General Mathematics Vol. 13, No. 4 (2005), 33–38

New Classes of Univalent Functions

Amelia Anca Holhoş

Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In this paper we define and study some properties of the classes $B_n(A)$ of univalent functions.

2000 Mathematics Subject Classification: 30C45.

Keywords: Univalent functions, Sălăgean differential operator.

1 Introduction

Let U denote the open unit disc: $U = \{z\ ;\ z \in \mathbb{C}\ ,\ |z| < 1\}$, let $\mathcal{A}$ denote the class of functions

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic in U, and let S denote the class of functions of this form wich are analytic and univalent in U.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$ in the unit disk U if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha$$

for all $z \in U$. The class of starlike functions of order α it is denoted by $S^*(\alpha)$ and $S^*(0) = S^*$.

For $f \in S$ we define the Sălăgean's differential operator D^n (see [2])

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z) \end{aligned}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad ; \quad n \in \mathbb{N}^* = \{1, 2, 3, ...\}.$$

For $\alpha \in [0,1)$ and $n \in \mathbb{N}$ Sălăgean introduced the class of n-starlike functions of order α

$$S_n(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, \quad z \in U \right\}$$

and for this class be obtained the following results:

Theorem 1.(see [2]) *For $n \in \mathbb{N}$ and $\alpha \in [0,1)$ we have $S_{n+1}(\alpha) \subset S_n(\delta(\alpha))$, where*

$$\delta(\alpha) = \begin{cases} \dfrac{2\alpha - 1}{2(1 - 2^{1-2\alpha})}, & \alpha \in [0,1) \setminus \left\{ \dfrac{1}{2} \right\} \\ \\ \dfrac{1}{2 \ln 2}, & \alpha = \dfrac{1}{2} \end{cases}$$

and the result is sharp.

Corollary 1.(see [2]) $S_{n+1}(\alpha) \subset S_n(\alpha)$, *for all $n \in \mathbb{N}$ and $\alpha \in [0,1)$.*

Remark 1. *Since*

$$S_n(\alpha) \subset S_{n-1}(\alpha) \subset ... \subset S_1(\alpha) \subset S_0(\alpha) = S^*(\alpha)$$

and $S^*(\alpha) \subseteq S^*(0) = S^*$, *all functions of* $S_n(\alpha)$, $n \in \mathbb{N}$ *and* $\alpha \in [0,1)$ *are starlike and univalent, and because* $S_1(\alpha) = K(\alpha) \subseteq K(0) = K$, *all functions of* $S_n(\alpha)$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\alpha \in [0,1)$ *are convex.*

Theorem 2.(see [1]) *Let* $-1 \leq B \leq 1$, *let* $A, \beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and* $A \neq B$ *such that* $\beta(A-B) \in \mathbb{R}$, $\operatorname{Re}(\beta+\gamma) > 0$,

$$\begin{cases} \operatorname{Re}\left[\beta(1+A) + \gamma(1+B)\right] \geq 0, & \text{if } B \neq -1 \\ \operatorname{Re}\left[\beta(1-A) + \gamma(1-B)\right] \geq 0, & \text{if } B \neq 1 \end{cases}$$

and

$$|\beta A + \gamma B| \leq |\beta + \gamma| .$$

Then the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz}$$

has a univalent solution q, *with* $q(0) = 1$, *given by*

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta\frac{A-B}{B}}}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta\frac{A-B}{B}}\,dt} - \dfrac{\gamma}{\beta}, & \text{if } \quad B \neq 0 \\ \\ \dfrac{z^{\beta+\gamma}e^{\beta Az}}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}e^{\beta At}dt} - \dfrac{\gamma}{\beta}, & \text{if } \quad B = 0 \end{cases}.$$

If the function p *is analytic in* U *and satisfy the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1+Az}{1+Bz}$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz}$$

and the function q *it is the best dominant.*

Definition 1. *Let $n \in \mathbb{N}^*$ and let $A \in (0,1)$. We define the class*

$$\mathcal{B}_n(A) = \left\{ f \in \mathcal{A} : \frac{D^{n+1}f(z)}{D^n f(z)} \prec 1 + Az, \quad z \in U \right\}.$$

