in C_2([0, \infty))$ by

$$(1.10) \quad (V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right),$$

for any $x \in [0, \infty)$.

The operators $(V_m)_{m \geq 1}$ are named Baskakov operators and they were introduced in 1957 by V. A. Baskakov in [2].

W. Meyer-König and K. Zeller have introduced in [10] a sequence of linear and positive operators. After a slight adjustment given by E. W. Cheney and A. Sharma in [6], these operators take the form $Z_m : B([0, 1]) \rightarrow C([0, 1])$, defined for any function $f \in B([0, 1])$ by

$$(1.11) \quad (Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right),$$

for any $m \in \mathbb{N}$ and for any $x \in [0, 1)$.

These operators are named the Meyer-König and Zeller operators.

Observe that $Z_m : C([0, 1]) \rightarrow C([0, 1])$, $m \in \mathbb{N}$.

In the paper [9], M. Ismail and C. P. May consider the operators $(R_m)_{m \geq 1}$.

For $m \in \mathbb{N}$, $R_m : C([0, \infty)) \rightarrow C([0, \infty))$ is defined for any function $f \in C([0, \infty))$ by

$$(1.12) \quad (R_m f)(x) = e^{-\frac{mx}{1+x}} \sum_{k=0}^{\infty} \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{m}\right)$$

for any $x \in [0, \infty)$.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the following functions sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I , $B(I) = \{f \mid f : I \rightarrow \mathbb{R}, f \text{ bounded on } I\}$, $C(I) = \{f \mid f : I \rightarrow \mathbb{R}, f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$, consider the function $\psi_x : I \rightarrow \mathbb{R}$ defined by $\psi_x(t) = t - x$, for any $t \in I$.

2 Preliminaries

The following construction is about the idea from [15]. Let I, J be real intervals with $I \cap J \neq \emptyset$ and $p_m = m$ for any $m \in \mathbb{N}$ (the finite case) or $p_m = \infty$ for any $m \in \mathbb{N}$ (the infinite case). For any $m \in \mathbb{N}$ and $k \in \{0, 1, \dots, p_m\} \cap \mathbb{N}_0$, consider the nodes $x_{m,k} \in I$ and the functions $\varphi_{m,k} : J \rightarrow \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$, for any $x \in J$. We suppose that for any compact $K \subset I \cap J$ there exists the sequence $(u_m(K))_{m \geq 1}$, depending on K such that

$$(2.1) \quad \lim_{m \rightarrow \infty} u_m(K) = 0$$

uniformly on K and

$$(2.2) \quad \left| \sum_{k=0}^{p_m} \varphi_{m,k}(x) - 1 \right| \leq u_m(K)$$

for any $x \in K$, any $m \in \mathbb{N}$ and we note $u(K) = \sup\{u_m(K) : m \in \mathbb{N}\}$.

Remark 2.1. From (2.1) it result that $\lim_{m \rightarrow \infty} \sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1$, for any $x \in J$.

Let a fixed function $w : I \rightarrow (0, \infty)$, called the weight function and the set functions

$$(2.3) \quad E_w(I) = \{f|f : I \rightarrow \mathbb{R} \text{ such that } wf \text{ is bounded on } I\}.$$

For $f \in E_w(I)$ there exists a positive constant $M(f)$, depending on f , such that $w(x)|f(x)| \leq M(f)$ for any $x \in I$. Then, for $m \in \mathbb{N}$ and $x \in J$, and taking in the end (2.2) into account, we have

$$\begin{aligned} \left| \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k}) \right| &\leq \sum_{k=0}^{p_m} \varphi_{m,k}(x) |f(x_{m,k})| \leq \frac{M(f)}{w(x)} \sum_{k=0}^{p_m} \varphi_{m,k}(x) \leq \\ &\leq \frac{M(f)}{w(x)} (1 + u_m(K)) \leq \frac{M(f)}{w(x)} (1 + u(K)), \end{aligned}$$

from where it results that the sum $\sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k})$ exists.

We consider the operators $(L_m)_{m \geq 1}$ defined by

$$(2.4) \quad (L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k})$$

for any $f \in E_w(I)$, $x \in J$ and $m \in \mathbb{N}$.

Proposition 2.1. The operators $(L_m)_{m \geq 1}$ are linear and positive on $E_w(I)$.

