Educaţia Matematică Vol. 1, Nr. 2 (2005), 19–26

Proving some geometric inequalities by using complex numbers

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Abstract

Let ABC be a triangle and let R and r be its circumradius and inradius, respectively. One of the most important result in Triangle Geometry is Euler's inequality $R \geq 2r$. There are many proofs for this inequality (geometric, trigonometric, analytic etc.). We refer to the books [3] and [4] for some useful discussions on this inequality.

In this note we will give other proofs by using complex numbers. The method of complex numbers in Geometry is a powerful technique. For other applications we refer to our new book [2].

2000 Mathematical Subject Classification: 26D99, 97D50

Theorem 1. *Let P be an arbitrary point in the plane of triangle ABC. Then*

$$\alpha PB \cdot PC + \beta PC \cdot PA + \gamma PA \cdot PB \geq \alpha\beta\gamma,$$

where α, β, γ are the side lengths of triangle ABC.

Proof. Let us consider the origin of the complex plane at P and let a, b, c be the affixes of vertices of triangle ABC. From the algebraic identity

$$\frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)} + \frac{ab}{(c-a)(c-b)} = 1 \tag{1}$$

by passing to moduli, it follows that

$$\frac{|b||c|}{|a-b||a-c|} + \frac{|c||a|}{|b-c||b-a|} + \frac{|a||b|}{|c-a||c-b|} \geq 1. \tag{2}$$

Taking into account that $|a| = PA$, $|b| = PB$, $|c| = PC$ and $|b-c| = \alpha$, $|c-a| = \beta$, $|a-b| = \gamma$, (2) is equivalent to

$$\frac{PB \cdot PC}{\beta\gamma} + \frac{PC \cdot PA}{\gamma\alpha} + \frac{PA \cdot PB}{\alpha\beta} \geq 1,$$

i.e. the desired inequality.

Remarks. 1) If P is the circumcenter O of triangle ABC we can derive Euler's inequality $R \geq 2r$. Indeed, in this case the inequality is equivalent to $R^2(\alpha+\beta+\gamma) \geq \alpha\beta\gamma$. Therefore we can write

$$R^2 \geq \frac{\alpha\beta\gamma}{\alpha+\beta+\gamma} = \frac{\alpha\beta\gamma}{2s} = \frac{4R}{2s} \cdot \frac{\alpha\beta\gamma}{4R} = 2R \cdot \frac{area[ABC]}{s} = 2Rr,$$

hence $R \geq 2r$.

2) We can obtain the inequality

$$R^2(\alpha+\beta+\gamma) \geq \alpha\beta\gamma \tag{3}$$

by a different argument, but also by using complex numbers. This alternative proof is given in our book [1]. Indeed, with the notations in the proof of Theorem 1, we have the identity

$$a^2(b-c) + b^2(c-a) + c^2(a-b) = (a-b)(b-c)(c-a). \tag{4}$$

Passing to moduli and using the well-known triangle inequality, we obtain

$$|a-b||b-c||c-a| \leq |a|^2|b-c| + |b|^2|c-a| + |c|^2|a-b|. \tag{5}$$

Suppose that the circumcenter O of triangle ABC is the origin of the complex plane. Then $|a| = |b| = |c| = R$ and (5) is equivalent to inequality (3).

3) If P is the centroid G of triangle ABC, we derive the following inequality involving the medians $m_\alpha, m_\beta, m_\gamma$:

$$\frac{m_\alpha m_\beta}{\alpha\beta} + \frac{m_\beta m_\gamma}{\beta\gamma} + \frac{m_\gamma m_\alpha}{\gamma\alpha} \geq \frac{9}{4},$$

with equality if and only if triangle ABC is equilateral.

Some Olympiad-calliber problems are directly connected to the result contained in Theorem 1. The first such problem deals with the case of equality when triangle ABC is acute-angled.

Problem 1. *Let ABC be an acute-angled triangle and let P be a point in its interior. Prove that*

$$\alpha \cdot PB \cdot PC + \beta \cdot PC \cdot PA + \gamma \cdot PA \cdot PB = \alpha\beta\gamma,$$

if and only if P is the orthocenter of triangle ABC.