Remark 2. $\mathcal{B}_n(A) \subset S_n(1-A)$. *From this we deduce that all functions of $B_n(A)$ are univalent.*

2 Some properties of the classes $\mathcal{B}_n(A)$.

Theorem 3. *Let $n \in N^*$ and $A \in (0,1)$, The $\mathcal{B}_{n+1}(A) \subset \mathcal{B}_n(A)$.*

Proof. Let $f \in \mathcal{B}_{n+1}(A)$. Then from definition of the class $\mathcal{B}_n(A)$ we have

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec 1 + Az.$$

Let

$$(1) \qquad p(z) = \frac{D^{n+1}f(z)}{D^n f(z)}, \quad z \in \dot{U} \quad \text{and} \quad \mathrm{p}(0) = 1.$$

We observe that p is analityc in U, and from definition of the Sălăgean's differential operator D^n we have

$$(2) \qquad p(z) + \frac{zp'(z)}{p(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec 1 + Az.$$

Since the conditions by the hypothessis of Theorem 4 are satisfied for $B = 0$, $\beta = 1$, $\gamma = 0$ and $A \in (0,1)$, then the differential equation

$$q(z) + \frac{zq'(z)}{q(z)+1} = 1 + Az$$

has a univalent solution given by

$$q(z) = \frac{ze^{Az}}{\dfrac{e^{Az}}{A} - \dfrac{1}{A}}$$

and because p is analytic in U and satisfies (2) we have

$$p(z) \prec q(z) \prec 1 + Az \tag{3}$$

and q is the best dominant. From (3) and (1) we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec \frac{ze^{Az}}{\dfrac{e^{Az}}{A} - \dfrac{1}{A}} \prec 1 + Az$$

therefore $f \in \mathcal{B}_n(A)$ and hence $\mathcal{B}_{n+1}(A) \subset \mathcal{B}_n(A)$.

Theorem 4. *If $A \in (0,1)$, $c \geq A - 1$ and $f \in \mathcal{B}_n(A)$, then $I(f) \in \mathcal{B}_n(A)$ where*

$$I(f)(z) = \frac{c+1}{z^c} \int_0^z f(t)\, t^{c-1} dt.$$

Proof. Since $f \in \mathcal{B}_n(A)$ we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec 1 + Az. \tag{4}$$

If we denote

$$g(z) = I(f)(z) = \frac{c+1}{z^c} \int_0^z f(t)\, t^{c-1} dt$$

then

$$f(z) = \frac{cg(z)}{c+1} + \frac{zg'(z)}{c+1}$$

and

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{D^{n+1}g(z)}{D^n g(z)} \frac{c + \dfrac{D^{n+2}g(z)}{D^{n+1}g(z)}}{c + \dfrac{D^{n+1}g(z)}{D^n g(z)}}.$$

If we get

$$p(z) = \frac{D^{n+1}g(z)}{D^n g(z)}$$

then

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{D^{n+2}g(z)}{D^{n+1}g(z)}$$

and

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} = p(z) + \frac{zp'(z)}{c+p(z)}. \tag{5}$$

From (4) and (5) and because the conditions of Theorem 4 are satisfied for $B=0$, $\beta=1$, $\gamma=c\geq A-1>-1$ then we have that the differential equation

$$q(z) + \frac{zq'(z)}{q(z)+c} = 1+Az$$

has a univalent solution given by

$$q(z) = \frac{z^{1+c}e^{Az}}{\int_0^z t^c e^{At}dt} - c.$$

Since p is analityc in U, then

$$p(z) \prec q(z) \prec 1+Az$$

and q is the best dominant. Hence

$$p(z) = \frac{D^{n+1}g(z)}{D^{n}g(z)} \prec \frac{z^{1+c}e^{Az}}{\int_0^z t^c e^{At}dt} - c \prec 1+Az$$

then $g\in\mathcal{B}_n(A)$ and $I(f)\in\mathcal{B}_n(A)$.

References

[1] S. S. Miller, P. T. Mocanu, *Univalent solutions of Briot - Bouquet differential equations*, J.of Differential Equations, 56, 3(1985), 297 - 309.

[2] G. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math. (Springer-Verlag) 1013(1983), 362-372.