Proof. The proof follows immediately.

3 Main results

In the following, let s be fixed natural number, s even. For any $x \in I \cap J$ we suppose that $\psi_x^i \in E_w(I)$, where $i \in \{0, 1, \dots, s+2\}$. For $m \in \mathbb{N}$ and $i \in \{0, 1, \dots, s+2\}$ define

$$(3.1) \quad (T_i L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{p_m} (x_{m,k} - x)^i \varphi_{m,k}(x)$$

for any $x \in I \cap J$.

Theorem 3.1. *Let $x \in I \cap J$ and we suppose that there exist $\alpha_{s+2} \geq 0$ and $m(s) \in \mathbb{N}$ such that $\frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(s)$. If $\gamma \in \mathbb{R}$ verify $\gamma < s+2 - \alpha_{s+2}$ and $\delta > 0$, then*

$$(3.2) \quad \lim_{m \rightarrow \infty} m^\gamma \sum_{|x_{m,k} - x| \geq \delta} (x_{m,k} - x)^s \varphi_{m,k}(x) = 0.$$

If for the compact interval $K \subset I \cap J$ exist $m(s) \in \mathbb{N}$ and the constant $k_{s+2}(K) \in \mathbb{R}$, depending on K , such that for any $m \in \mathbb{N}$, $m \geq m(s)$ and $x \in K$ we have

$$(3.3) \quad \frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}} \leq k_{s+2}(K),$$

then the convergence given in (3.2) is uniform on K .

Proof. We have

$$\begin{aligned} \sum_{|x_{m,k} - x| \geq \delta} (x_{m,k} - x)^s \varphi_{m,k}(x) &\leq \frac{1}{\delta^2} \sum_{|x_{m,k} - x| \geq \delta} (x_{m,k} - x)^{s+2} \varphi_{m,k}(x) \leq \\ &\leq \sum_{k=0}^{p_m} (x_{m,k} - x)^{s+2} \varphi_{m,k}(x) = \frac{1}{\delta^2 m^{s+2}} (T_{s+2} L_m)(x), \end{aligned}$$

so

$$(3.4) \quad m^\gamma \sum_{|x_{m,k} - x| \geq \delta} (x_{m,k} - x)^s \varphi_{m,k}(x) \leq \frac{1}{\delta^2 m^{s+2-\gamma}} (T_{s+2} L_m)(x).$$

But

$$\frac{1}{\delta^2 m^{s+2-\gamma}} (T_{s+2} L_m)(x) = \frac{1}{\delta^2 m^{s+2-\alpha_{s+2}-\gamma}} \cdot \frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}}$$

and because $\gamma < s+2-\alpha_{s+2}$, we get $s+2-\alpha_{s+2}-\gamma > 0$. Because $\frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, it results that

$$\lim_{m \rightarrow \infty} \frac{1}{\delta^2 m^{s+2-\alpha_{s+2}-\gamma}} \cdot \frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}} = 0.$$

Considering the limit compute above, the fact that s is even and (3.4), we obtain (3.2).

Remark 3.1. In Theorem 3.1 we choose the smallest α_{s+2} and the bigger γ , if they exists.

In the following, we suppose that exists $M > 0$ such that the inequality

$$(3.5) \quad \sum_{k=0}^{p_m} \varphi_{m,k}(x) \leq M$$

holds for any $x \in J$ and any $m \in \mathbb{N}$.

Theorem 3.2. If $f \in E_w(I)$ is a s times differentiable function at $x \in I \cap J$ (if $s = 0$ we consider that f is continuous on $I \cap J$) and we suppose that exists $\alpha_{s+2} \geq 0$ and $m(s) \in \mathbb{N}$ such that $\frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, then for any γ which verify

$$(3.6) \quad \gamma < s + 2 - \alpha_{s+2}$$

we have

$$(3.7) \quad \lim_{m \rightarrow \infty} m^\gamma \left[(L_m f)(x) - \sum_{i=0}^s \frac{1}{m^i i!} (T_i L_m)(x) f^{(i)}(x) \right] = 0.$$

If $f \in E_w(I)$ is a s times differentiable function on I and for the compact interval $K \subset I \cap J$ exist $m(s) \in \mathbb{N}$ and the constant $k_{s+2}(K) \in \mathbb{R}$, depending on K , such that for any $m \in \mathbb{N}$, $m \geq m(s)$ and $x \in K$ we have

$$(3.8) \quad \frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}} \leq k_{s+2}(K),$$

then the convergence given in (3.7) is uniform on K .