(1998 Chinese Mathematical Olympiad)

Solution. Let P be the origin of the complex plane and let a, b, c be the affixes of A, B, C, respectively. The relation in the problem is equivalent to

$$|ab(a-b)| + |bc(b-c)| + |ca(c-a)| = |(a-b)(b-c)(c-a)|.$$

Let

$$z_1 = \frac{ab}{(a-c)(b-c)}, \quad z_2 = \frac{bc}{(b-a)(c-a)}, \quad z_3 = \frac{ca}{(c-b)(a-b)}.$$

It follows that

$$|z_1| + |z_2| + |z_3| = 1 \quad \text{and} \quad z_1 + z_2 + z_3 = 1,$$

the latter from identity (1) in the previous problem.

We will prove that P is the orthocenter of triangle ABC if and only if z_1, z_2, z_3 are positive real numbers. Indeed, if P is the orthocenter, then, since the triangle ABC is acute-angled, it follows that P is in the interior of ABC. Hence there are positive real numbers r_1, r_2, r_3 such that

$$\frac{a}{b-c} = -r_1 i, \quad \frac{b}{c-a} = -r_2 i, \quad \frac{c}{a-b} = -r_3 i,$$

implying $z_1 = r_1 r_2 > 0$, $z_2 = r_2 r_3 > 0$, $z_3 = r_3 r_1 > 0$ and we are done. Conversely, suppose that z_1, z_2, z_3 are all positive real numbers. Because

$$-\frac{z_1 z_2}{z_3} = \left(\frac{b}{c-a}\right)^2, \quad -\frac{z_2 z_3}{z_1} = \left(\frac{c}{a-b}\right)^2, \quad -\frac{z_3 z_1}{z_2} = \left(\frac{a}{b-c}\right)^2$$

it follows that

$$\frac{a}{b-c}, \ \frac{b}{c-a}, \ \frac{c}{a-b}$$

are pure imaginary numbers, thus $AP \perp BC$ and $BP \perp CA$, showing that P is the orthocenter of triangle ABC.

Problem 2. *Let G be the centroid of triangle ABC and let R_1, R_2, R_3 be the circumradii of triangles GBC, GCA, GAB, respectively. Then*

$$R_1 + R_2 + R_3 \geq 3R,$$

where R is the circumradius of triangle ABC.

Solution. In Theorem 1, let P be the centroid G of triangle ABC. Then

$$\alpha \cdot GB \cdot GC + \beta \cdot GC \cdot GA + \gamma \cdot GA \cdot GB \geq \alpha\beta\gamma, \tag{6}$$

where α, β, γ are the side lengths of triangle ABC.

But

$$\alpha \cdot GB \cdot GC = 4R_1 \cdot area[GBC] = 4R_1 \cdot \frac{1}{3} area[ABC]$$

and the other two relations:

$$\beta \cdot GC \cdot GA = 4R_2 \cdot \frac{1}{3} area[ABC], \quad \gamma \cdot GA \cdot GB = 4R_3 \cdot \frac{1}{3} area[ABC].$$

Hence (6) is equivalent to

$$\frac{4}{3}(R_1 + R_2 + R_3) \cdot area[ABC] \geq 4R \cdot area[ABC],$$

i.e. $R_1 + R_2 + R_3 \geq 3R$, as desired.

Problem 3. *Let ABC be a triangle and let P be a point in its interior. Let R_1, R_2, R_3 be the radii of the circumcircles of triangles PBC, PCA, PAB, respectively. Lines PA, PB, PC intersect sides BC, CA, AB at A_1, B_1, C_1, respectively. Denote*

$$k_1 = \frac{PA_1}{AA_1}, \quad k_2 = \frac{PB_1}{BB_1}, \quad k_3 = \frac{PC_1}{CC_1}.$$

Prove that

$$k_1R_1 + k_2R_2 + k_3R_3 \geq R,$$

where R is the circumradius of triangle ABC.

(2004 Romanian IMO Team Selection Test)

Solution. Note that

$$k_1 = \frac{area[PBC]}{area[ABC]}, \quad k_2 = \frac{area[PCA]}{area[ABC]}, \quad k_3 = \frac{area[PAB]}{area[ABC]}.$$

But $area[ABC] = \dfrac{\alpha\beta\gamma}{4R}$ and $area[PBC] = \dfrac{\alpha \cdot PB \cdot PC}{4R_1}$. Other two similar relations for $area[PCA]$ and $area[PAB]$ hold.

The desired inequality is equivalent to

$$R\frac{\alpha \cdot PB \cdot PC}{\alpha\beta\gamma} + R\frac{\beta \cdot PC \cdot PA}{\alpha\beta\gamma} + R\frac{\gamma \cdot PA \cdot PB}{\alpha\beta\gamma} \geq R,$$

which reduces to the inequality in Theorem 1.

