

# A Class of Holomorphic Functions Defined Using a Differential Operator

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*Dedicated to Professor Dumitru Acu on his 60th anniversary*

## Abstract

By using the differential operator  $D^n f(z)$ ,  $z \in U$  (Definition 1), we introduce a class of holomorphic functions. We let  $M_n(\alpha)$  denote this class and we obtain some inclusion relations.

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## 1 Introduction and preliminaries

Denote by  $U$  the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let  $\mathcal{H}[U]$  be the space of holomorphic function in  $U$ .

We let:

$$A_n = \{f \in \mathcal{H}[U], f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$$

with  $A_1 = A$ .

Let

$$K = \left\{ f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}$$

be the class of normalized convex functions in  $U$ .

If  $f$  and  $g$  are analytic functions in  $U$ , then we say that  $f$  is subordinate to  $g$ , written  $f \prec g$ , or  $f(z) \prec g(z)$ , if there is a function  $w$  analytic in  $U$  with  $w(0) = 0$ ,  $|w(z)| < 1$ , for all  $z \in U$  such that  $f(z) = g[w(z)]$  for  $z \in U$ . If  $g$  is univalent, then  $f \prec g$  if and only if  $f(0) = g(0)$  and  $f(U) \subset g(U)$ .

We use the following subordination results.

**Lemma A.** (Hallenbeck and Ruscheweyh [1, p. 71]) *Let  $h$  be a convex function with  $h(0) \equiv a$  and let  $\gamma \in \mathbb{C}^*$  be a complex number with  $\operatorname{Re} \gamma \geq 0$ . If  $p \in \mathcal{H}[U]$  with  $p(0) = a$  and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1} dt.$$

The function  $q$  is convex and is the best dominant. (The definition of best dominant is given in [2]).

**Lemma B.** (S. S. Miller and P. T. Mocanu [2]) *Let  $g$  be a convex function in  $U$  and let*

$$h(z) = g(z) + n\alpha z g'(z)$$

where  $\alpha > 0$  and  $n$  is a positive integer.

If  $p(z) = g(0) + p_n z^n + \dots$  is holomorphic in  $U$  and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

**Definition 1.** (G. S. Sălăgean [4]). For  $f \in A$  and  $n \geq 0$  we define the operator  $D^n f$  by

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^{n+1} f(z) &= z[D^n f(z)]', \quad z \in U. \end{aligned}$$

## 2 Main results

**Definition 2.** For  $\alpha < 1$  and  $n \in \mathbb{N}$ , we let  $M_n(\alpha)$  denote the class of functions  $f \in A$  which satisfy the inequality:

$$\operatorname{Re} [D^n f(z)]' > \alpha.$$

**Theorem 1.** If  $\alpha < 1$  and  $n \in \mathbb{N}$ , then we have

$$M_{n+1}(\alpha) \subset M_n(\delta),$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha) \ln 2.$$

The result is sharp.

**Proof.** Let  $f \in M_{n+1}(\alpha)$ . We have

$$(1) \quad D^{n+1} f(z) = z[D^n f(z)]', \quad z \in U.$$

Differentiating (1) we obtain

$$(2) \quad [D^{n+1} f(z)]' = [D^n f(z)]' + z[D^n f(z)]''.$$

If we let  $p(z) = [D^n f(z)]'$ , then (2) becomes

$$[D^{n+1} f(z)]' = p(z) + zp'(z).$$

Since  $f \in M_{n+1}(\alpha)$ , by using Definition 2 we have

$$\operatorname{Re} [p(z) + zp'(z)] > \alpha$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1 + (2\alpha - 1)z}{1 + z} \equiv h(z).$$

By using Lemma A, we have:

$$p(z) \prec q(z) \prec h(z)$$

where

$$\begin{aligned} q(z) &= \frac{1}{z} \int_0^z h(t) dt = \frac{1}{z} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} dt \\ &= \frac{1}{z} \int_0^z \left[ 2\alpha - 1 + 2(1 - \alpha) \frac{1}{1 + t} \right] dt = 2\alpha - 1 + 2(1 - \alpha) \frac{\ln(1 + z)}{z}. \end{aligned}$$

Function  $q$  is convex and it is the best dominant.

From  $p(z) \prec q(z)$ , it results that

$$\operatorname{Re} p(z) > \operatorname{Re} q(1) = \delta = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha) \ln 2$$

from which we deduce  $M_{n+1}(\alpha) \subset M_n(\delta)$ .

**Theorem 2.** *Let  $g$  be a convex function,  $g(0) = 1$  and let  $h$  be a function such that*

$$h(z) = g(z) + zg'(z).$$

*If  $f \in M_n(\alpha)$  and verifies the differential subordination*

$$(3) \quad [D^{n+1}f(z)]' \prec h(z)$$

*then*

$$[D^n f(z)]' \prec g(z)$$

*and this result is sharp.*

**Proof.** From  $D^{n+1}f(z) = z[D^n f(z)]'$ , we obtain

$$[D^{n+1}f(z)]' = [D^n f(z)]' + z[D^n f(z)]''.$$

If we let  $p(z) = [D^n f(z)]'$ ,  $p'(z) = [D^n f(z)]''$  then we obtain

$$[D^{m+1}f(z)]' = p(z) + zp'(z)$$

and (3) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z) \equiv h(z).$$

By using Lemma B, we have

$$p(z) \prec g(z), \text{ i.e., } [D^m f(z)]' \prec g(z)$$

and this result is sharp.

**Theorem 3.** *Let  $g$  be a convex function,  $g(0) = 1$ , and*

$$h(z) = g(z) + zg'(z).$$

*If  $f \in M_n(\alpha)$  and verifies the differential subordination*

$$(4) \quad [D^n f(z)]' \prec h(z), \quad z \in U,$$

*then*

$$\frac{D^n f(z)}{z} \prec g(z), \quad z \in U$$

*and this result is sharp.*

**Proof.** We let  $p(z) = \frac{D^n f(z)}{z}$ ,  $z \in U$ , and we obtain

$$D^n f(z) = zp(z).$$

By differentiating, we obtain

$$[D^n f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (4) becomes

$$p(z) + zp'(z) \prec h(z) \equiv g(z) + zg(z).$$

By using Lemma B, we have

$$p(z) \prec g(z),$$

i.e.

$$\frac{D^n f(z)}{z} \prec g(z).$$

We remark that in the case of meromorphic functions a similar results was obtained by M. Pap in [3].

## References

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