Proof. According to Taylor's formula for the function f around x , we have

$$(3.9) \quad f(t) = \sum_{i=0}^s \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x)$$

where μ is a bounded function and $\lim_{t \rightarrow x} \mu(t-x) = 0$. Then exists a neighborhood $V = [-a, a]$ of the point 0 such that for any $\epsilon > 0$, exists $\delta_\epsilon > 0$, for any $h \in V$ with $|h| < \delta_\epsilon$, we have

$$(3.10) \quad |\mu(h)| < \epsilon.$$

If we replace t with $x_{m,k}$ in (3.9), multiply by $\varphi_{m,k}(x)$ and sum after k , when $k \in \{0, 1, \dots, p_m\}$, we obtain

$$\begin{aligned} (L_m f)(x) &= \sum_{k=0}^{p_m} \sum_{i=0}^s \frac{(x_{m,k} - x)^i}{i!} \varphi_{m,k}(x) f^{(i)}(x) + \\ &+ \sum_{k=0}^{p_m} (x_{m,k} - x)^s \varphi_{m,k}(x) \mu(x_{m,k} - x) = \\ &= \sum_{i=0}^s \frac{1}{m^i i!} \left[m^i \sum_{k=0}^{p_m} (x_{m,k} - x)^i \varphi_{m,k}(x) \right] f^{(i)}(x) + \\ &+ \sum_{k=0}^{p_m} (x_{m,k} - x)^s \varphi_{m,k}(x) \mu(x_{m,k} - x), \end{aligned}$$

or

$$(L_m f)(x) - \sum_{i=0}^s \frac{1}{m^i i!} (T_i L_m)(x) f^{(i)}(x) = \sum_{k=0}^{p_m} (x_{m,k} - x)^s \varphi_{m,k}(x) \mu(x_{m,k} - x)$$

and thus

$$(3.11) \quad m^\gamma \left[(L_m f)(x) - \sum_{i=0}^s \frac{1}{m^i i!} (T_i L_m)(x) f^{(i)}(x) \right] = (R_m f)(x),$$

where

$$(3.12) \quad (R_m f)(x) = m^\gamma \sum_{k=0}^{p_m} (x_{m,k} - x)^s \varphi_{m,k}(x) \mu(x_{m,k} - x).$$

Consider δ_ϵ from (3.10), $I_m = \{0, 1, \dots, p_m\} \cap \mathbb{N}$, $I_{m,1} = \{k \in I_m : |x_{m,k} - x| < \delta_\epsilon\}$ and $I_{m,2} = \{k \in I_m : |x_{m,k} - x| \geq \delta_\epsilon\}$. Then

$$\begin{aligned} |(R_m f)(x)| &\leq m^\gamma \sum_{k=0}^{p_m} (x_{m,k} - x)^s \varphi_{m,k}(x) |\mu(x_{m,k} - x)| = \\ &= m^\gamma \sum_{k \in I_{m,1}} (x_{m,k} - x)^s \varphi_{m,k}(x) |\mu(x_{m,k} - x)| + \\ &+ m^\gamma \sum_{k \in I_{m,2}} (x_{m,k} - x)^s \varphi_{m,k}(x) |\mu(x_{m,k} - x)| \end{aligned}$$

and taking (3.10) into account, and considering the fact that μ is bounded, so $\sup_{t \in V} |\mu(t)| = \eta$, we have

$$(3.13) \quad \begin{aligned} |(R_m f)(x)| &\leq m^\gamma \epsilon \sum_{k \in I_{m,1}} (x_{m,k} - x)^s \varphi_{m,k}(x) + \\ &+ m^\gamma \eta \sum_{k \in I_{m,2}} (x_{m,k} - x)^s \varphi_{m,k}(x). \end{aligned}$$

But $(x_{m,k} - x)^s \leq (2a)^s$, so

$$(3.14) \quad \sum_{k \in I_{m,1}} (x_{m,k} - x)^s \varphi_{m,k}(x) \leq (2a)^s \sum_{k \in I_{m,1}} \varphi_{m,k}(x) \leq (2a)^s \sum_{k=0}^{p_m} \varphi_{m,k}(x).$$