In the case when triangle ABC is acute-angled, from Problem 1 it follows that equality holds if and only if P is the orthocenter of ABC.

Theorem 2. *Let P be an arbitrary point in the plane of triangle ABC. Then*

$$\alpha \cdot PA^2 + \beta \cdot PB^2 + \gamma \cdot PC^2 \geq \alpha\beta\gamma. \tag{7}$$

Proof. Let us consider the origin of the complex plane at the point P and let a, b, c be the affixes of the vertices of triangle ABC. The following identity is easy to verify:

$$\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} = 1. \tag{8}$$

By passing to moduli it follows that

$$1 = \left|\sum_{cyc} \frac{a^2}{(a-b)(a-c)}\right| \le \sum_{cyc} \frac{|a|^2}{|a-b||a-c|}$$

Taking into account that $|a| = PA$, $|b| = PB$, $|c| = PC$ and $|b-c| = \alpha$, $|c-a| = \beta$, $|a-b| = \gamma$, the previous inequality is equivalent to (7).

Remarks. 1) If P is the circumcenter O of triangle ABC, then $PA = PB = PC = R$ and from (8) we derive again inequality (3), which is equivalent to Euler's inequality $R \ge 2r$.

2) If P is the centroid G of triangle ABC, then

$$PA^2 = \frac{1}{9}[2(\beta^2+\gamma^2)-\alpha^2], \quad PB^2 = \frac{1}{9}[2(\gamma^2+\alpha^2)-\beta^2],$$

$$PC^2 = \frac{1}{9}[2(\alpha^2+\beta^2)-\gamma^2]$$

and (7) is equivalent to

$$2\sum_{cyc}(\beta^2+\gamma^2) \ge 9\alpha\beta\gamma + \alpha^3+\beta^3+\gamma^3. \tag{9}$$

3) If P is the incenter I of triangle ABC, then

$$PA = \frac{r}{\sin\frac{A}{2}}, \quad PB = \frac{r}{\sin\frac{B}{2}}, \quad PC = \frac{r}{\sin\frac{C}{2}}$$

and is not difficult to see that we have equality in (7).

4) A different proof for (7), by using a variant of Lagrange's identity, is given in the book [4].

Theorem 3. *Let P be an arbitrary point in the plane of triangle ABC. Then*

$$\alpha \cdot PA^3 + \beta \cdot PB^3 + \gamma \cdot PC^3 \geq 3\alpha\beta\gamma PG, \tag{10}$$

where G is the centroid of triangle ABC.

Proof. The identity

$$x^3(y-z) + y^3(z-x) + z^3(x-y) = (x-y)(y-z)(z-x)(x+y+z) \tag{11}$$

holds for any complex numbers x, y, z. Passing to moduli, we obtain

$$|x|^3|y-z| + |y|^3|z-x| + |z|^3|x-y| \geq |x-y||y-z||z-x||x+y+z| \tag{12}$$

Let a, b, c, z_P be the affixes of points A, B, C, P, respectively. In (12) consider $x = z_P - a$, $y = z_P - b$, $z = z_P - c$ and obtain inequality (10).

Remarks. 1) If P is the circumcenter O of triangle ABC, after some elementary transformations, (10) becomes

$$\frac{R^2}{6r} \geq OG. \tag{13}$$

2) Squaring both sides of (13), we obtain

$$R^2 \geq 36r^2 \cdot OG^2. \tag{14}$$

Using the relation $OG^2 = R^2 - \frac{1}{9}(\alpha^2 + \beta^2 + \gamma^2)$, (14) is equivalent to

$$R^2(R^2 - 4r^2) \geq 4r^2[8R^2 - (\alpha^2 + \beta^2 + \gamma^2)]. \tag{15}$$

The inequality (15) improves Euler's inequality for the class of obtuse triangles. This is equivalent to proving that $\alpha^2 + \beta^2 + \gamma^2 < 8R^2$ in any such triangle. The last relation can be written as $\sin^2 A + \sin^2 B + \sin^2 C < 2$, or $\cos^2 A + \cos^2 B - \sin^2 C > 0$. That is

$$\frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} - 1 + \cos^2 C > 0,$$

which reduces to $\cos(A + B)\cos(A - B) + \cos^2 C > 0$. This is equivalent to $\cos C[\cos(A - B) - \cos(A + B)] > 0$, i.e. $\cos A \cos B \cos C < 0$, which is clearly true.

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