Taking (3.5) and (3.14) into account, we have that

$$(3.15) \quad m^\gamma \epsilon \sum_{k \in I_{m,1}} (x_{m,k} - x)^s \varphi_{m,k}(x) \leq m^\gamma (2a)^s M.$$

From (3.6), we have that $\gamma < s + 2 - \alpha_{s+2}$ and then from Theorem 3.1 we obtain $\lim_{m \rightarrow \infty} m^\gamma \sum_{k \in I_{m,2}} (x_{m,k} - x)^s \varphi_{m,k}(x) = 0$, thus for ϵ from (3.10), there exists $m(\epsilon) \in \mathbb{N}$, for any $m \in \mathbb{N}$, $m \geq m(\epsilon)$, we have

$$(3.16) \quad m^\gamma \eta \sum_{k \in I_{m,2}} (x_{m,k} - x)^s \varphi_{m,k}(x) < \epsilon.$$

Choose $\epsilon = \frac{1}{m[m^\gamma(2a)^s M+1]}$ and there exists $m(\epsilon) \in \mathbb{N}$, for any $m \in \mathbb{N}$, $m \geq m(\epsilon)$, from (3.13)-(3.16) it results that $|(R_m f)(x)| < \frac{1}{m}$, and so

$$(3.17) \quad \lim_{m \rightarrow \infty} (R_m f)(x) = 0.$$

From (3.11) and (3.13), (3.7) follows. For the second afirmation from Theorem 3.2, we apply in the proof above the Theorem 3.1.

For $s = 0$, respectively $s = 2$ in Theorem 3.2 we obtain the Corollary 3.1.

Corollary 3.1. *If $f \in E_w(I)$ is a s times differentiable function at $x \in I \cap J$ and we suppose that exist $\alpha_{s+2} \geq 0$ and $m(s) \in \mathbb{N}$ such that $\frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, then for any γ which verify*

$$(3.18) \quad \gamma < s + 2 - \alpha_{s+2},$$

we have

$$(3.19) \quad \lim_{m \rightarrow \infty} m^\gamma [(L_m f)(x) - (T_0 L_m)(x)f(x)] = 0$$

if $s = 0$, and

$$(3.20) \quad \lim_{m \rightarrow \infty} m^\gamma \left[(L_m f)(x) - (T_0 L_m)(x)f(x) - \frac{1}{m}(T_1 L_m)(x)f^{(1)}(x) - \frac{1}{2m^2}(T_2 L_m)(x)f^{(2)}(x) \right] = 0,$$

if $s = 2$.

If $f \in E_w(I)$ is a s times differentiable function on I and for the compact $K \subset I \cap J$ exist $m(s) \in \mathbb{N}$ and the constant $k_{s+2}(K) \in \mathbb{R}$, depending on K such that for any $m \in \mathbb{N}$, $m \geq m(s)$ and $x \in K$ we have

$$(3.21) \quad \frac{(T_{s+2}L_m)(x)}{m^{\alpha_{s+2}}} \leq k_{s+2}(K),$$

where $s \in \{0, 2\}$, then the convergences given in (3.19) and (3.20) are uniform on K .

Remark 3.2. *The relation (3.20) from Corollary 3.1 is a Voronovskaja-type identity.*

In the following, in every application we have $\sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1$, so $(T_0 L_m)(x) = 1$ for any $x \in J$, $m \in \mathbb{N}$, $u_m(K) = 0$ for any $K \subset I \cap J$ and $m \in \mathbb{N}$, $\alpha_2 = 1$, $\alpha_4 = 2$, $\gamma = 0$ if $s = 0$ and $\gamma = 1$ if $s = 2$.

In the following, by particularization of the sequence $x_{m,k}$, $m \in \mathbb{N}$, $k \in \{0, 1, \dots, p_m\} \cap \mathbb{N}_0$ and applying Corollary 3.1, we can obtain convergence theorems and Voronovskaja-type theorems for the operators from the first section of this paper. Because every application is a simple substitute in the Corollary 3.1, we won't replace anything.

Application 3.1. We study a particular case of the Stancu operators. Let $\alpha = 10$ and $\beta = -\frac{1}{2}$. We obtain $I = [0, 22]$, $K = J = [0, 1]$ and for any $f \in C([0, 22])$, $x \in [0, 1]$ and $m \in \mathbb{N}$

$$(P_m^{(10, -1/2)} f)(x) = \sum_{k=0}^m p_{m,k}(x) f\left(\frac{2k+20}{2m-1}\right),$$

where $\varphi_{m,k}(x) = p_{m,k}(x)$ and $x_{m,k} = \frac{2k+20}{2m-1}$, $k \in \{0, 1, \dots, m\}$. We obtain $(T_1 P_m^{(10, -1/2)})(x) = \frac{m(20+x)}{2m-1}$, $(T_2 P_m^{(10, -1/2)})(x) = m^2 \frac{4mx(1-x) + (20+x)^2}{(2m-1)^2}$ for any $m \in \mathbb{N}$ and $x \in [0, 1]$, $k_2(K) = \frac{5}{4}$, $k_4(K) = \frac{19}{16}$ (see [19]).

For the Bleimann-Butzer-Hahn operators and for the Meyer-König and Zeller operators we only give the convergence theorems.

Application 3.2. We consider $I = J = [0, \infty)$, $E_w(I) = C_B([0, \infty))$, $w(x) = 1$ for any $x \in [0, \infty)$, $K = [0, b]$, $b > 0$, $p_m = m$, $x_{m,k} = \frac{k}{m+1-k}$, $\varphi_{m,k}(x) = \frac{1}{(1+x)^m} \binom{m}{k} x^k$, $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$, $x \in [0, \infty)$ and in this case we obtain the Bleimann-Butzer-Hahn operators. We have $(T_1 L_m)(x) = -mx \left(\frac{x}{1+x}\right)^m$, $x \in K$ and $k_2(K) = 4b(1+b)^2$ for $m \geq 24(1+b)$ (see [17]).

Application 3.3. If $I = J = [0, 1]$, $w(x) = 1$ for any $x \in [0, 1]$, $E_w(I) = B([0, 1])$, $K = [0, 1]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m+k}$, $(\varphi_{m,k})(x) = \binom{m+k}{k} (1-x)^{m+1} x^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x \in [0, 1]$, we obtain the Meyer-König and Zeller operators and we have $(T_1 Z_m)(x) = 0$, $m \in \mathbb{N}$, $x \in [0, 1]$, and $k_2(K) = 2$ (see [16]).

Application 3.4. If $I = J = [0, \infty)$, $w(x) = \frac{1}{1+x^2}$ for any $x \in [0, \infty)$, $E_w(I) = C_2([0, \infty))$, $K = [0, b]$, $b > 0$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = e^{-mx} \frac{(mx)^k}{k!}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x \in [0, \infty)$, we obtain the Mirakjan-Favard-Szász operators. We have $(T_1 S_m)(x) = 0$, $(T_2 S_m)(x) = mx$, $m \in \mathbb{N}$, $x \in [0, \infty)$, $k_2(K) = b$ and $k_4(K) = 3b^2 + b$ (see [16]).

Application 3.5. Let $I = J = [0, \infty)$, $w(x) = \frac{1}{1+x^2}$ for any $x \in [0, \infty)$, $E_w(I) = C_2([0, \infty))$, $K = [0, b]$, $b > 0$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = (1+x)^{-m} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $x \in [0, \infty)$. In this case we obtain the Baskakov operators and we have $(T_1 V_m)(x) = 0$, $(T_2 V_m)(x) = mx(1+x)$, $m \in \mathbb{N}$, $x \in [0, \infty)$, $k_2(K) = b(1+b)$ and $k_4(K) = 9b^4 + 10b^3 + 10b^2 + b$ (see [16]).

Application 3.6. If $I = J = [0, \infty)$, $w(x) = 1$ for any $x \in [0, \infty)$, $E_w(I) = C([0, \infty))$, $K = [0, b]$, $b > 0$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{-\frac{(k+m)x}{1+x}}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x \in [0, \infty)$, we obtain the Ismail-May operators. We have $(T_1 R_m)(x) = 0$, $(T_2 R_m)(x) = mx(1+x)^2$, $m \in \mathbb{N}$, $x \in [0, \infty)$, $k_2(K) = 1 + b(1+b)^2$ and $k_4(K) = 1 + b^2(1+b)^4$ (see [18]).

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